KEY TO ADVANCED TRIGONOMETRY

BY

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PREFACE

Solutions in this Key have been given in considerable detail, and it is hoped that clearness has nowhere been sacrificed to brevity. Also, in order to make the book as useful as possible, both to busy teachers and to private students, alternative methods of solution have been supplied where suitable. The Key therefore forms to some extent a supplementary teaching manual.

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ABBREVIATIONS, ETC.

To save space a number of obvious abbreviations are used, such as **eqn.** for equation, **numr.** for numerator, and **l.h.s.** for left-hand side.

References to figures (**Fig.**) and pages (**p.**) are to the *Advanced Trigonometry* itself. *E.T.* refers to Durell and Wright's Elementary Trigonometry.

The following conventions are also used :

(i) In the text of *Advanced Trigonometry*, certain results are numbered; e.g. in Ch. X, exp (z_1) exp $(z_2) = \exp(z_1 + z_2)$ is numbered (7). In the Key this would be referred to as **eqn. (7) of Ch. X**, or if the reference occurred in Ch. X of the Key, simply as **eqn. (7)**.

(ii) Similarly the *illustrative* examples which occur in the text will be referred to as **Ex. 5 of Ch. XIV**, or as **Ex. 5** if the reference occurs in Ch. XIV.

(iii) A reference such as **No. 16** or **(Ex.) XI. a, No. 16** is to a question in one of the exercises, and if it has no prefix the example belongs to the exercise in which the reference occurs.

CHAPTER I

EXERCISE I. a. (p. 1.)

1. The cos A formula may be used when it is not necessary to introduce logarithms. For logarithmic work, the other formulae are better; to find all the angles of a triangle, the formulae for $\tan \frac{1}{2}A$, $\tan \frac{1}{2}B$, $\tan \frac{1}{2}C$ are best (with $A + B + C = 180°$ as a check) as they only require the logarithms of s, $s - a$, $s - b$, $s - c$.

13. Impossible because $12{\cdot}3 < 16{\cdot}9 \sin 51°$; cf. No. 14.

14. The length of the perpendicular from B to AC $= c \sin A \simeq 5{\cdot}1$; $\therefore$ there are no solutions if $a < 5{\cdot}1$, one solution if $a = 5{\cdot}1$, two solutions if $5{\cdot}1 < a < 14{\cdot}5$, and one if $a \geqslant 14{\cdot}5$.

15. Draw AN perp. to BC; since $AC_1 = AC_2$, N is mid-point of C_1C_2; $a_1 + a_2 = 2BN$; $a_1a_2 = BC_1 . BC_2 = BN^2 - CN^2 = BA^2 - CA^2$; $A_1 + A_2 = 2\angle BAN$.

16. (i) $a_1 \sim a_2 = C_1C_2 = 2CN = 2\sqrt{(AC^2 - AN^2)}$;

(ii) $\sin \frac{1}{2}(A_1 - A_2) = \sin NAC_1 = \dfrac{NC_1}{AC_1} = \dfrac{\frac{1}{2}(a_1 - a_2)}{b}$.

17. $(a_1 - a_2)^2 = 4NC^2$; $(a_1 + a_2)^2 \tan^2 B = 4NB^2 \tan^2 B = 4AN^2$; add.

18. (i) Eliminate a_1, a_2 from $a_1 + a_2 = 2c\cos B$, $a_1a_2 = c^2 - b^2$, $a_1 = 3a_2$; the first and third give $a_1 = 3a_2 = \frac{3}{2}c \cos B$;

$$\therefore\ c^2 - b^2 = a_1a_2 = \tfrac{3}{4}c^2 \cos^2 B = \tfrac{3}{4}c^2(1 - \sin^2 B).$$

(ii) $C_1 + C_2 = 180°$; $\therefore$ $C_1 = 60°$; $\therefore$ $c \sin B = AN = b \sin 60°$.

19. $\triangle ABC_1 : \triangle ABC_2 = BC_1 : BC_2 = a_1 : a_2$; $\therefore$ $2a_1 = 3a_2$; eliminate as in No. 18 (i).

20. From No. 15, $A_1 = 2A_2 = \frac{1}{3}(360° - 4B)$;

$$\therefore\ \angle NAC_1 = \tfrac{1}{2}(A_1 - A_2) = \tfrac{1}{3}(90° - B);$$

$\therefore$ $c \sin B = AN = b \cos \dfrac{90° - B}{3}$; use $4\cos^3\theta - 3\cos\theta = \cos 3\theta$;

$$\therefore\ 4\left(\frac{c \sin B}{b}\right)^3 - 3\left(\frac{c \sin B}{b}\right) = \cos(90° - B) = \sin B;\ \text{simplify.}$$

22. $b \sin A = a \sin \theta$; $\therefore$ $a^2 - b^2 \sin^2 A = a^2 - a^2 \sin^2\theta = a^2 \cos^2\theta$; also $b \cos A = \dfrac{a \sin\theta}{\sin A} . \cos A$; $\therefore$ formula becomes

$$c = \frac{a \sin\theta}{\sin A} . \cos A \pm a \cos\theta = \frac{a}{\sin A}(\sin\theta \cos A \pm \cos\theta \sin A).$$

23. Put $2bc(1-\cos A)=(b-c)^2\tan^2\theta$, then $\tan^2\theta=\dfrac{2bc\,.\,2\sin^2\frac{A}{2}}{(b-c)^2}$,

$$\therefore\ \tan\theta=\frac{2\sin\frac{A}{2}\,.\,\sqrt{(bc)}}{b\sim c}\quad\text{and}$$

$$a^2=(b-c)^2+(b-c)^2\tan^2\theta=(b-c)^2\sec^2\theta\,;$$

$$\therefore\ a=(b\sim c)\sec\theta.$$

24. $\dfrac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}=\dfrac{b-c}{b+c}=\dfrac{1-\tan\theta}{1+\tan\theta}=\tan(45^\circ-\theta).$

25. Put $b=a\tan\phi$; $a\cos\theta-b\sin\theta$

$$=\frac{1}{\cos\phi}(a\cos\phi\cos\theta-a\sin\phi\sin\theta)=a\sec\phi\cos(\phi+\theta).$$

26. Produce BC to D so that CD = 5; $\therefore$ BD = 9; $\therefore$ $BA^2=BC\,.\,BD$;

$\therefore\ \angle BAC=\angle CDA=\angle CAD=\frac{1}{2}\angle ACB.$

Also, see solution of No. 38; $6^2-4^2=4\times5$.

27. Expression $=\dfrac{a}{bc}+\dfrac{b^2+c^2-a^2}{2abc}=\dfrac{a^2+b^2+c^2}{2abc}$.

28. $b^2(\cot A+\cot B)=\dfrac{b^2\sin(A+B)}{\sin A\sin B}=\dfrac{b^2\sin C}{\sin A\sin B}=\dfrac{b\,.\,c\sin B}{\sin A\sin B}=\dfrac{bc}{\sin A}$,

and similarly.

29. $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}$, $\therefore\ =\dfrac{a\cos B-b\cos A}{\sin A\cos B-\sin B\cos A}$

$$=\operatorname{cosec}(A-B)\,.\,[a\cos B-b\cos A].$$

30. $\dfrac{b^2-c^2}{a^2}=\dfrac{\sin^2B-\sin^2C}{\sin^2A}=\dfrac{\sin(B+C)\sin(B-C)}{\sin^2A}=\dfrac{\sin(B-C)}{\sin A}$.

31. Left side $=\dfrac{b\cos C+c\cos B}{\sin B\cos C+\sin C\cos B}=\dfrac{a}{\sin(B+C)}=\dfrac{a}{\sin A}$,

and similarly.

32. $\sin B\cos B=\sin C\cos C$; $\therefore\ \sin 2B=\sin 2C$;

$\therefore\ 2B=2C$ or $180^\circ-2C$; $\therefore\ B=C$ or $B+C=90^\circ$.

33. $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}=\dfrac{bc\sin A}{abc}=\dfrac{2\Delta}{abc}$;

$$\therefore\ \text{left side}=\frac{4\Delta^2}{a^2b^2c^2}(a^2+bc\cos A)=\frac{2\Delta^2}{a^2b^2c^2}[2a^2+(b^2+c^2-a^2)].$$

34. Left side $=\dfrac{2-2\cos(A-B)\cos(A+B)}{2-2\cos(A-C)\cos(A+C)}=\dfrac{2-\cos 2A-\cos 2B}{2-\cos 2A-\cos 2C}$

$$=\frac{2\sin^2A+2\sin^2B}{2\sin^2A+2\sin^2C}=\frac{a^2+b^2}{a^2+c^2}.$$

35. Left side $=\cos C(a\cos B+b\cos A)+c\cos A\cos B$

$$=-\cos(A+B)\,.\,c+c\cos A\cos B$$

$$=c(-\cos A\cos B+\sin A\sin B+\cos A\cos B)$$

$$=c\sin A\sin B=\sin A\,.\,\frac{ac\sin B}{a}=\sin A\,.\,\frac{2\Delta}{a}.$$

36. Expression $=\dfrac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}=\dfrac{2\cos\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B)}{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A+B)}$

$$=\frac{\sin A+\sin B}{\sin(A+B)}=\frac{\sin A+\sin B}{\sin C}=\frac{a+b}{c}.$$

37. If $b+c=2a$, $2s=a+b+c=3a$; $\therefore\ s-a=\frac{1}{2}a$;

$$\therefore\ \Delta=s(s-a)\,.\,\tan\frac{A}{2}=\frac{3a}{2}\cdot\frac{a}{2}\tan\frac{A}{2}.$$

38. $a^2-b^2=bc$; $\therefore\ \sin^2A-\sin^2B\equiv\sin(A+B)\sin(A-B)=\sin B\sin C$;

$\therefore\ \sin(A-B)=\sin B$; $\therefore\ A-B=B$, since $A-B\neq180^\circ-B$.

Or produce CA to D so that AD = AB; $\therefore$ CD $=b+c$;

$\therefore\ CB^2=CA\,.\,CD$; $\therefore$ CB touches circle BAD;

$\therefore\ \angle CBA=\angle BDA=\frac{1}{2}\angle BAC$, since AB = AD.

39. $c^2=(a\cos B+b\cos A)^2=(a\cos B+b\cos A)^2-(a\sin B-b\sin A)^2$;

expand.

40. Left side $=\tan\frac{1}{2}(B-C)+\cot\frac{1}{2}(B-C)$; but

$$\tan\theta+\cot\theta=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}=\frac{1}{\frac{1}{2}\sin 2\theta}=2\operatorname{cosec}2\theta\,;$$

for θ write $\frac{1}{2}(B-C)$.

41. $1+2\cos 2A=1+2(1-2\sin^2A)=3-4\sin^2A=\dfrac{\sin 3A}{\sin A}$;

$$\therefore\ \text{left side}=\frac{a}{\sin A}\,.\,\sin 3A\cos 3B+\frac{b}{\sin B}\,.\,\sin 3B\cos 3A$$

$$=\frac{c}{\sin C}\,.\,(\sin 3A\cos 3B+\sin 3B\cos 3A)$$

$$=\frac{c}{\sin C}\,.\,\sin(3A+3B)=\frac{c}{\sin C}\,.\,\sin 3C=c(1+2\cos 2C).$$

42. $\cos A\cos B+\sin A\sin B\sin C\leqslant\cos A\cos B+\sin A\sin B$,

i.e. $\cos(A-B)$, and $\therefore$ only $=1$ if $A=B$ and $\sin C=1$;

$\therefore\ A=B=45^\circ$.

EXERCISE I. b. (p. 7.)

1. Use eqn. (3). **2.** Use eqns. (4) and (5).

3. $\Delta=\sqrt{\{s(s-a)(s-b)(s-c)\}}=84$; use eqn. (5).

4. Since $b=c$, $B=59°$; $\therefore\ r=\frac{23\cdot5}{2}\tan\frac{1}{2}B=11\cdot75\tan 29°\,30'$.

5. (i) $\angle BAI_3=\frac{1}{2}$ ext. $\angle$ at $A=\frac{1}{2}(180°-A)$;

$\angle I_3I_1I_2=180°-\angle BIC=\angle IBC+\angle ICB=\frac{1}{2}B+\frac{1}{2}C$.

(ii) $II_1=$ diam. of cyclic quad. $IBI_1C=\frac{BC}{\sin BI_1C}=\frac{a}{\cos\frac{A}{2}}=\frac{2R\sin A}{\cos\frac{A}{2}}$,

Or $II_1=QQ_1\sec\frac{A}{2}=(AQ_1-AQ)\sec\frac{A}{2}$; but $AQ_1=s$ and $AQ=s-a$.

(iii) $I_2I_3=2R'\sin I_2I_1I_3$ where $R'=$ circumradius of $\triangle I_1I_2I_3=2R$;

$\therefore\ I_2I_3=4R\cos\frac{A}{2}$.

Or $I_2I_3=AI_2+AI_3$

$=(s-c)\operatorname{cosec}\frac{A}{2}+(s-b)\operatorname{cosec}\frac{A}{2}=a\operatorname{cosec}\frac{A}{2}=$ etc.

6. In $4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, put $R=\frac{abc}{4\Delta}$, $\sin\frac{A}{2}=\sqrt{\left[\frac{(s-b)(s-c)}{bc}\right]}$, etc.

and use $\Delta^2=s(s-a)(s-b)(s-c)$; expression $\frac{\Delta}{s}$.

7. Expression $=a[-\cos(B+C)+\cos B\cos C]$

$=2R\sin A\,.\,[\sin B\sin C]$.

8. As in No. 6. **Or** right side $=\frac{R\sin A\sin B\sin C}{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$

$=\frac{2R^2\sin A\sin B\sin C}{r}=\frac{\frac{1}{2}ab\sin C}{r}=\frac{\Delta}{r}=s$.

9. As in No. 6. **Or** as in No. 8,

right side $=\frac{R\sin A\sin B\sin C}{2\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{2R^2\sin A\sin B\sin C}{r_1}=$ etc.

10. Left side $=\frac{\Delta}{s-b}\cdot\frac{\Delta}{s-c}\cdot\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}=\frac{\Delta\,.\,\Delta}{\Delta}$.

11. Left side $=\sum\frac{\Delta^2}{(s-b)(s-c)}$

$=\Sigma s(s-a)=s\{(s-a)+(s-b)+(s-c)\}=s^2$.

12. Left side $=4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

$+4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}=4R\cos\frac{A}{2}\sin\frac{B+C}{2}$.

13. $r_1-r=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$=4R\sin\frac{A}{2}\cos\frac{B+C}{2}=4R\sin^2\frac{A}{2}$; from No. 12,

$r_2+r_3=4R\cos^2\frac{A}{2}$; subtract,

left side $=4R\left(\cos^2\frac{A}{2}-\sin^2\frac{A}{2}\right)=4R\cos A=\frac{2a\cos A}{\sin A}$.

14. $\triangle AIB$ is similar to $\triangle ACI_1$; $\therefore\ \frac{AI}{AB}=\frac{AC}{AI_1}$.

Or $AI=(s-a)\sec\frac{A}{2}$,

$AI_1=s\sec\frac{A}{2}$, but $\sec^2\frac{A}{2}=\frac{bc}{s(s-a)}$.

15. $IA\,.\,IB=r\operatorname{cosec}\frac{A}{2}\,.\,r\operatorname{cosec}\frac{B}{2}=r\,.\,4R\sin\frac{C}{2}$.

16. $IA\,.\,IB\,.\,IC=r^3\operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}=r^3\,.\,\frac{4R}{r}$

$=4R\,.\,r^2=\frac{abc}{\Delta}\cdot\frac{\Delta^2}{s^2}$.

17. From No. 5, $II_1=4R\sin\frac{A}{2}$;

$\therefore$ left side $=64R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=64R^3\,.\,\frac{r}{4R}$.

18. $\triangle ABI=\frac{1}{2}r\,.\,c$; $\triangle ACI=\frac{1}{2}r\,.\,b$.

19. $AD(\cot B+\cot C)=BD+DC=a=\frac{2\Delta}{AD}$.

20. $AD=c\sin B=2R\sin C\sin B=2\sin B\sin C\,.\,\frac{r}{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$.

21. $\angle OAI_3=\angle OAB+\angle BAI_3$

$=(90°-C)+\frac{1}{2}(180°-A)=90°+\frac{1}{2}(B-C)$;

$\therefore$ perp. from O to $I_2I_3=R\sin[90°+\frac{1}{2}(B-C)]=R\cos\frac{1}{2}(B-C)$;

also from No. 5, $I_2I_3=4R\sin\frac{1}{2}(B+C)$;

$\therefore\ \triangle OI_2I_3=\frac{1}{2}\,.\,R\cos\frac{1}{2}(B-C)\,.\,4R\sin\frac{1}{2}(B+C)$

$=R^2(\sin B+\sin C)=\frac{1}{2}R(b+c)$;

similarly $\triangle OI_3I_1=\frac{1}{2}R(c+a)$.

22. $AH = AE \operatorname{cosec} C = c \cos A \cdot \dfrac{1}{\sin C} = \dfrac{a \cos A}{\sin A}$

$$= a \cot A = 2BX \cot BOX = 2OX.$$

23. By No. 22, $\Sigma AH = 2\Sigma OX = 2R \,.\, \Sigma \cos A$

$$= 2R + 2R(\cos A + \cos B + \cos C - 1)$$

$$= 2R + 8R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 2(R + r).$$

24. As in No. 3, $\Delta = 84$; $\therefore\ \frac{1}{2} \,.\, 14 \,.\, AD = 84$.

25. $\dfrac{b}{\sin B} = \dfrac{c}{\sin C} = \dfrac{BE \operatorname{cosec} A}{\sin C}$;

$$\therefore\ b = 9{\cdot}3 \operatorname{cosec} 83^\circ \operatorname{cosec} 46^\circ \sin 37^\circ.$$

26. $BP = s - b$; $\therefore\ (s-b)(s-c) = \sqrt{\{s(s-a)(s-b)(s-c)\}}$

$$\therefore\ \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}} = 1;\quad \therefore\ \tan\frac{A}{2} = 1;\quad \therefore\ \frac{A}{2} = 45^\circ.$$

27. (i) As on p. 5, $\dfrac{EF}{BC} = \dfrac{AF}{AC} = \cos FAC = \cos(180^\circ - A) = -\cos A$; $\dfrac{FD}{AC} = \dfrac{BF}{BC} = \cos B$, etc. (ii) $\angle FDA = \angle FCA$, cyclic quad., $= A - 90^\circ$; similarly $\angle EDA = A - 90^\circ$; add.

(iii) $AH = AE \operatorname{cosec} AHE = c \cos EAB \operatorname{cosec} DCE$

$$= c\cos(180^\circ - A)\operatorname{cosec} C = -2R\cos A.$$

(iv) $HE = HA \sin HAE = (-2R\cos A) \,.\, \sin(90^\circ - C) = -2R\cos A\cos C.$

28. Use eqn. (13); $\cos A = -\frac{7}{10}$, $\cos B = \frac{113}{130}$, $\cos C = \frac{25}{28}$, $R = \dfrac{65}{\sqrt{51}}$.

29. Circumradius of ΔII_2I_3 is $2R$; the angles of ΔII_2I_3 are $90^\circ + \frac{1}{2}A$, $\frac{1}{2}C$, $\frac{1}{2}B$; $\therefore$ by eqn. (13),

$$\rho_1^2 = -4(2R)^2 \cos(90^\circ + \tfrac{1}{2}A)\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C$$

$$= 16R^2 \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 4Rr_1.$$

30. $\angle BHC = 180^\circ - A$.

31. Circumradius $= \dfrac{BC}{2\sin BOC} = \dfrac{2R\sin A}{2\sin 2A} = \frac{1}{2}R\sec A \geqslant \frac{1}{2}R$.

32. ΔAEF is similar to ΔABC;

$$\therefore\ \text{inradius of } \Delta AEF : r = AF : AC = \cos A.$$

33. By eqns. (9), (10), $\Delta DEF = \frac{1}{2}b\cos B \,.\, c\cos C \,.\, \sin(180^\circ \sim 2A)$

$$= \pm bc\cos B\cos C\cos A\sin A.$$

34. $r = (s-b)\tan\dfrac{B}{2} = \frac{1}{2}(a+c-b)\tan\dfrac{B}{2}$; $\therefore\ a = 2r\cot\dfrac{B}{2} + b - c$;

substitute for a in $a^2 - 2ac\cos B + c^2 - b^2 = 0$; then

$$\left(2r\cot\frac{B}{2} + b - c\right)^2 - 2c\cos B\left(2r\cot\frac{B}{2} + b - c\right) + c^2 - b^2 = 0$$

or $$4\cot^2\frac{B}{2} \,.\, r^2 + (\ldots)r + 2c(c-b)(1+\cos B) = 0;$$

$\therefore$ product of values of r

$$= \frac{2c(c-b) \,.\, 2\cos^2\dfrac{B}{2}}{4\cot^2\dfrac{B}{2}} = c(c-b)\sin^2\frac{B}{2}.$$

35. $\angle$s of $\Delta I_1I_2I_3$ are $90^\circ - \frac{1}{2}A$, $90^\circ - \frac{1}{2}B$, $90^\circ - \frac{1}{2}C$; circumradius $= 2R$;

$$\therefore\ \text{inradius} = 4 \,.\, 2R\sin\left(45^\circ - \frac{A}{4}\right)\sin\left(45^\circ - \frac{B}{4}\right)\sin\left(45^\circ - \frac{C}{4}\right)$$

$$= 4R\sin\left(45^\circ - \frac{A}{4}\right)\left[\cos\frac{B-C}{4} - \sin\frac{B+C}{4}\right]$$

$$= 2R\left\{2\sin\frac{B+C}{4}\cos\frac{B-C}{4} - 2\sin^2\frac{B+C}{4}\right\}$$

$$= 2R\left(\sin\frac{B}{2} + \sin\frac{C}{2} - 1 + \cos\frac{B+C}{2}\right).$$

36. If Δ is acute-angled,

perimeter $= \Sigma(a\cos A) = \Sigma(R\sin 2A) = R\Sigma(\sin 2A)$

$= 4R\sin A\sin B\sin C.$

If $\angle BAC > 90^\circ$,

perimeter $= (-a\cos A + b\cos B + c\cos C)$

$= R(-\sin 2A + \sin 2B + \sin 2C)$

$= 4R\sin A\cos B\cos C.$

37. (i) Inradius $= 4 \,.\, \frac{1}{2}R \,.\, \sin(90^\circ - A)\sin(90^\circ - B)\sin(90^\circ - C)$

$= 2R\cos A\cos B\cos C.$

(ii) Inradius $= 4 \,.\, \frac{1}{2}R \,.\, \sin(A - 90^\circ)\sin B\sin C$, by No. 27 (ii)

$= -2R\cos A\sin B\sin C$;

38. $a\sin B\sin C = a \,.\, \dfrac{b}{2R} \,.\, \sin C = \dfrac{\Delta}{R}$; similarly $b\sin C\sin A = \dfrac{\Delta}{R}$, etc.

39. Expression $= \frac{1}{2}c\sin B + 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}\cos A - R\cos^2 A$

$$= R\sin C\sin B + R\cos A\left\{2\sin\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) - \cos A\right\}$$

$$=\mathrm{R}\{\cos\mathrm{B}\cos\mathrm{C}-\cos(\mathrm{B}+\mathrm{C})\}$$
$$+\mathrm{R}\cos\mathrm{A}\left\{2\cos\frac{\mathrm{B}+\mathrm{C}}{2}\cos\frac{\mathrm{B}-\mathrm{C}}{2}-2\sin^2\frac{\mathrm{A}}{2}-\cos\mathrm{A}\right\}$$
$$=\mathrm{R}\{\cos\mathrm{B}\cos\mathrm{C}+\cos\mathrm{A}\}$$
$$+\mathrm{R}\cos\mathrm{A}\{\cos\mathrm{B}+\cos\mathrm{C}-1\}$$
$$=\mathrm{R}(\cos\mathrm{B}\cos\mathrm{C}+\cos\mathrm{C}\cos\mathrm{A}+\cos\mathrm{A}\cos\mathrm{B}).$$

40. (i) Using eqns. (9), (10), apply $a^2=b^2+c^2-2bc\cos\mathrm{A}$ to the pedal triangle;

(ii) Apply the method of (i) to the pedal triangle of the pedal triangle;

(iii) Using the results in No. 5, apply $a^2=b^2+c^2-2bc\cos\mathrm{A}$ to $\Delta\,\mathrm{I}_1\mathrm{I}_2\mathrm{I}_3$.

EXERCISE I. c. (p. 12.)

1. By eqn. (17), $\dfrac{\mathrm{BK}}{\mathrm{KC}}=\dfrac{\Delta\mathrm{BOA}}{\Delta\mathrm{AOC}}=\dfrac{\frac{1}{2}\mathrm{R}^2\sin 2\mathrm{C}}{\frac{1}{2}\mathrm{R}^2\sin 2\mathrm{B}}$.

2. By eqn. (14), $\dfrac{\mathrm{BK}}{\mathrm{KC}}=\dfrac{\sin 15^\circ}{\sin 45^\circ}=\dfrac{\sqrt{3}-1}{2\sqrt{2}}\div\dfrac{1}{\sqrt{2}}$.

3. $\mathrm{AB}=\mathrm{AC}$; $\therefore$ by eqn. (14), $\dfrac{\mathrm{BK}}{\mathrm{KC}}=\dfrac{\sin 30^\circ}{\sin 90^\circ}=\frac{1}{2}$.

4. $\cot\mathrm{B}=\dfrac{\cos\mathrm{B}}{\sin\mathrm{B}}=\dfrac{a^2+c^2-b^2}{2ac\sin\mathrm{B}}=\dfrac{a^2+c^2-b^2}{4\Delta}$; from I. b, No. 3, $\Delta=84$;

$\therefore\ \cot\mathrm{B}=\dfrac{13^2+15^2-14^2}{4\times 84}$; $\cot\mathrm{C}=\dfrac{13^2+14^2-15^2}{4\times 84}$

then use eqn. (20).

5. $\mathrm{A}=90^\circ$; in eqn. (15), $z:y=3:2$ and $\cot\mathrm{B}=\frac{60}{11}$, $\cot\mathrm{C}=\frac{11}{60}$;
$\therefore\ 5\cot\theta=2\times\frac{60}{11}-3\times\frac{11}{60}$.
Or $\mathrm{AD}=\frac{660}{61}$, $\mathrm{DC}=\frac{121}{61}$;
$\therefore\ \mathrm{KD}=\frac{2}{5}\times 61-\frac{121}{61}$.

6. $\mathrm{A}=90^\circ$; in eqn. (15), if $z:y=3:-2$, $\theta=180^\circ-\angle\mathrm{AKC}$,
$\cot\mathrm{B}=\frac{84}{13}$, $\cot\mathrm{C}=\frac{13}{84}$; $\therefore\ \cot\mathrm{AKC}=2\times\frac{84}{13}+3\times\frac{13}{84}$.
Or $\mathrm{AD}=\dfrac{84\,.\,13}{85}$, $\mathrm{DK}=\mathrm{DC}+\mathrm{CK}=\dfrac{13^2}{85}+2\times 85$.

7. By eqn. (20),
$$2\cot\mathrm{AXC}=\cot\mathrm{B}-\cot\mathrm{C}=\frac{a^2+c^2-b^2}{4\Delta}-\frac{a^2+b^2-c^2}{4\Delta},$$
by No. 4, $$=\frac{2c^2-2b^2}{4\Delta}=\frac{c^2-b^2}{2\Delta}.$$

8. $\mathrm{BK}\,.\,\mathrm{DC}+\mathrm{BD}\,.\,\mathrm{CK}+\mathrm{BC}\,.\,\mathrm{KD}$
$=\mathrm{BK}(\mathrm{BC}-\mathrm{BD})+\mathrm{BD}(\mathrm{BK}-\mathrm{BC})+\mathrm{BC}(\mathrm{BD}-\mathrm{BK})$
with the usual sign conventions, $=0$.

Hence $\dfrac{\mathrm{KD}\,.\,\mathrm{BC}}{\mathrm{AD}}=\dfrac{\mathrm{BD}\,.\,\mathrm{KC}}{\mathrm{AD}}-\dfrac{\mathrm{BK}\,.\,\mathrm{DC}}{\mathrm{AD}}$;

$\therefore\ \cot\theta\,.\,\mathrm{BC}=\cot\mathrm{B}\,.\,\mathrm{KC}-\cot\mathrm{C}\,.\,\mathrm{BK}$; hence result.

9. As in No. 7, $\cot\mathrm{AXB}=\frac{1}{2}(\cot\mathrm{C}-\cot\mathrm{B})=\dfrac{b^2-c^2}{4\Delta}$;

but $4\Delta=2bc\sin\mathrm{A}=\dfrac{abc}{\mathrm{R}}$.

10. By eqn. (14) or by cross-ratios,
$$\frac{\mathrm{BK}}{\mathrm{KC}}\div\frac{\mathrm{BK}'}{\mathrm{K}'\mathrm{C}}=(\mathrm{BKCK}')=\frac{\sin\mathrm{BAK}\,.\,\sin\mathrm{CAK}'}{\sin\mathrm{BAK}'\,.\,\sin\mathrm{CAK}}$$
$$=\frac{\sin\frac{\mathrm{A}}{3}\,.\,\sin\left(-\frac{\mathrm{A}}{3}\right)}{\sin\frac{2\mathrm{A}}{3}\,.\,\sin\left(-\frac{2\mathrm{A}}{3}\right)}=\left(\frac{\sin\frac{\mathrm{A}}{3}}{2\sin\frac{\mathrm{A}}{3}\cos\frac{\mathrm{A}}{3}}\right)^2=\frac{1}{4}\sec^2\frac{\mathrm{A}}{3}.$$

11. By eqn. (20), $2\cot\theta=\cot\mathrm{OAB}-\cot\mathrm{OBA}=$, by No. 7,
$\dfrac{\mathrm{OA}^2-\mathrm{OB}^2}{2\Delta\mathrm{OAB}}=\dfrac{10^2-7^2}{2\Delta}$ where $\Delta=\sqrt{(\frac{29}{2}\,.\,\frac{9}{2}\,.\,\frac{15}{2}\,.\,\frac{5}{2})}=\frac{15}{4}\sqrt{87}$.

12. Let perps. at A, B to OA, OB meet at K; bisect AB at M; then MK is vertical; $\therefore$ required $\angle$ is $\angle\mathrm{KMB}=\theta$, say. By eqn. (16), with $y=z$, $2\cot\theta=\cot\mathrm{AKM}-\cot\mathrm{MKB}=\cot\alpha-\cot\beta$.

13. In eqn. (18), $y=2$, $z=3$; $\therefore\ 2\,.\,6^2+3\,.\,4^2=5\mathrm{AK}^2+2\mathrm{BK}^2+3\mathrm{KC}^2$,

where $\dfrac{\mathrm{BK}}{3}=\dfrac{\mathrm{KC}}{2}=\dfrac{\mathrm{BC}}{5}=1$;

$\therefore\ 5\mathrm{AK}^2=2\,.\,6^2+3\,.\,4^2-2\,.\,3^2-3\,.\,2^2=90$; $\therefore\ \mathrm{AK}^2=18$.

14. Altitude BE is parallel to XA; $\therefore\ \mathrm{CA}=\mathrm{AE}$;

$\therefore\ \tan\mathrm{C}=\dfrac{\mathrm{BE}}{\mathrm{EC}}=\dfrac{\mathrm{BE}}{2\mathrm{EA}}=\frac{1}{2}\tan(180^\circ-\mathrm{A})=-\frac{1}{2}\tan\mathrm{A}$.

15. $\frac{1}{2}b\,.\,\mathrm{AK}\sin\dfrac{\mathrm{A}}{2}=\Delta\mathrm{CAK}$; $\frac{1}{2}c\,.\,\mathrm{AK}\sin\dfrac{\mathrm{A}}{2}=\Delta\mathrm{KAB}$; add;

$\therefore\ \mathrm{AK}\,.\,(b+c)\sin\dfrac{\mathrm{A}}{2}=2\Delta=bc\sin\mathrm{A}=2\,bc\sin\dfrac{\mathrm{A}}{2}\cos\dfrac{\mathrm{A}}{2}$;

$$\mathrm{AK}^2=\frac{4b^2c^2}{(b+c)^2}\cdot\cos^2\frac{\mathrm{A}}{2}=\frac{4b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}$$
$$=\frac{bc}{(b+c)^2}\cdot(a+b+c)(b+c-a)=\frac{bc}{(b+c)^2}\{(b+c)^2-a^2\}.$$

16. $\frac{1}{2}c \,.\, \mathsf{AK}' \sin\left(90° + \frac{\mathsf{A}}{2}\right) = \triangle\mathsf{BAK}'$; $\frac{1}{2}b \,.\, \mathsf{AK}' \sin\left(90° - \frac{\mathsf{A}}{2}\right) = \triangle\mathsf{CAK}'$; subtract;

$$\therefore \mathsf{AK}' \,.\, (c-b)\cos\frac{\mathsf{A}}{2} = 2\Delta = bc\sin\mathsf{A} = 2bc\sin\frac{\mathsf{A}}{2}\cos\frac{\mathsf{A}}{2};$$

$$\therefore \mathsf{AK}' = \frac{2bc}{c-b}\cdot\sin\frac{\mathsf{A}}{2}; \quad \therefore \mathsf{AK}'^2 = \frac{4b^2c^2}{(c-b)^2}\cdot\frac{(s-b)(s-c)}{bc}$$

$$= \frac{bc}{(c-b)^2}\cdot(a+c-b)(a+b-c) = \frac{bc}{(c-b)^2}\{a^2-(c-b)^2\}.$$

17. (i) $\frac{\mathsf{AI}}{\mathsf{IK}} = \frac{\mathsf{AC}}{\mathsf{CK}} = \frac{\mathsf{AB}}{\mathsf{BK}}$, $\therefore\ = \frac{\mathsf{AC}+\mathsf{AB}}{\mathsf{CK}+\mathsf{BK}} = \frac{b+c}{a}$;

(ii) $\mathsf{PP}_1 = a - 2(s-c) = c - b$, and $\frac{\mathsf{PK}}{\mathsf{KP}_1} = \frac{\mathsf{IP}}{\mathsf{I}_1\mathsf{P}_1} = \frac{r}{r_1} = \frac{s-a}{s}$;

$$\therefore \frac{\mathsf{PK}}{s-a} = \frac{\mathsf{KP}_1}{s}, \quad \therefore = \frac{\mathsf{PP}_1}{s-a+s} = \frac{c-b}{c+b}, \text{ but } \frac{\mathsf{PD}}{\mathsf{PK}} = \frac{\mathsf{IA}}{\mathsf{IK}} = \frac{b+c}{a};$$

$$\therefore \mathsf{PD} = \frac{b+c}{a}\cdot\frac{(s-a)(c-b)}{c+b};$$

(iii) $\tan\mathsf{APC} = \frac{\mathsf{AD}}{\mathsf{PD}} = \frac{2\Delta}{a}\cdot\frac{a}{(s-a)(c-b)} = \frac{2r_1}{c-b}$.

18. LM, LN are perps. to AB, AC, then $\mathsf{AM} = \mathsf{AN} = \frac{1}{2}(\mathsf{AB}+\mathsf{AC})$;

$$\therefore \mathsf{AL} = \mathsf{AM}\sec\frac{\mathsf{A}}{2} = \frac{1}{2}(b+c)\sec\frac{\mathsf{A}}{2}; \ \triangle\mathsf{ACK} \text{ is similar to } \triangle\mathsf{ALB};$$

$$\therefore \frac{\mathsf{AC}}{\mathsf{AK}} = \frac{\mathsf{AL}}{\mathsf{AB}}; \quad \therefore \mathsf{AK}\,.\,\mathsf{AL} = bc; \quad \therefore \frac{\mathsf{AL}}{\mathsf{KL}} = \frac{\mathsf{AL}\,.\,\mathsf{AK}}{\mathsf{KL}\,.\,\mathsf{AK}} = \frac{bc}{\mathsf{BK}\,.\,\mathsf{KC}},$$

but $\mathsf{BK}\,.\,\mathsf{KC} = \frac{ac}{b+c}\cdot\frac{ab}{b+c}$.

Or $\frac{\mathsf{AL}}{\mathsf{KL}} = \frac{\mathsf{AL}^2}{\mathsf{KL}\,.\,\mathsf{AL}} = \frac{\mathsf{AL}^2}{\mathsf{CL}^2}$,

but $\mathsf{CL} = \frac{1}{2}a\sec\frac{\mathsf{A}}{2}$.

19. By eqn. (12), $\triangle\mathsf{BHC} = \frac{1}{2}a\,.\,2\mathsf{R}\cos\mathsf{B}\cos\mathsf{C}$; $\therefore 2\triangle\mathsf{BNC}$
$= \triangle\mathsf{BOC} + \triangle\mathsf{BHC} = \frac{1}{2}a\,.\,\mathsf{R}\cos\mathsf{A} + \frac{1}{2}a\,.\,2\mathsf{R}\cos\mathsf{B}\cos\mathsf{C}$
$= \frac{1}{2}a\mathsf{R}\,[-\cos(\mathsf{B}+\mathsf{C}) + 2\cos\mathsf{B}\cos\mathsf{C}] = \frac{1}{2}a\mathsf{R}\,.\,\cos(\mathsf{B}-\mathsf{C})$.
$\mathsf{BK} : \mathsf{KC} = \triangle\mathsf{BNA} : \triangle\mathsf{CNA} = c\mathsf{R}\,.\cos(\mathsf{A}-\mathsf{B}) : b\mathsf{R}\,.\cos(\mathsf{C}-\mathsf{A})$.

20. $\triangle\mathsf{BIC} : \triangle\mathsf{CIA} : \triangle\mathsf{AIB} = \frac{1}{2}ar : \frac{1}{2}br : \frac{1}{2}cr = a : b : c$.

21. $\triangle\mathsf{BI}_1\mathsf{C} : \triangle\mathsf{CI}_1\mathsf{A} : \triangle\mathsf{AI}_1\mathsf{B} = -\frac{1}{2}ar_1 : \frac{1}{2}br_1 : \frac{1}{2}cr_1$.

22. By eqn. (12), $\triangle\mathsf{BHC} = \frac{1}{2}a\,.\,2\mathsf{R}\cos\mathsf{B}\cos\mathsf{C}$
$= 2\mathsf{R}^2\cos\mathsf{A}\cos\mathsf{B}\cos\mathsf{C}\,.\,\tan\mathsf{A}$;

$$\therefore x : y : z = \triangle\mathsf{BHC} : \triangle\mathsf{CHA} : \triangle\mathsf{AHB} = \tan\mathsf{A} : \tan\mathsf{B} : \tan\mathsf{C}.$$

23. OG : GN : NH = 2 : 1 : 3; (i) equivalent to 3 at G and 1 at H, i.e. to 4 at N; (ii) 2 at O and 1 at H is equivalent to 3 at G; $\therefore$ 3 at G and -2 at O is equivalent to 1 at H.

24. Use eqn. (21) and similar equations for BY^2, CZ^2 and add.

25. $\angle\mathsf{BGC} = 90°$; $\therefore a^2 = \mathsf{BG}^2 + \mathsf{GC}^2 = \frac{4}{9}(\mathsf{BY}^2 + \mathsf{CZ}^2) =$, by eqn. (21), $\frac{2}{9}(c^2 + a^2 - \frac{1}{2}b^2 + a^2 + b^2 - \frac{1}{2}c^2) = \frac{1}{9}(b^2 + c^2 + 4a^2)$.

26. $2\mathsf{GB}\,.\,\mathsf{GC}\cos\mathsf{BGC} = \mathsf{GB}^2 + \mathsf{GC}^2 - a^2 =$, as in No. 25,

$$\tfrac{1}{9}(b^2 + c^2 + 4a^2) - a^2 = \tfrac{1}{9}(b^2 + c^2 - 5a^2);$$

also $2\mathsf{GB}\,.\,\mathsf{GC}\sin\mathsf{BGC} = 4\triangle\mathsf{BGC} = \frac{4}{3}\triangle\mathsf{ABC} = \frac{4\Delta}{3}$; divide,

$$\tan\mathsf{BGC} = \frac{4\Delta}{3} \div \tfrac{1}{9}(b^2 + c^2 - 5a^2).$$

27. From $\triangle\mathsf{ADC}$, transversal YTB, $\frac{\mathsf{AT}}{\mathsf{TD}}\cdot\frac{\mathsf{DB}}{\mathsf{BC}}\cdot\frac{\mathsf{CY}}{\mathsf{YA}} = -1$;

$$\therefore \frac{\mathsf{AT}}{\mathsf{TD}} = \frac{\mathsf{BC}}{\mathsf{BD}}; \quad \therefore \frac{\mathsf{AT}}{\mathsf{AD}} = \frac{\mathsf{BC}}{\mathsf{BC}+\mathsf{BD}};$$

$$\therefore \mathsf{AT} = \frac{2\Delta}{\mathsf{BC}+\mathsf{BD}} = \frac{2\Delta}{a + c\cos\mathsf{B}}.$$

28. Use eqn. (20), $\cot\mathsf{AXC} = \frac{1}{2}(\cot 55° - \cot 23°\,30')$;

$$\therefore \angle\mathsf{AXC} = 128°\,39'; \quad \therefore \angle\mathsf{BAX} = 73°\,39';$$

similarly $\angle\mathsf{BYC} = 141°\,23'$, $\angle\mathsf{YBA} = 39°\,53'$;

$$\therefore \frac{\mathsf{BY}}{\mathsf{AX}} = \frac{\mathsf{BG}}{\mathsf{AG}} = \frac{\sin 73°\,39'}{\sin 39°\,53'} = 1{\cdot}496;$$

$$\therefore \mathsf{BY} = 40 \times 1{\cdot}496 = 59{\cdot}8.$$

29. $\mathsf{AT}_1^2 + \mathsf{AT}_2^2 = 2\mathsf{AX}^2 + 2\mathsf{XT}_1^2 = 2(\frac{1}{2}a)^2 + 2(\frac{1}{6}a)^2 = \frac{a^2}{2} + \frac{a^2}{18}$.

30. $\mathsf{AK}^2 + \mathsf{BK}^2 = 2\mathsf{KX}^2 + 2\mathsf{AX}^2 = 2\mathsf{KX}^2 + 2\mathsf{CX}^2 = 2\mathsf{KC}^2$.

31. $$\frac{4m^2 - a^2}{4ah} = \frac{2(b^2 + c^2 - \frac{1}{2}a^2) - a^2}{8\Delta} = \frac{2(b^2 + c^2 - a^2)}{8\Delta}$$

$$= \frac{2\,.\,2bc\cos\mathsf{A}}{4bc\sin\mathsf{A}} = \cot\mathsf{A}.$$

32. In $\triangle\mathsf{YXA}$, $\mathsf{YA} = \frac{b}{2}$, $\mathsf{YX} = \frac{c}{2}$, $\angle\mathsf{YXA} = \beta$, $\angle\mathsf{YAX} = \gamma$; apply the formula, $\tan\frac{\mathsf{B}-\mathsf{C}}{2} = \frac{b-c}{b+c}\tan\frac{\mathsf{B}+\mathsf{C}}{2}$, to $\triangle\mathsf{YXA}$.

33. $\theta = B + \frac{1}{2}A$, etc.; $\therefore \Sigma(a \sin 2\theta) = \Sigma[a \sin(2B + A)]$

$$= \Sigma[2R \sin A \,.\, \sin(2B + A)]$$
$$= 2R \,.\, \Sigma[\sin A \,.\, \sin(C - B)] = 0.$$

34. From No. 26, $3 \cot BGC = \dfrac{b^2 + c^2 - 5a^2}{4\Delta}$;

as in No. 31, $\cot A = \dfrac{b^2 + c^2 - a^2}{4\Delta}$; subtract.

35. $QC = \dfrac{4a}{3}$; take K on CB, so that $AK = AC = b$; then $\angle AKC = \angle ACK = 2\angle ABC$; $\therefore KB = KA = b$; $\therefore KC = a - b$; altitude AD bisects KC;

$$\therefore DC = \tfrac{1}{2}(a - b); \quad \therefore QA^2 = QC^2 + CA^2 - 2QC \,.\, CD$$
$$= \left(\frac{4a}{3}\right)^2 + b^2 - 2\left(\frac{4a}{3}\right) \cdot \tfrac{1}{2}(a - b) = \left(b + \frac{2a}{3}\right)^2;$$
$$\therefore QA = b + \frac{2a}{3} = AC + \tfrac{1}{2}QC.$$

36. Let p be length of perp. from O to ABCD; then

$$p \,.\, AB = 2\Delta OAB = OA \,.\, OB \sin AOB; \quad \therefore AB = \frac{OA \,.\, OB \sin AOB}{p};$$

similarly for CD, AD, CB; substitute.

37. $\dfrac{BU}{UC} = \dfrac{\Delta BAU}{\Delta UAC} = \dfrac{BA \,.\, AU \sin BAU}{UA \,.\, AC \sin UAC}$; $\therefore \dfrac{\sin BAU}{\sin UAC} = \dfrac{b}{c} \cdot \dfrac{BU}{UC}$; similarly for $\dfrac{\sin CBV}{\sin VBA}$ and $\dfrac{\sin ACW}{\sin WCB}$; multiply ratios together and use Ceva's theorem $\dfrac{BU}{UC} \cdot \dfrac{CV}{VA} \cdot \dfrac{AW}{WB} = 1$.

38. As in No. 36,

$$\frac{(\alpha + \beta)(\beta + \gamma)}{\alpha\gamma} = \frac{AC \,.\, BD}{AB \,.\, CD} = \frac{\sin APC \,.\, \sin BPD}{\sin APB \,.\, \sin CPD}$$
$$= \frac{\sin^2 2\theta}{\sin^2 \theta} = 4\cos^2 \theta.$$

39. (i) $\Sigma(a \,.\, TA^2) = \Sigma(a \,.\, IA^2) + (a + b + c) \,.\, TI^2$ is least when $TI = 0$, i.e. $T \equiv I$; (ii) G is centroid of 1, 1, 1 at A, B, C;

$$\therefore \Sigma(TA^2) = \Sigma(GA^2) + 3TG^2 \text{ is least when } T \equiv G.$$

40. O is centroid of sin 2A, sin 2B, sin 2C at A, B, C (see Ex. 5, p. 12); $\therefore \Sigma(TA^2 \,.\, \sin 2A) = \Sigma(OA^2 \,.\, \sin 2A) + OT^2 \,.\, \Sigma(\sin 2A)$; $\therefore$ OT is constant; $\therefore$ locus of T is circle, centre O.

EXERCISE I. d. (p. 17.)

1. $OX = IP$; $\therefore R \cos A = r = 4R \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$;

$$\therefore \cos A = 2 \sin \frac{A}{2} \{\cos \tfrac{1}{2}(B - C) - \cos \tfrac{1}{2}(B + C)\}$$
$$= \cos B + \cos C - (1 - \cos A).$$

2. r = perp. from G to $BC = \frac{1}{3}AD$; $\therefore ar = \frac{1}{3}a \,.\, AD = \frac{2}{3}\Delta = \frac{2}{3}r \,.\, s$;

$$\therefore \frac{a}{2} = \frac{s}{3} = \frac{s - a}{1}; \quad \therefore r_1 = \frac{\Delta}{s - a} = \frac{3\Delta}{s} = 3r.$$

3. $OI^2 = R^2 - 2Rr = R^2 - 8R^2 \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$

$$= R^2\{1 - 2(\cos B + \cos C - 1 + \cos A)\} \text{ as in No. 1.}$$

4. Circumcircle of ΔABC is 9 point circle of $\Delta I_1I_2I_3$ (orthocentre I) and so bisects II_1, I_2I_3 at M, N, say; M, N are mid-points of arcs BC of circumcircle; $\therefore$ MN is a diameter.

$$\therefore II_1 = 2IM = 2BM = 2MN \sin BNM = 2(2R) \sin \frac{A}{2}; \text{ also}$$
$$II_1 = I_1A - IA = r_1 \operatorname{cosec} \frac{A}{2} - r \operatorname{cosec} \frac{A}{2} = (r_1 - r) \operatorname{cosec} \frac{A}{2};$$
$$\therefore II_1^2 = 4R \sin \frac{A}{2} \cdot (r_1 - r) \operatorname{cosec} \frac{A}{2}.$$

Since $\angle I_2BI_3 = 90°$, $I_2I_3 = 2NI_3 = 2NB = 2(2R) \cos \dfrac{A}{2}$; also

$$I_2I_3 = I_2A + AI_3 = r_2 \sec \frac{A}{2} + r_3 \sec \frac{A}{2} = (r_2 + r_3) \sec \frac{A}{2};$$
$$\therefore I_2I_3^2 = 4R \cos \frac{A}{2} \cdot (r_2 + r_3) \sec \frac{A}{2}.$$

Or $II_1^2 = 4IM^2 = 4CM^2 = 4MX \,.\, MN = 4 \,.\, \frac{1}{2}(r_1 - r) \,.\, 2R$, and
$I_2I_3^2 = 4NI_3^2 = 4NB^2 = 4NX \,.\, NM = 4 \,.\, \frac{1}{2}(r_2 + r_3) \,.\, 2R$.

5. Use No. 4. **Or** $II_2^2 - II_3^2 = AI_2^2 - AI_3^2$ (Pythagoras) $= I_1I_2^2 - I_1I_3^2$.

6. $\Sigma(OA^2) = \Sigma(GA^2) + 3OG^2$; but $\Sigma(GA^2) = \frac{4}{9}\Sigma(AX^2) =$, by Ex. I. c, No. 24, $\frac{1}{3}(a^2 + b^2 + c^2)$; also $OH = 3OG$; $\therefore OH^2 = 9OG^2$

$$= 3[\Sigma(OA^2) - \tfrac{1}{3}(a^2 + b^2 + c^2)] = 3(3R^2) - (a^2 + b^2 + c^2).$$

7. $AH = 2R \cos A = R = AO$; $\therefore$ AI, which bisects $\angle OAH$, bisects OH at right angles, i.e. it cuts OH at N;

$$\therefore OH^2 = 4ON^2 = 4(OI^2 - NI^2) = 4[(R^2 - 2Rr) - (\tfrac{1}{2}R - r)^2]$$
$$= 4R^2 - 8Rr - R^2 + 4Rr - 4r^2 = 3R^2 - 4Rr - 4r^2.$$

8. $\tan \text{IAX} = \tan \frac{1}{2}(\text{BAX} - \text{CAX}) =$, by Ex. I. c, No. 32, $\frac{b-c}{b+c}\tan\frac{A}{2}$;
but $\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)} = \tan\frac{1}{2}(B-C)\tan\frac{A}{2}$.

9. GI passes through P and XP : PD = XG : GA = ½;

$\therefore\ BP = \frac{1}{3}(2BX + BD);\quad \therefore\ s - b = \frac{1}{3}(a + c\cos B);$

$\therefore\ 3a(2s - 2b) = 2a^2 + 2ac\cos B;$

$\therefore\ 3a(a - b + c) = 2a^2 + a^2 + c^2 - b^2;$

$\therefore\ 3a(c - b) = c^2 - b^2;$ but $c \neq b$; $\therefore\ 3a = c + b;$

$$\therefore\ 3\sin A = \sin C + \sin B = 2\sin\frac{C+B}{2}\cos\frac{C-B}{2};$$

$$\therefore\ 6\sin\frac{A}{2}\cos\frac{A}{2} = 2\cos\frac{A}{2}\cos\frac{C-B}{2};$$

$$\therefore\ 3\sin\frac{A}{2} = \cos\frac{C-B}{2} = 2\sin\frac{B}{2}\sin\frac{C}{2} + \cos\frac{C+B}{2}$$

$$= 2\sin\frac{B}{2}\sin\frac{C}{2} + \sin\frac{A}{2}.$$

10. $IO = r;\ \therefore\ r^2 = IO^2 = R^2 - 2Rr;\ \therefore\ 2R^2 = R^2 + 2Rr + r^2 = (R + r)^2;$
$\therefore\ R\sqrt{2} = R + r =$, as in No. 1,
$R + R(\cos B + \cos C - 1 + \cos A) = R(\cos A + \cos B + \cos C).$

11. $HD - OX = 2R\cos B\cos C - R\cos A = R[2\cos B\cos C + \cos(B + C)]$

$= R[3\cos B\cos C - \sin B\sin C];\quad XD = XC - DC = \frac{a}{2} - b\cos C$

$= R(\sin A - 2\sin B\cos C) = R[\sin(B + C) - 2\sin B\cos C]$
$= R[\cos B\sin C - \sin B\cos C] = R\cos B\cos C(\tan C - \tan B);$

but $\tan\phi = \frac{HD - OX}{XD}$.

12. NX = radius of 9 point circle $= \frac{1}{2}R$; $\therefore\ NB^2 + NC^2 = 2NX^2 + 2XB^2$
$= 2(\frac{1}{2}R)^2 + 2(\frac{1}{2}a)^2 = \frac{1}{2}(R^2 + a^2)$; similarly for $NC^2 + NA^2$, etc.;

$\therefore\ 2NA^2 \equiv (NA^2 + NB^2) + (NA^2 + NC^2) - (NB^2 + NC^2)$

$= \frac{1}{2}(R^2 + c^2) + \frac{1}{2}(R^2 + b^2) - \frac{1}{2}(R^2 + a^2) = \frac{1}{2}(R^2 + b^2 + c^2 - a^2);$

$\therefore\ 4NA^2 = R^2(1 + 4\sin^2 B + 4\sin^2 C - 4\sin^2 A);$

but $4\sin^2 B = 2(1 - \cos 2B)$, etc.

13. From Ex. I. b, No. 8, $s^2 - r^2$

$$= 16R^2\left\{\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} - \sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}\right\}$$

$$= 2R^2\{(1 + \cos A)(1 + \cos B)(1 + \cos C) - (1 - \cos A)(1 - \cos B)(1 - \cos C)\}$$

$$= 4R^2(\cos A\cos B\cos C + \cos A + \cos B + \cos C);$$

but from eqn. (13), $\rho^2 = -4R^2\cos A\cos B\cos C$, and from No. 1, $r = R(\cos A + \cos B + \cos C - 1)$; $\therefore\ s^2 - r^2 = -\rho^2 + 4R(r + R)$;

$\therefore\ \rho^2 = 4R(r + R) + r^2 - s^2.$

14. ON^2 = sum of squares of radii $= R^2 + (\frac{1}{2}R)^2$; but $ON^2 = \frac{1}{4}OH^2 = \frac{1}{4}R^2(1 - 8\cos A\cos B\cos C)$, eqn. (24).

15. $r = AH = 2R\cos A$; $\therefore$ from No. 1,

$2R\cos A = R(\cos A + \cos B + \cos C - 1);$

$\therefore\ 2R = R(\cos B + \cos C - \cos A + 1)$

$$= R\left(2\cos\frac{B+C}{2}\cos\frac{B-C}{2} + 2\sin^2\frac{A}{2}\right)$$

$$= 2R\sin\frac{A}{2}\left(\cos\frac{B-C}{2} + \cos\frac{B+C}{2}\right)$$

$$= 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = r_1;$$

$\therefore\ R^2 + 2Rr_1 = R^2 + r_1^2$, i.e. $OI_1^2 = R^2 + r_1^2$ = sum of squares of radii.

16. Since $\angle OAT = C - B$, $\angle OAI = \angle IAT = \frac{1}{2}(C - B)$; $\therefore\ \frac{OT}{IT} =$, as in eqn. (14),

$$\frac{OA\sin OAT}{IA\sin IAT} = \frac{R\sin(C-B)}{r\operatorname{cosec}\frac{A}{2}\sin\frac{1}{2}(C-B)} = \frac{2R\cos\frac{1}{2}(C-B)}{4R\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$= \frac{\cos\frac{1}{2}(C-B)}{2\sin\frac{B}{2}\sin\frac{C}{2}};\quad \therefore\ \frac{OT}{OI} = \frac{\cos\frac{1}{2}(C-B)}{\cos\frac{1}{2}(C-B) - 2\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$= \frac{\cos\frac{1}{2}(C-B)}{\cos\frac{1}{2}(C+B)} = \frac{\cos\frac{1}{2}(C-B)}{\sin\frac{1}{2}A}$$

17. From No. 16, $\frac{\Delta OIH + \Delta OIA}{\Delta OAH} = \frac{OI}{OT} = \frac{1}{\cos\frac{1}{2}(C-B)\operatorname{cosec}\frac{A}{2}}$; but

$\Delta OAH = \frac{1}{2}AO\,.\,AH\sin OAH = \frac{1}{2}R\,.\,2R\cos A\,.\,\sin(C - B)$ and

ΔOIA = similarly $\frac{1}{2}R\,.\,r\operatorname{cosec}\frac{A}{2}\,.\,\sin\frac{1}{2}(C - B)$

$$= 2R^2\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{1}{2}(C-B);$$

$$\therefore\ \Delta OIH = \frac{R^2\cos A\sin(C-B)}{\cos\frac{1}{2}(C-B)\operatorname{cosec}\frac{A}{2}} - 2R^2\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{1}{2}(C-B)$$

$$= 2R^2\sin\frac{1}{2}(C-B)\left\{\cos A\sin\frac{A}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right\};$$

but $\cos A \sin\frac{A}{2} - \sin\frac{B}{2}\sin\frac{C}{2} = \cos A\cos\frac{B+C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}$

$$=\tfrac{1}{2}\left[\cos\frac{2A+B+C}{2}+\cos\frac{2A-B-C}{2}-\cos\frac{B-C}{2}+\cos\frac{B+C}{2}\right]$$

$$=\tfrac{1}{2}\left[\cos\left(180^\circ-\frac{B+C}{2}\right)+2\sin\frac{A-C}{2}\sin\frac{B-A}{2}+\cos\frac{B+C}{2}\right]$$

$$=\sin\frac{C-A}{2}\sin\frac{A-B}{2}.$$

18. The orthocentre of $\triangle BHC$ is A, and the circumradius $=R$, and the angles are $180^\circ - A$, $90^\circ - C$, $90^\circ - B$; $\therefore$ from eqn. (24), $SA^2 = R^2\{1-8\cos(180^\circ - A)\cos(90^\circ - C)\cos(90^\circ - B)\}$.

19. In $\triangle$s AOI, AHI, $OI = HI$, AI is common, $\angle OAI = \angle HAI$; $\therefore$ either $\angle AOI = \angle AHI$ or $\angle AOI = 180^\circ - \angle AHI$; $\therefore$ either $AO = AH$ or A, O, I, H are concyclic; similarly $BO = BH$ or B, O, I, H are concyclic, etc.; but it is impossible for circle OIH to pass through A and B and C; $\therefore$ $AO = AH$ or $BO = BH$ or $CO = CH$. If $AO = AH$, $R = 2R\cos A$; $\therefore$ $\cos A = \frac{1}{2}$.

20. Use eqns. (27), (28);

left side $=\frac{1}{2}R - r + \Sigma(\frac{1}{2}R + r_1) = 2R - r + r_1 + r_2 + r_3$;

but $\quad r_2 + r_3 + r_1 - r = 4R\cos^2\frac{A}{2} + 4R\sin^2\frac{A}{2} = 4R$,

see Ex. I. b, Nos. 12, 13.

21. (i) $\angle BHD = \angle ACB$, from cyclic quad. HEDC, $=\angle BTD$, cyclic quad. ACTB; $\therefore$ $\triangle BDH \equiv \triangle BDT$; $\therefore$ $HD = DT$;

(ii) $HA \cdot HT = 2HA \cdot HD = 2\rho^2$, p. 6;

(iii) $HO^2 = (\text{tangent})^2 + (\text{radius})^2 = HA \cdot HT + R^2 = 2\rho^2 + R^2$.

22. (i) Left side $=(R^2 - 2Rr) + \Sigma(R^2 + 2Rr_1)$

$= 4R^2 + 2R(r_1 + r_2 + r_3 - r) = 4R^2 + 2R \cdot (4R)$ as in No. 20;

(ii) In (i), O is the 9-point centre and I is the orthocentre of $\triangle I_1I_2I_3$, whose circumradius $=2R$; also $12R^2 = 3(2R)^2$. Apply this result to $\triangle ABC$ and we have (ii). **Or** N is centroid of 1, 1, 1, 1 at A, B, C, H, by Ex. I. c, No. 23; $\therefore$ by eqn. (19), $OA^2 + OB^2 + OC^2 + OH^2 = 4ON^2 + NA^2 + NB^2 + NC^2 + NH^2$; but $OH^2 = 4ON^2$ and $OA^2 = OB^2 = OC^2 = R^2$.

23. (i) N is centroid of 1, 1, 1, 1 at A, B, C, H; $\therefore$ by eqn. (19), (see No. 22), $DA^2 + DB^2 + DC^2 + DH^2 - 4DN^2$

$= OA^2 + OB^2 + OC^2 + OH^2 - 4ON^2 = 3R^2$;

but $DN =$ radius of 9-point circle $=\frac{1}{2}R$; (ii) this is the same result for $\triangle I_1I_2I_3$ as (i) is for $\triangle ABC$, see No. 22.

24. As in No. 23, $HA^2 + HB^2 + HC^2 + HH^2 - 4HN^2$

$= OA^2 + OB^2 + OC^2 + OH^2 - 4ON^2 = 3R^2$;

but $HH = 0$ and $HN = \frac{1}{2}HO$.

25. By eqn. (19), $OA^2 + OB^2 + OC^2 = 3OG^2 + GA^2 + GB^2 + GC^2$; but $OG^2 = \frac{1}{9}OH^2 = \frac{1}{9}R^2(1 - 8\cos A\cos B\cos C)$, eqn. (24), and

$OA^2 = OB^2 = OC^2 = R^2$.

26. By Ex. I. c, No. 20, and eqn. (19),

$\Sigma(a \cdot OA^2) = \Sigma(a \cdot IA^2) + (a + b + c)OI^2$;

$\therefore$ $R^2(a + b + c) = \Sigma(a \cdot IA^2) + (a + b + c)(R^2 - 2Rr)$;

$\therefore$ $\Sigma(a \cdot IA^2) = 2Rr \cdot (2s)$.

Similarly, as in Ex. I. c, No. 21, I_3 is centroid of a, b, $-c$ at A, B, C;

$\therefore$ $a \cdot OA^2 + b \cdot OB^2 - c \cdot OC^2$

$= a \cdot I_3A^2 + b \cdot I_3B^2 - c \cdot I_3c^2 + (a + b - c) \cdot OI_3^2$;

$\therefore$ $a \cdot I_3A^2 + b \cdot I_3B^2 - c \cdot I_3C^2$

$=(a + b - c)R^2 - (a + b - c)(R^2 + 2Rr_3)$

$=2(s - c)(-2Rr_3)$.

27. By Example 5, p. 12, and eqn. (19),

$\Sigma(TA^2 \cdot \sin 2A) = \Sigma(OA^2 \cdot \sin 2A) + OT^2 \cdot \Sigma(\sin 2A)$

$=(R^2 + OT^2) \cdot \Sigma(\sin 2A)$

$=(R^2 + OT^2) \cdot 4\sin A\sin B\sin C$.

28. By No. 26,

$\Sigma(a \cdot TA^2) = \Sigma(a \cdot IA^2) + IT^2 \cdot \Sigma(a) = 4Rrs + r^2 \cdot 2s$

$=2rs(2R + r) = 2\Delta(2R + r)$.

29. If the circles cut at K,

$$\cos OKI_1 = \frac{R^2 + r_1^2 - OI_1^2}{2Rr_1} = \frac{R^2 + r_1^2 - (R^2 + 2Rr_1)}{2Rr_1} = \frac{r_1 - 2R}{2R};$$

$$\therefore \sin OKI_1 = \sqrt{\left\{1 - \left(\frac{r_1 - 2R}{2R}\right)^2\right\}} = \frac{\sqrt{\{r_1(4R - r_1)\}}}{2R};$$

but $OI_1 \cdot$ (common chord)

$=4\triangle OKI_1 = 2Rr_1\sin OKI_1 = r_1 \cdot \sqrt{\{r_1(4R - r_1)\}}$

and $OI_1 = \sqrt{\{R^2 + 2Rr_1\}} = \sqrt{\{R(R + 2r_1)\}}$.

30. $t_1^2 = I_1O^2 - R^2 = (R^2 + 2Rr_1) - R^2 = 2Rr_1$; similarly $t_2^2 = 2Rr_2$; etc.;

$\therefore$ $\dfrac{t_1^2}{t_2^2} = \dfrac{2Rr_1}{2Rr_2}$; also $t_1^2t_2^2t_3^2 = 2Rr_1 \cdot 2Rr_2 \cdot 2Rr_3 = 8R^3 \cdot r_1r_2r_3$;

but $\quad rr_1r_2r_3 = \dfrac{\Delta^4}{s(s-a)(s-b)(s-c)} = \Delta^2 = \left(\dfrac{abc}{4R}\right)^2$,

$\therefore$ $r_1r_2r_3 = \dfrac{a^2b^2c^2}{16R^2r}$; $\therefore$ $t_1^2t_2^2t_3^2 = \dfrac{a^2b^2c^2R}{2r}$.

EXERCISE I. e. (p. 19).

1. $\frac{a-b}{c}=\frac{\sin A-\sin B}{\sin C}=\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}=\frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}$;

$$\therefore\ \sin\frac{A-B}{2}=\frac{19{\cdot}8}{22{\cdot}2}\cos 14^\circ\,38';$$

hence $\frac{A-B}{2}=59^\circ\,39'$;

but $\frac{A+B}{2}=90^\circ-14^\circ\,38'$.

2. $\frac{a+c}{b}=\frac{\sin A+\sin C}{\sin B}=\frac{\cos\frac{1}{2}(A-C)}{\sin\frac{1}{2}B}$, as in No. 1;

$$\therefore\ \cos\tfrac{1}{2}(A-C)=\frac{9{\cdot}28}{3{\cdot}36}\sin 18^\circ\,42\tfrac{1}{2}',$$

hence $\frac{1}{2}(A-C)=\pm 27^\circ\,38{\cdot}4'$; but $\frac{1}{2}(A+C)=90^\circ-18^\circ\,42\frac{1}{2}'$.

3. $a\,.\,AD=2\Delta=b\,.\,BE=c\,.\,CF$; $\therefore\ a:b\,.\,c=\frac{1}{8}:\frac{1}{6}:\frac{1}{9}=12:9:8$;

$$\therefore\ \cos A=\frac{9^2+8^2-12^2}{144}=\tfrac{1}{144};\ \cos B=\frac{12^2+8^2-9^2}{192},$$

hence $B=48^\circ\,35'$; $\therefore\ a=CF\operatorname{cosec}B=9\operatorname{cosec}48^\circ\,35'$.

4. $\cos A=-\cos AYX=\frac{AX^2-AY^2-YX^2}{2AY\,.\,YX}=\frac{7^2-4^2-5^2}{2\,.\,5\,.\,4}=\frac{1}{5}$.

5. From AC cut off AK = AB, then $CK=b-c$;

$$\therefore\ BC=a=2(b-c)=2CK;$$

but $\angle BKA=\angle KBA=67\frac{1}{2}^\circ$;

$$\therefore\ \frac{\sin CBK}{\sin 67^\circ\,30'}=\frac{CK}{CB}=\tfrac{1}{2};$$

hence $\angle CBK=27^\circ\,31'$.

6. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$, hence $r_3=15$. Now use Ex. (iv), p. 19.

7. As in No. 2, $2=\frac{a+c}{b}=\frac{\cos\frac{1}{2}(A-C)}{\sin\frac{1}{2}B}$, this gives $\frac{1}{2}(A-C)$; also $\frac{1}{2}(A+C)=90^\circ-\frac{1}{2}B$ is known, thus A, C are found.

8 $abc=2R\sin A\,.\,bc=4R\Delta$; $b^2+c^2=a^2+2bc\cos A$

$$=a^2+2\cdot\frac{4R\Delta}{a}\cdot\cos A=a^2\pm\frac{8R\Delta}{a}\sqrt{(1-\sin^2 A)}$$

$$=a^2\pm\frac{8R\Delta}{a}\sqrt{\left(1-\frac{a^2}{4R^2}\right)}=a^2\pm\frac{4\Delta}{a}\sqrt{(4R^2-a^2)}.$$

The negative sign does not give real values for b, c unless $\Delta\leqslant\frac{a}{4}[2R-\sqrt{(4R^2-a^2)}]$; there are no real values unless $R\geqslant\frac{a}{2}$ and $\Delta\leqslant\frac{a}{4}[2R+\sqrt{(4R^2-a^2)}]$.

9. $\sin^2\frac{1}{2}A=\frac{(s-b)(s-c)}{bc}=\frac{(s-b)(s-c)}{\{(s-a)+(s-c)\}\{(s-a)+(s-b)\}}$

$$=\frac{\frac{1}{r_2}\cdot\frac{1}{r_3}}{\left(\frac{1}{r_1}+\frac{1}{r_3}\right)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}=\frac{r_1^2}{(r_3+r_1)(r_2+r_1)},$$

10. $\tan B=\frac{CF}{FB}=\frac{b\sin A}{AB-AF}=\frac{b\sin A}{c-b\cos A}$.

11. $(a^2+b^2-c^2)^2+(4\Delta)^2=(2ab\cos C)^2+(2ab\sin C)^2=4a^2b^2$;

$$\therefore\ (a^2+b^2-c^2)^2=4(a^2b^2-4\Delta^2);$$

$$\therefore\ a^2+b^2-c^2=\pm 2\sqrt{(a^2b^2-4\Delta^2)}.$$

12. $\frac{c^2-a^2}{b^2}=\frac{\sin^2 C-\sin^2 A}{\sin^2 B}=\frac{\sin(C+A)\,.\,\sin(C-A)}{\sin^2 B}=\frac{\sin(C-A)}{\sin B}$

gives $C-A$; also $C+A=180^\circ-B$ is known.

13. $b\,.\,BE=2\Delta=c\,.\,CF$; $\therefore\ c=2b$;

$$\therefore\ \tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}=\tfrac{1}{3}\cot 26^\circ\,30';$$

hence $\frac{1}{2}(C-B)=33^\circ\,46'$; but $\frac{1}{2}(C+B)=90^\circ-26^\circ\,30'$.

14. The least value occurs when $B=C$, and in this case $\frac{a}{2}=r\cot\frac{C}{2}$ where $C=\frac{1}{2}(180^\circ-42^\circ)=69^\circ$; $\therefore\ a=2\times 3{\cdot}5\cot 34^\circ\,30'$.

15. $bc\sin A=2\Delta$; $\therefore\ bc\cdot\frac{\sqrt{3}}{2}=4\sqrt{3}$; $\therefore\ bc=8$;

$$b^2+c^2-2bc\cos A=a^2;\ \therefore\ b^2+c^2-bc=57;$$

$\therefore\ b^2+c^2=65$; hence $b=8$ or 1, $c=1$ or 8.

If $b=8$, $c=1$, then $\cos B=\frac{a^2+c^2-b^2}{2ac}=-\frac{3}{\sqrt{57}}$;

if $b=1$, $c=8$, then $\cos B=\frac{15}{2\sqrt{57}}$.

16. $a^2=b^2+c^2-2bc\cos 60°=b^2+c^2-bc=(b-c)^2+bc=16+bc$; also

$$bc\frac{\sqrt{3}}{2}=2\Delta=a\,.\,AD=11a;\quad \therefore\ bc=\frac{22a}{\sqrt{3}};\quad \therefore\ a^2=16+\frac{22a}{\sqrt{3}};$$

hence $a=8\sqrt{3}\left(\text{or } -\frac{2\sqrt{3}}{3}\right)$. Also as in No. 1,

$$\frac{b-c}{a}=\frac{\sin\frac12(B-C)}{\cos\frac12 A};\quad \therefore\ \sin\tfrac12(B-C)=\frac{4}{8\sqrt{3}}\cdot\cos 30°=\tfrac14.$$

17. $5=\frac{BC}{AD}=\frac{BD+DC}{AD}=\cot B+\cot C$; also $7=\frac{\cot B\cot C+1}{\cot C-\cot B}$;
hence $7(5-2\cot B)=\cot B(5-\cot B)+1$; hence

$$\cot B=17 \text{ or } 2.$$

18. $\frac{2s-a}{a}=\frac{b+c}{a}=\frac{\cos\frac12(B-C)}{\sin\frac12 A}$ as in Nos. 1, 2; this gives $\frac12(B-C)$; also $\frac12(B+C)=90°-\frac12 A$ is known.

19. $a(p-2r)=2\Delta-2ar=2sr-2ar=2r(s-a)=2r\,.\,r\cot\frac12 A$.

EXERCISE I. f. (p. 21.)

1. $2\Delta=a\,.\,AD=a^2/(\cot B+\cot C)$; $\therefore\ 2\frac{\partial\Delta}{\partial a}=2a/(\cot B+\cot C)=\frac{4\Delta}{a}$.

2. From Ex. I. e, No. 11, $16\Delta^2=4b^2c^2-(b^2+c^2-a^2)^2$;

$$\therefore\ 32\Delta\frac{\partial\Delta}{\partial a}=4a(b^2+c^2-a^2)=4a\,.\,2bc\cos A=32R\Delta\cos A.$$

3. $2bc\cos A=b^2+c^2-a^2$; $\therefore\ 2c\cos A\,.\,\delta b-2bc\sin A\,.\,\delta A\simeq 2b\,.\,\delta b$;
$\therefore\ 2bc\sin A\,.\,\delta A\simeq -2\delta b\,.\,(b-c\cos A)=-2\delta b\,.\,a\cos C$;

$$\therefore\ ba\sin C\,.\,\delta A=-y\,.\,a\cos C;\quad \therefore\ \delta A\simeq -\frac{y\cot C}{b}.$$

4. $4R\Delta=abc$; $\therefore\ 4\Delta.\delta R+4R\,.\,\delta\Delta\simeq bc\,.\,\delta a$; but from No. 2, $\delta\Delta\simeq R\cos A\,.\,\delta a$; $\therefore\ 4\Delta\,.\,\delta R\simeq\delta a\,.\,(bc-4R^2\cos A)$;

$$\therefore\ 2bc\sin A\,.\,\delta R\simeq x\,.\left(bc-\frac{bc\cos A}{\sin B\sin C}\right);$$

$$\therefore\ \sin A\,.\,\delta R\simeq\tfrac12 x\left[1+\frac{\cos(B+C)}{\sin B\sin C}\right]$$

$$=\tfrac12 x\,.\,\frac{\cos B\cos C}{\sin B\sin C}=\tfrac12 x\cot B\cot C.$$

5. $c^2=a^2+b^2-2ab\cos C$;

$$\therefore\ 0\simeq 2a\,.\,\delta a+2b\,.\,\delta b-2\cos C(b\delta a+a\delta b);$$
$$\therefore\ \delta a\,.\,(a-b\cos C)+\delta b\,.\,(b-a\cos C)\simeq 0;$$
$$\therefore\ \delta a\,.\,c\cos B+\delta b\,.\,c\cos A\simeq 0.$$

6. If α, β are the observed angles, the calculated distance $=x=50(\tan\alpha-\tan\beta)$; $\therefore\ \delta x\simeq 50(\sec^2\alpha\,.\,\delta\alpha-\sec^2\beta\,.\,\delta\beta)$; $\leqslant 50(\sec^2\alpha+\sec^2\beta)\,.\,m$, where m is the number of radians in the greatest possible errors in α and β, the error in δx being greatest when $\delta\alpha$ and $\delta\beta$ are of opposite sign. Here

$$m=\tfrac{1}{60}\cdot\frac{\pi}{180},\ \text{since } 1'=\tfrac{1}{60}\cdot\frac{\pi}{180}\ \text{radians};$$

$$\therefore\ \delta x\leqslant 50(2+\tfrac43)\cdot\tfrac{1}{60}\cdot\frac{\pi}{180}\simeq 0\cdot05.$$

7. $c^2=a^2+b^2-2ab\cos C$;
$\therefore\ c\delta c\simeq a\,.\,\delta a+b\,.\,\delta b-\cos C(a\delta b+b\delta a)+ab\sin C\,.\,\delta C$;
$\therefore$ as in No. 5, $c\delta c\simeq\delta a\,.\,c\cos B+\delta b\,.\,c\cos A+ac\sin B\,.\,\delta C$
$=x\,.\,c\cos B+y\,.\,c\cos A+\gamma\,.\,ac\sin B$.

8. $2\Delta=ab\sin C$; $\therefore\ 2\delta\Delta\simeq b\sin C\,.\,\delta a+a\sin C\,.\,\delta b+ab\cos C\,.\,\delta C$,

$$\therefore\ \frac{\delta\Delta}{\Delta}\simeq\frac{1}{ab\sin C}\{b\sin C\,.\,x+a\sin C\,.\,y+ab\cos C\,.\,\gamma\}$$

$$=\frac{x}{a}+\frac{y}{b}+\cot C\,.\,\gamma.$$

9. $2ab\cos C=a^2+b^2-c^2$; $\therefore\ -2ab\sin C\,.\,\frac{\partial C}{\partial c}=-2c$:

$$\therefore\ \frac{\partial C}{\partial c}=\frac{c}{ab\sin C}=\frac{c}{ac\sin B}=\frac{1}{a}\operatorname{cosec}B;$$

also $-2ab\sin C\,.\,\frac{\partial C}{\partial a}+2b\cos C=2a=2(b\cos C+c\cos B)$;

$$\therefore\ \frac{\partial C}{\partial a}=-\frac{2c\cos B}{2ab\sin C}=-\frac{c\cos B}{ac\sin B}=-\frac{1}{a}\cot B;$$

similarly $\frac{\partial C}{\partial b}=-\frac{1}{b}\cot A$; error $\simeq\frac{\partial C}{\partial c}\,.\,z+\frac{\partial C}{\partial a}\,.\,x+\frac{\partial C}{\partial b}\,.\,y$.

10. $\sin B=\frac{b}{2R}$; $\therefore$ B may be regarded as correct;

$$a^2+c^2-2ac\cos B=b^2;$$
$$\therefore\ 2a\,.\,\delta a+2c\,.\,\delta c-2\cos B(c\,\delta a+a\,\delta c)\simeq 0;$$
$$\therefore\ \delta c(c-a\cos B)\simeq -x(a-c\cos B);$$
$$\therefore\ \delta c\,.\,b\cos A\simeq -x\,.\,b\cos C.$$

11. $CF(\cot A+\cot B)=c$; $\therefore\ \Delta=\frac{\frac12 c^2}{\cot A+\cot B}$;

$$\therefore\ \frac{\partial\Delta}{\partial A}=\frac{\frac12 c^2\operatorname{cosec}^2A}{(\cot A+\cot B)^2}=\frac{2}{c^2\sin^2A}\cdot\Delta^2$$

$$=\frac{2}{c^2\sin^2A}\cdot(\tfrac12 bc\sin A)^2=\tfrac12 b^2;\ \text{similarly for } \frac{\partial\Delta}{\partial B}.$$

From $\dfrac{\partial\Delta}{\partial A}=\dfrac{2\Delta^2}{c^2\sin^2 A}$,

$$\frac{\partial^2\Delta}{\partial A\,\partial B}=\frac{4\Delta}{c^2\sin^2 A}\cdot\frac{\partial\Delta}{\partial B}=\frac{4\Delta}{a^2\sin^2 C}\cdot\tfrac{1}{2}a^2=\frac{2\Delta}{\sin^2 C}.$$

The small area is roughly that of a parallelogram, of sides $2a\operatorname{cosec} C\,.\,\beta$, $2b\operatorname{cosec} C\,.\,\alpha$, and angle C;

$$\therefore\ \text{area} \simeq 4ab\operatorname{cosec}^2 C\,.\,\alpha\beta\,.\,\sin C=4\alpha\beta\,.\,2\Delta\operatorname{cosec}^2 C.$$

[The result can be obtained in a more general manner by applying the Second Mean Value Theorem in the calculus to the function

$$\Delta(A+\alpha, B+\beta)-\Delta(A-\alpha, B+\beta)$$
$$-\Delta(A+\alpha, B-\beta)+\Delta(A-\alpha, B-\beta),$$

where $\Delta(\theta, \phi)$ denotes the area of the triangle with base angles θ, ϕ.]

12. If error in C is γ radians,

$$\tfrac{1}{2}ab-z=\tfrac{1}{2}(a-x)(b-y)\sin\left(\frac{\pi}{2}\pm\gamma\right)\simeq\tfrac{1}{2}(ab-bx-ay)\cos\gamma;$$

$$\therefore\ ab(1-\cos\gamma)\simeq 2z-(bx+ay)\cos\gamma;$$

$$\therefore\ 2ab\sin^2\frac{\gamma}{2}\simeq 2z-(bx+ay);\quad \therefore\ 2ab\left(\frac{\gamma}{2}\right)^2\simeq 2z-bx-ay;$$

$$\therefore\ \text{error in C}=\gamma\simeq\sqrt{\left\{\frac{2(2z-bx-ay)}{ab}\right\}}\ \text{radians.}$$

EXERCISE I. g. (p. 22.)

1. $(s-a)+(s-b)=2(s-c)$; $\therefore\ r\cot\tfrac{1}{2}A+r\cot\tfrac{1}{2}B=2r\cot\tfrac{1}{2}C$.

2. $\Sigma(a\cos B\cos C)=2R\,.\,\Sigma(\sin A\cos B\cos C)=2R\sin A\sin B\sin C$ since $\sin(A+B+C)=0$, (expand);
$\Sigma(ab\sin^2 C)=\Sigma(c^2\sin A\sin B)=\Sigma(2R\,.\,c\,.\,\sin A\sin B\sin C)$
$=2R\sin A\sin B\sin C\,.\,\Sigma c=2R\sin A\sin B\sin C\,.\,2s$.

3. Left side $=4R\sin A\,.\,R\sin(B-C)=4R^2\sin(B+C)\,.\,\sin(B-C)$
$=4R^2(\sin^2 B-\sin^2 C)=b^2-c^2$.

4. Right side $=r^2\,.\,\dfrac{s-a}{r}\cdot\dfrac{s-b}{r}\cdot\dfrac{s-c}{r}=\dfrac{\Delta^2}{rs}=\Delta$.

5. See solution of Ex. I. b, Nos. 12, 13.

6. As in No. 5, $r_1-r=4R\sin^2\tfrac{1}{2}A$;
$\therefore$ right side $=4R\sin^2\tfrac{1}{2}A\,.\,(4R-4R\sin^2\tfrac{1}{2}A)$
$=4R\sin^2\tfrac{1}{2}A\,.\,4R\cos^2\tfrac{1}{2}A=4R^2\sin^2 A=a^2$.

7. $IA=r\operatorname{cosec}\dfrac{A}{2}$; $II_1=4R\sin\dfrac{A}{2}$ from Ex. I. b, No. 5 (ii).

8. If O′ is circumcentre of $\Delta I_1I_2I_3$, $O'I_1$ is perp. to BC, [this is the same as OA is perp. to EF];
$\therefore$ area of quad. $O'BI_1C=\tfrac{1}{2}O'I_1\,.\,BC=\tfrac{1}{2}(2R)\,.\,a$;
$\therefore\ \Delta I_1I_2I_3=\Sigma O'BI_1C=R\,.\,\Sigma a=2Rs=\dfrac{2R\Delta}{r}=\dfrac{abc}{2r}$.

9. Circumradius $=\dfrac{BC}{2\sin BIC}=\dfrac{a}{2\sin(90^\circ+\tfrac{1}{2}A)}=\dfrac{R\sin A}{\cos\dfrac{A}{2}}=2R\sin\dfrac{A}{2}$;
B, I, C, I_1 are concyclic; $\therefore$ Δs have same circumcircle.

10. Draw IV perp. to AD, $\angle IAD=\tfrac{1}{2}(B\sim C)$;
$\therefore\ PD=IV=IA\sin\tfrac{1}{2}(B\sim C)=r\operatorname{cosec}\tfrac{1}{2}A\sin\tfrac{1}{2}(B\sim C)$
$=4R\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\,.\,\sin\tfrac{1}{2}(B\sim C)$.

11. $\Delta OAI=\tfrac{1}{2}AO\,.\,AI\sin\tfrac{1}{2}(B\sim C)=\tfrac{1}{2}R\,.\,r\operatorname{cosec}\tfrac{1}{2}A\,.\,\sin\tfrac{1}{2}(B\sim C)$
$=R^2\cdot\dfrac{\sin A}{a}\cdot\dfrac{r\sin\tfrac{1}{2}(B\sim C)}{\sin\tfrac{1}{2}A}=\dfrac{R^2r}{a}\cdot 2\cos\tfrac{1}{2}A\sin\tfrac{1}{2}(B\sim C)$
$=\dfrac{R^2r}{a}\cdot 2\sin\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}(B\sim C)$.

12. Incircle cuts AD at U, W; V is mid-point of UW;
cosine of required angle $=\sin IUV=\dfrac{IV}{r}$
$=\operatorname{cosec}\tfrac{1}{2}A\,.\,\sin\tfrac{1}{2}(B\sim C)$ from No. 10.

13. $2R=r_1-r=$, from I. b, No. 13,
$4R\sin^2\dfrac{A}{2}=2R(1-\cos A)$; $\therefore\ \cos A=0$.

14. $a-b=2R(\sin A-\sin B)=2R(\sin 54^\circ-\sin 18^\circ)$
$=2R\left(\dfrac{\sqrt5+1}{4}-\dfrac{\sqrt5-1}{4}\right)=R$.

15. $\dfrac{2r}{R}=8\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C=2(\cos A+\cos B+\cos C-1)$
$=2(\tfrac{3}{2}+\cos C-1)=1+2\cos C$.

16. $r=(s-b)\tan\tfrac{1}{2}B$;
$\therefore$ difference $=\tan\tfrac{1}{2}B\,.\,(s_1-s_2)=\tan\tfrac{1}{2}B\,.\,\tfrac{1}{2}(c_1-c_2)$
$=\tan\tfrac{1}{2}B\,.\,b\cos A_1=\tan\tfrac{1}{2}B\,.\,\sqrt{(b^2-b^2\sin^2 A_1)}$
$=\tan\tfrac{1}{2}B\,.\,\sqrt{(b^2-a^2\sin^2 B)}$.

17. Left side $=8R^3 . \Sigma(\sin^3 A \cos A)=2R^3 . \Sigma(3\sin A-\sin 3A)\cos A$
$=R^3 . \Sigma[3\sin 2A-(\sin 4A+\sin 2A)]=R^3 . \Sigma(2\sin 2A-\sin 4A)$
$=$, *E.T.*, p. 271, $R^3 . (8\sin A\sin B\sin C+4\sin 2A\sin 2B\sin 2C)$
$=R^3 . 8\sin A\sin B\sin C(1+4\cos A\cos B\cos C)$.

18. From the solution of Ex. I. b, No. 13,

$$\frac{r_1-r}{4R}=\sin^2\frac{A}{2};\ \text{but } r_1=s . \tan\frac{A}{2};$$

$$\therefore\ \frac{r_1-r}{4R}=\frac{r_1^2}{r_1^2+s^2}=\frac{r^2r_1^2}{r^2r_1^2+\Delta^2};$$

$\therefore$ r_1 satisfies the equation $(x-r)(r^2x^2+\Delta^2)=4Rr^2x^2$; similarly r_2 and r_3 do so.

19. From the solution of No. 8,

$$\frac{a+b+c}{\Delta I_1I_2I_3}=\frac{1}{R}=\frac{4R\sin A\sin B\sin C}{2\Delta}$$

$$=\frac{R . \Sigma(\sin 2A)}{2\Delta}=\frac{R . \Sigma(2\sin A\cos A)}{2\Delta}=\frac{\Sigma(a\cos A)}{2\Delta}.$$

20. X is centre and EF is a chord of circle on BC as diameter $\therefore\ \angle FXT=\frac{1}{2}\angle FXE=\angle FBE=90°-A$, and

$$XF=\frac{a}{2};\ \therefore\ XT=\frac{a}{2}\cos(90°-A).$$

21. $EF=a\cos A$, etc.; $\therefore\ EF^2+FD^2=DE^2$; $\therefore\ \angle EFD=90°$;
$\therefore\ 180°-2C=90°$.

22. $BC=2BX$; $\therefore$ by parallels $EC=2EA$; also $a^2-c^2=BC^2-BA^2$
$=EC^2-EA^2$ (Pythagoras) $=4EA^2-EA^2=3EA^2=\frac{3}{2} . EA . EC$
$=\frac{3}{2}[c\cos(180°-A) . a\cos C]=-\frac{3}{2}ac\cos A\cos C$.

23. $a+c=2b$; $\therefore\ 2s=a+b+c=3b$;

$$\therefore\ \tan\frac{A}{2}\tan\frac{C}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\cdot\frac{(s-a)(s-b)}{s(s-c)}\right\}}$$

$$=\frac{s-b}{s}=\frac{s-\frac{2}{3}s}{s}=\frac{1}{3}.$$

Then $1+\cos A\cos C-\cos A-\cos C=(1-\cos A)(1-\cos C)$

$$=4\sin^2\frac{A}{2}\sin^2\frac{C}{2}=\sin A\sin C . \tan\frac{A}{2}\tan\frac{C}{2}=\frac{1}{3}\sin A\sin C.$$

24. N is mid-point of AB, $AN=NB=k$; $CN=k\sqrt{3}=DN$;
$\therefore\ CD=\sqrt{2} . k\sqrt{3}=k . \sqrt{6}$;

but $CA=DA=2k$; $\therefore\ \cos CAD=\dfrac{4+4-6}{8}=\frac{1}{4}$.

25. $\dfrac{AB}{IB}=\dfrac{\sin BIX}{\sin BAI}=\dfrac{\sin 54°}{\sin 18°}$; but $4\sin 18° . \cos 36°$

$$=\frac{2\sin 36°}{\cos 18°} . \cos 36°=\frac{\sin 72°}{\cos 18°}=1;$$

thus $\dfrac{AB}{IB}=\dfrac{\cos 36°}{\sin 18°}=4\cos^2 36°$

$$=2(1+\cos 72°)=2\left(1+\frac{BX}{AB}\right)=\frac{2s}{c}.$$

26. AD is diameter of circle ATDT′;

$$\therefore\ AD=\frac{TT'}{\sin A};\ \therefore\ TT'=AD . \sin A=\frac{2\Delta . \sin A}{a}=\frac{\Delta}{R}.$$

27. Let triangle be A′B′C′; then $A'B=A'C=R\tan A$;

$$\therefore\ \Delta A'B'C'=\Sigma(\text{quad. }OBA'C)=\Sigma(R^2\tan A)$$
$$=R^2 . \tan A\tan B\tan C.$$

28. Suppose K is centre and x radius of either circle.
$OA=R$, $KA=x\operatorname{cosec}\frac{1}{2}A$, $OK=R\mp x$, $\angle OAK=\frac{1}{2}(B\sim C)$;
$\therefore\ R^2+x^2\operatorname{cosec}^2\frac{1}{2}A-2Rx\operatorname{cosec}\frac{1}{2}A\cos\frac{1}{2}(B-C)=(R\mp x)^2$.

Taking the upper sign,

$x^2(\operatorname{cosec}^2\frac{1}{2}A-1)=2Rx\operatorname{cosec}\frac{1}{2}A\{\cos\frac{1}{2}(B-C)-\cos\frac{1}{2}(B+C)\}$;
$\therefore\ x^2\cot^2\frac{1}{2}A=4Rx\operatorname{cosec}\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$;
$\therefore\ x=4R\tan^2\frac{1}{2}A\operatorname{cosec}^2\frac{1}{2}A\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$;
$\therefore\ x=\sec^2\frac{1}{2}A . 4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C=r\sec^2\frac{1}{2}A$.

Taking the lower sign,

$x^2(\operatorname{cosec}^2\frac{1}{2}A-1)=2Rx\operatorname{cosec}\frac{1}{2}A\{\cos\frac{1}{2}(B-C)+\cos\frac{1}{2}(B+C)\}$;
$\therefore\ x^2\cot^2\frac{1}{2}A=4R\operatorname{cosec}\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$;
$\therefore\ x=4R\tan^2\frac{1}{2}A\operatorname{cosec}^2\frac{1}{2}A\sin\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$;
$\therefore\ x=\sec^2\frac{1}{2}A . 4R\sin\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C=r_1\sec^2\frac{1}{2}A$.

29. Take C as origin and the line through C as x-axis; then CA, CB may be taken as making angles α, $\alpha+C$ with the x-axis (where α may be reflex). Then l, m are the y-coordinates of A, B and so $l=b\sin\alpha$,

$$m=a\sin(\alpha+C)=a(\sin\alpha\cos C+\cos\alpha\sin C)$$

$$=a\left(\frac{l}{b}\cos C+\cos\alpha\sin C\right);\ \therefore\ \frac{m}{a}-\frac{l}{b}\cos C=\cos\alpha\sin C,$$

also $\frac{l}{b}\sin C = \sin \alpha \sin C$; square and add,

$$\left(\frac{m}{a}-\frac{l}{b}\cos C\right)^2+\frac{l^2}{b^2}\sin^2 C=\sin^2 C=\frac{4\Delta^2}{a^2b^2};$$

$$\therefore \frac{m^2}{a^2}+\frac{l^2}{b^2}-\frac{2ml}{ab}\cos C=\frac{4\Delta^2}{a^2b^2}.$$

30. P, Q are points on CA, CB such that $CP = x$, $CQ = y$. If $\triangle CPQ = \frac{1}{2}\triangle CAB$, $xy = \frac{1}{2}ab$; also

$$PQ^2 = x^2+y^2-2xy\cos C = x^2+y^2-2xy\left(1-2\sin^2\frac{C}{2}\right)$$

$$=(x-y)^2+4xy\sin^2\frac{C}{2}=(x-y)^2+2ab\sin^2\frac{C}{2};$$

$\therefore$ PQ is least if $x=y$ and then

$$PQ=\sqrt{\left(2ab\sin^2\frac{C}{2}\right)}$$

$$=\sqrt{\left(ab\sin C\,.\,\tan\frac{C}{2}\right)}=\sqrt{\left(2\Delta\,.\,\tan\frac{C}{2}\right)};$$

this is less than the corresponding lines across the angles A, B because $C<B<A$. Also P, Q are internal points for CA, CB if $x<b$; but for $x=y$, $x^2=\frac{1}{2}ab$; $\therefore$ condition is $\frac{1}{2}ab<b^2$ or $a<2b$, which is true, because $a<b+c<2b$.

31. $2bc\cos A = b^2+c^2-a^2$;

$$\therefore 2c\cos A\,.\,\delta b+2b\cos A\,.\,\delta c-2bc\sin A\,.\,\delta A = 2b\,\delta b+2c\,\delta c-2a\,\delta a;$$

$$\therefore 2\Delta\,.\,\delta A = a\,.\,\delta a-\delta b(b-c\cos A)-\delta c(c-b\cos A)$$

$$= a\,.\,\delta a-\delta b\,.\,a\cos C-\delta c\,.\,a\cos B.$$

32. $(a+b+c)r=2\Delta$; $\therefore (a+b+c)\,.\,\delta r+r\,.\,\delta a=2\frac{\partial\Delta}{\partial a}\,.\,\delta a=$, by Ex. I. f, No. 2, $2R\cos A\,.\,\delta a$; $\therefore 2s\,.\,\delta r=x(2R\cos A-r)$.

CHAPTER II.

EXERCISE II. a. (p. 26.)

1. Substitute in eqn. (4); $s=11$; $\Delta=\sqrt{(7\,.\,6\,.\,5\,.\,4)}=\sqrt{840}$.

2. Substitute in eqn. (5); $x=\sqrt{\left(\frac{58\times 59}{62}\right)}$; $y=\sqrt{\left(\frac{62\times 59}{58}\right)}$.

3. Substitute in eqn. (7); $S=\sqrt{840}$; $\therefore 2R=\frac{\sqrt{(62\times 59\times 58)}}{2\sqrt{840}}$.

4. Use eqn. (8); $\cos B=\frac{4^2+5^2-6^2-7^2}{2(20+42)}=-\frac{11}{31}$.

5. $\tan^2\frac{B}{2}=\frac{1-\cos B}{1+\cos B}=\frac{2(ab+cd)-(a^2+b^2-c^2-d^2)}{2(ab+cd)+(a^2+b^2-c^2-d^2)}$ from eqn. (8)

$$=\frac{(c+d)^2-(a-b)^2}{(a+b)^2-(c-d)^2}=\frac{(c+d+a-b)(c+d-a+b)}{(a+b+c-d)(a+b-c+d)}.$$

6. From No. 5, $(s-d)^2\tan^2\frac{B}{2}=\frac{(s-a)(s-b)(s-d)}{s-c}$

$=$ similarly $(s-b)^2\tan^2\frac{A}{2}$; take square root of each;

$\tan\frac{A}{2}$, $\tan\frac{B}{2}$ are both positive.

7. From No. 5, $\tan\frac{C}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{(s-a)(s-d)}\right\}}$;

$$\tan\frac{D}{2}=\sqrt{\left\{\frac{(s-c)(s-d)}{(s-a)(s-b)}\right\}}.$$

8. From eqn. (8), $\tan A=\frac{2S}{ad+bc}\div\frac{a^2+d^2-b^2-c^2}{2(ad+bc)}$.

9. Quad. becomes triangle. Results are

(i) $\Delta=\sqrt{\{s(s-a)(s-b)(s-c)\}}$

(ii) $\cos(\angle a, b)=\frac{a^2+b^2-c^2}{2ab}$; (iii) $4R\Delta=abc$.

10. $a+c=b+d$, $\therefore\ =s$; $\therefore S=\sqrt{(abcd)}$, $\tan^2\frac{B}{2}=\frac{cd}{ab}$. A circle can be inscribed in the quadrilateral.

11. If D is $\angle$ opposite 60°,

$$4^2+3^2-2\,.\,4\,.\,3\cos D=7^2+4^2-2\,.\,7\,.\,4\cos 60°=37;$$

$\therefore \cos D=-\frac{1}{2}$, $D=120°$; $\therefore$ sum of opposite $\angle$s is 180°.;

radius $=\frac{\sqrt{37}}{2\sin 60°}$.

12. $b^2=OB^2+OC^2+2OB\,.\,OC\cos\theta$, etc.;

$$\therefore 2xy\cos\theta=2(AO+OC)(BO+OD)\cos\theta$$

$$=2\cos\theta(AO\,.\,BO+AO\,.\,DO+CO\,.\,BO+CO\,.\,DO)$$

$$=(AO^2+BO^2-a^2)+(d^2-OA^2-OD^2)+(b^2-OB^2-OC^2)+(OD^2+OC^2-c^2)=d^2+b^2-a^2-c^2;$$

$$\therefore \cos\theta=\frac{b^2+d^2-a^2-c^2}{2xy}=\frac{b^2+d^2-a^2-c^2}{2(bd+ac)};$$

then as in No. 5.

13. $S = \frac{1}{2}xy \sin\theta$, see *E.T.*, pp. 178, 179; but $xy = ac + bd$.

14. $\triangle$OAD is similar to $\triangle$OBC; $\therefore \frac{OA}{d} = \frac{OB}{b}$; $\therefore \frac{OA}{ad} = \frac{OB}{ab}$ = similarly

$$\frac{OC}{bc} = \frac{OD}{cd} = \frac{OA + OC}{ad + bc} = \frac{x}{ad + bc};$$ use eqn. (5).

15. Use values of BO, OD obtained in No. 14.

16. $\triangle$QAB is similar to $\triangle$QCD;

$$\therefore \frac{QA}{QC} = \frac{a}{c} = \frac{QB}{QD}; \quad \therefore \frac{QA}{QB + b} = \frac{a}{c} = \frac{QB}{QA + d};$$

solve for QA, QB.

$$QB = \frac{a(ab + cd)}{c^2 - a^2}; \quad QC = \frac{c(bc + ad)}{c^2 - a^2};$$

$$PA = \frac{d(ad + bc)}{d^2 - b^2}; \quad PB = \frac{b(ab + cd)}{d^2 - b^2}.$$

17. $\angle QKB = 180° - \angle QAB = 180° - \angle BCD = 180° - \angle PKB$; $\therefore$ PKQ is a st. line. Use the results found in No. 16.

EXERCISE II. b. (p. 28.)

1. Use eqn. (9); this gives $\cos^2\frac{B + D}{2} = \frac{1}{26}$; $\therefore \cos(B + D) = \frac{1}{13} - 1$.

2. Use eqn. (13).

3. From eqn. (10), $xy \sin\theta = 2S = 276$; from No. 2,

$$\sin\theta = \frac{46}{\sqrt{(46^2 + 3^2)}}.$$

4. From eqn. (13), $\tan\theta = \frac{4 \times 33}{55}$.

5. From eqn. (13), $S = \frac{1}{4}(55)\tan 60°$; from eqn. (10), $xy = 2S \operatorname{cosec}\theta$.

6. From eqn. (6), $xy = 26$; from eqn. (11), $52\cos\theta = 28$.

7. (i) DACB is a st. line; (ii) ABDC is a st. line. In each case, $xy = DB \cdot AC = AB \cdot CD \sim BC \cdot AD$.

8. (i) $BX = a - d = b - c = BY$;
(ii) Since AXD, BXY, CYD are isosceles, the bisectors of their angles A, B, C bisect the bases DX, XY, YD at right angles; $\therefore$ they meet at the circumcentre of DXY, and this point is equidistant from the sides of ABCD.

9. Put $B + D = 90°$ in eqn. (12).

10. (i) $a + c = b + d$; $\therefore a^2 + 2ac + c^2 = b^2 + 2bd + d^2$;
$\therefore b^2 + d^2 - a^2 - c^2 = 2(ac - bd)$; hence by eqn. (13);
(ii) $ac - bd = a(b + d - a) - bd = (b - a)(a - d)$;
(iii) From eqns. (10) and (11),
$$4x^2y^2 = 16S^2 + (b^2 + d^2 - a^2 - c^2)^2 = 16S^2 + 4(ac - bd)^2,$$
from work in (i).

11. Radius $= \frac{S}{s}$; $s = \frac{1}{2}(a + b + c + d) = a + c$; also from eqn. (15),
$$S = \sqrt{(abcd)} \cdot \sin\frac{B + D}{2} = \sqrt{(abcd)} \cdot \sin\frac{A + C}{2}.$$

12. By the formula $\sin\frac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{bc}}$ applied to $\triangle$ABC,
$$\sqrt{(ab)} \cdot \sin\tfrac{1}{2}B = \tfrac{1}{2}\sqrt{\{(x - a + b)(x + a - b)\}} = \tfrac{1}{2}\sqrt{\{x^2 - (a - b)^2\}}$$
$$= \tfrac{1}{2}\sqrt{\{x^2 - (c - d)^2\}} = \text{similarly } \sqrt{(cd)} \cdot \sin\tfrac{1}{2}D.$$

13. From eqns. (10), (11), $\tan\theta = \frac{4S}{b^2 + d^2 - a^2 - c^2}$; from eqn. (15),
$S = \sqrt{(abcd)} \cdot \sin\frac{A + C}{2}$. Also as in No. 10,
$$b^2 + d^2 - a^2 - c^2 = 2(ac - bd).$$

14. $a + b = c + d = s$; $\therefore s - a = b$, etc. From eqn. (9),
$$S^2 = badc - abcd\cos^2\frac{B + D}{2} = abcd \cdot \sin^2\frac{B + D}{2}.$$

15. Suppose P, R, Q, S are points in order on minor arc of a circle; tangent at P cuts tangents at S, Q in A, B; tangent at R cuts tangents at S, Q in D, C. Consider quad. ABCD.
$\alpha, \beta, \gamma, \delta$ being the lengths of the tangents from A, B, C, D, $a = \alpha - \beta$, $b = \beta - \gamma$, $c = \delta - \gamma$, $d = \alpha - \delta$; $\therefore a + b = c + d$.

16. Quad. is circumscribable because $4 + 5 = 3 + 6$.
(i) From eqn. (16), $S = \sqrt{(4 \cdot 3 \cdot 5 \cdot 6)}$;
(ii) Inradius $= \frac{S}{s} = \frac{\sqrt{360}}{9}$; (iii) Use eqn. (8);
(iv) $x^2 = \frac{39 \times 38}{42}$; $y^2 = \frac{42 \times 38}{39}$;
(v) From eqn. (7), $R = \frac{\sqrt{(39 \times 38 \times 42)}}{4\sqrt{360}}$.

17. $12 + 11 + x = 13 + 8 + 9$; $r = \frac{60}{s} = \frac{60}{30}$.

18. For n-gon, $2n - 3$ independent elements are required.
Pentagon ABCDE; AB = 39, BC = 52, etc.; AC = 65.

$AB : BC : CA = 3 : 4 : 5$; $\therefore \angle ABC = 90°$.

Area of $\triangle ABC = \frac{1}{2} \times 39 \times 52 = 1014$.

For quad. ACDE, $s = \frac{1}{2}(65 + 39 + 33 + 25) = 81$.

Area of ACDE $= \sqrt{(16 \times 42 \times 48 \times 56)} = 1344$.

Total area $= 1014 + 1344$.

EXERCISE II. c. (p. 29.)

1. $1^2 + 3^2 - 6 \cos \alpha = 6^2 + 4^2 + 48 \cos \alpha$.
2. Eqn. (4) shows area does not depend on order of sides; Eqn. (7) shows radius is also independent. For possible diagonals, $x^2 = \frac{194 \times 197}{202}$; $y^2 = \frac{202 \times 197}{194}$; $z^2 = \frac{194 \times 202}{197}$.
3. Since $BA = BD$, $\sin(\frac{1}{2}\angle ABD) = \frac{39}{65} = \frac{3}{5}$; similarly $\sin(\frac{1}{2}\angle DBC) = \frac{5}{13}$; $\therefore \cos(\frac{1}{2}\angle ABD) = \frac{4}{5}$, $\cos(\frac{1}{2}\angle DBC) = \frac{12}{13}$; $\therefore \sin(\frac{1}{2}\angle ABC) = \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \frac{5}{13} = \frac{56}{65}$;
   $\therefore AC = 2 \times 65 \times \sin(\frac{1}{2}\angle ABC) = 112$.
   Also perpendicular from B to AD $= \sqrt{(65^2 - 39^2)} = 52$; perp. from B to CD $= \sqrt{(65^2 - 25^2)} = 60$;
   $\therefore$ area $= \frac{1}{2}(52 \times 78 + 60 \times 50)$.
   Since the centre of circle ADC is B, the quad. is not cyclic.
4. Produce AB, DC to meet at E; then $EB = BC = 1 = CE$;
   $\therefore d^2 = AE^2 + ED^2 - 2AE \cdot ED \cos E = 5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cdot \cos 60°$.
5. Produce CB, DA to meet at E, then $\angle AEB = 30°$; $\therefore EA = AB = 1$, and $EB = 2 \cos 30° = \sqrt{3}$, $EC = 2\sqrt{3}$, $\sin D / \sin 30° = 2\sqrt{3}/2$; $\therefore D = 60°$ or $120°$, and
   $$d = 2\sqrt{3} \cos 30° \pm 2 \cos 60° - 1 = 3 \text{ or } 1.$$
6. $\triangle BCD$ is determined uniquely. In $\triangle ABD$, we know AB, $\angle BAD$, and BD from $\triangle BCD$. Ambiguous case.
7. $BD^2 = 12^2 + 5^2$; $\therefore BD = 13$; $\therefore 13^2 = 7^2 + d^2 - 14d \cos 60°$;
   $\therefore d^2 - 7d - 120 = 0$.
8. $BD^2 = 12^2 + 5^2$; $\therefore BD = 13$; $\therefore 13^2 = 14^2 + d^2 - 28d \cos 60°$;
   $\therefore d^2 - 14d + 27 = 0$.
9. AC is perp. to BC and $= \sqrt{3}$; hence
   $\angle ACD = 60°$ and $\angle CAD = 60°$;
   $\therefore \triangle CAD$ is equilateral and $AD = \sqrt{3} = CD$.
10. Produce DA, CB to cut at E; $\triangle EAB$ is equilateral;
   $\therefore EA = EB = \sqrt{3}$; $BC = EC - EB = 5 \sec 30° - \sqrt{3}$;
   $AD = ED - EA = 5 \tan 30° - \sqrt{3}$.

11. From eqn. (11), $2xy \cos \theta = 0$; $\therefore \theta = 90°$.
12. AC is diameter of circle ABCD; $\therefore x = 2 \times$ circumradius of $\triangle ABD = \frac{BD}{\sin 60°} = \frac{2y}{\sqrt{3}}$.
13. $d^2 - a^2 = BD^2 = b^2 + c^2 - 2bc \cos BCD = b^2 + c^2 + 2bc \cos BAD$;
   $$\cos BAD = \frac{a}{d}.$$
14. $a + d = b + c$, $\therefore = s$; $\therefore$ from eqn. (4), $S = \sqrt{(dcba)}$; also from II. a, No. 5; $\tan^2 \frac{A}{2} = \frac{(s-a)(s-d)}{(s-b)(s-c)} = \frac{da}{cb}$. As in II. b, No. 15; circle touches sides of quad. produced.
15. $a + c = b + d = s$; from eqn. (15), $S = \sqrt{(abcd)} \cdot \sin \frac{B + D}{2}$; also as in II. b, No. 12, $\frac{ab \sin^2 \frac{1}{2}B}{cd \sin^2 \frac{1}{2}D} = \frac{x^2 - (a-b)^2}{x^2 - (c-d)^2} = 1$.
16. (i) From eqn. (16), $S = \sqrt{(abcd)}$; from eqn. (10), $xy \sin \theta = 2S$; from eqn. (6), $xy = ac + bd$; $\therefore \sin \theta = \frac{2S}{xy} = \frac{2\sqrt{(abcd)}}{ac + bd}$, whence $\cos \theta = \frac{ac - bd}{ac + bd}$ and $\tan^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{bd}{ac}$.
   (iii) From eqn. (8), $\cos A = \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)}$; but $a - d = b - c$;
   $\therefore a^2 + d^2 - 2ad = b^2 + c^2 - 2bc$;
   $\therefore a^2 + d^2 - b^2 - c^2 = 2(ad - bc)$.
17. n-gon is $A_1A_2A_3 \ldots$; O is centre, R is radius of circle; $\angle A_1OA_2 = \frac{2\pi}{n}$; $\triangle OA_1A_2 = \frac{1}{2}R^2 \sin \frac{2\pi}{n}$; area of n-gon is $\frac{nR^2}{2} \sin \frac{2\pi}{n}$;
   $\therefore$ area of $2n$-gon is $\frac{2nR^2}{2} \sin \frac{2\pi}{2n}$;
   $$\text{ratio of areas} = \sin \frac{2\pi}{n} : 2 \sin \frac{\pi}{n}.$$
18. As in No. 17, area of inscribed n-gon is $\frac{nR^2}{2} \sin \frac{2\pi}{n}$; area of circumscribed n-gon is $nR^2 \tan \frac{\pi}{n}$;
   $$\text{ratio of areas} = \sin \frac{2\pi}{n} : 2 \tan \frac{\pi}{n}.$$
19. With the notation of No. 17, if M is the mid-point of A_1A_2, $C - B = \pi \cdot OA_1^2 - \pi \cdot OM^2 = \pi l^2$, and $A = nl \cdot OM$;
   $\therefore \pi A^2 = n^2 l^2 \cdot \pi OM^2 = n^2 l^2 B$.

20. With the notation of No. 19, if $2p$ is the perimeter, the area is $p\,.\,\mathrm{OM}$ and $\mathrm{A_1M} = \frac{p}{n}$ or $\frac{p}{2n}$, thus $\mathrm{OM} = \frac{p}{n}\cot\frac{\pi}{n}$ or $\frac{p}{2n}\cot\frac{\pi}{2n}$; thus the ratio is

$$\cot\frac{\pi}{n} : \tfrac{1}{2}\cot\frac{\pi}{2n} = \cos\frac{\pi}{n} : \sin\frac{\pi}{2n}\,.\,\cos\frac{\pi}{2n}\,.\,\cot\frac{\pi}{2n}.$$

21. Perimeter $2p$; $r_n = \frac{p}{n}\cot\frac{\pi}{n}$; $\mathrm{R}_n = \frac{p}{n}\operatorname{cosec}\frac{\pi}{n}$;

$$\therefore\ r_n + \mathrm{R}_n = \frac{p}{n}\left(\frac{\cos\frac{\pi}{n} + 1}{\sin\frac{\pi}{n}}\right) = \frac{p}{n}\cot\frac{\pi}{2n} = 2r_{2n}.$$

$$\text{Also } \mathrm{R}_n\,.\,r_{2n} = \frac{p}{n}\operatorname{cosec}\frac{\pi}{n}\,.\,\frac{p}{2n}\cot\frac{\pi}{2n} = \frac{p^2}{4n^2}\operatorname{cosec}^2\frac{\pi}{2n} = \mathrm{R}_{2n}^2.$$

22. Perimeter $12p$; area of square $= 9p^2$; side of hexagon $= 2p$; area of hexagon $= 6p^2\cot 30° = 6p^2\sqrt{3}$; ratio of areas $= 9 : 6\sqrt{3} = \sqrt{3} : 2 = 0{\cdot}8660\ldots$; $\frac{13}{15} = 0{\cdot}8666\ldots$.

EXERCISE II. d. (p. 30.)

1. If $\angle$B means reflex $\angle$ABC and if θ is $\angle$ between DB(D$\to$B) and CA (C$\to$A), eqns. (9) to (13) remain true and are proved in a similar way.

2. From equation (3) since the areas are equal, $\cos(90° + \mathrm{D}) = \cos(90° + \mathrm{B}')$; $\therefore\ \sin\mathrm{D} = \sin\mathrm{B}'$; also $ab + cd\sin\mathrm{D} = 2\mathrm{S} = ab\sin\mathrm{B}' + cd = ab\sin\mathrm{D} + cd$; $\therefore\ (ab - cd)(1 - \sin\mathrm{D}) = 0$.

3. $\mathrm{B} = 90°$; $\angle\mathrm{DAC} = 90°$; $\triangle\mathrm{ABC} = \tfrac{1}{2}\,.\,24\,.\,7$; $\triangle\mathrm{ADC} = \tfrac{1}{2}\,.\,60\,.\,25$; $\therefore\ \mathrm{S} = 84 + 750$; $\cos\mathrm{A} = -\sin\mathrm{BAC} = -\frac{7}{25}$; $\therefore\ y^2 = 24^2 + 60^2 + 2\times 24\times 60\times\frac{7}{25}$.

4. $13^2 + 14^2 - 2\,.\,13\,.\,14\cos\mathrm{B} = 9^2 + 12^2 - 2\,.\,9\,.\,12\cos\mathrm{D}$ gives
$$91\cos\mathrm{B} - 54\cos\mathrm{D} = 35;$$
$138 = \mathrm{S} = \tfrac{1}{2}\,.\,13\,.\,14\,.\,\sin\mathrm{B} + \tfrac{1}{2}\,.\,9\,.\,12\,.\,\sin\mathrm{D}$ gives
$$91\sin\mathrm{B} + 54\sin\mathrm{D} = 138;$$
$\therefore\ (54\cos\mathrm{D} + 35)^2 + (138 - 54\sin\mathrm{D})^2 = 91^2(\cos^2\mathrm{B} + \sin^2\mathrm{B})$, this reduces to $138\sin\mathrm{D} - 35\cos\mathrm{D} = 138$;
$$\therefore\ 138\left[1 - \cos\left(\frac{\pi}{2} - \mathrm{D}\right)\right] = -35\sin\left(\frac{\pi}{2} - \mathrm{D}\right);$$
$$\therefore\ 138\sin^2\left(\frac{\pi}{4} - \frac{\mathrm{D}}{2}\right) = -35\sin\left(\frac{\pi}{4} - \frac{\mathrm{D}}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\mathrm{D}}{2}\right);$$
$$\therefore\ \sin\left(\frac{\pi}{4} - \frac{\mathrm{D}}{2}\right) = 0 \text{ or } \tan\left(\frac{\pi}{4} - \frac{\mathrm{D}}{2}\right) = -\frac{35}{138};$$
$\therefore\ \mathrm{D} = 90°$ or $118°\,28'$; $x^2 = 81 + 144 - 2\,.\,9\,.\,12\cos\mathrm{D}$.

5. $\mathrm{S} \equiv \triangle\mathrm{ABC} - \triangle\mathrm{ADC} = \tfrac{1}{2}ab\sin\mathrm{B} - \tfrac{1}{2}cd\sin\mathrm{D}$ if $\mathrm{B} = \angle\mathrm{ABC}$, $\mathrm{D} = \angle\mathrm{ADC}$, and would $= \tfrac{1}{2}ab\sin\mathrm{B} + \tfrac{1}{2}cd\sin\mathrm{D}$, if, e.g. D were taken to be reflex.

Also $a^2 + b^2 - 2ab\cos\mathrm{B} = \mathrm{AC}^2 = c^2 + d^2 - 2cd\cos\mathrm{D}$ would hold because $\cos(360° - \mathrm{D}) = \cos\mathrm{D}$.

Thus (3) would hold, but $\cos(\mathrm{B} + \mathrm{D})$ would be $+1$; $\therefore$ (4) would not be true; instead, $\mathrm{S}^2 = (s - a)(s - b)(s - c)(s + d)$, where $2s = a + b + c - d$.

Taking $-d$ for d, and $180° - \mathrm{ADC}$ for D, all the results would hold; A would be $180° - \mathrm{DAB}$.

6. For x, $a^2 + b^2 - x^2 = 2ab\cos\mathrm{B}$, $c^2 + d^2 - x^2 = 2cd\cos\mathrm{D}$; but $\mathrm{B} = \mathrm{D}$;
$$\therefore\ \text{solving, } x^2 = \frac{(ac - bd)(bc - ad)}{ab - cd};$$
$$\text{similarly}\quad y^2 = \frac{(ac - bd)(ab - cd)}{bc - ad}.$$
These agree with eqn. (5) if the convention just given in No. 5 is used. Every arrangement of 8, 9, 10, 13 for a, b, c, d in any order, subject to $ac = bd + xy > bd$, makes x^2 negative. See No. 17.

7. AB, DC produced cut at P; $\angle\mathrm{PBC} = \mathrm{D}$;
$\therefore\ a + b\cos\mathrm{D} = \mathrm{AP} = d\sin\mathrm{D}$; and $c + b\sin\mathrm{D} = \mathrm{PD} = d\cos\mathrm{D}$; solve for $\sin\mathrm{D}$, $\cos\mathrm{D}$; $\therefore\ (d^2 - b^2)\cos\mathrm{D} = ab + cd$, $(d^2 - b^2)\sin\mathrm{D} = ad + bc$. Square and add.

8. Tangent at A cuts DB at K; $\triangle$ABK is similar to $\triangle$DAK;
$\therefore\ \frac{d^2}{a^2} = \frac{\triangle\mathrm{DAK}}{\triangle\mathrm{ABK}} = \frac{\mathrm{DK}}{\mathrm{BK}}$. If tangent at C cuts DB at K$'$, $\frac{c^2}{b^2} = \frac{\mathrm{DK}'}{\mathrm{BK}'}$;
$\therefore$ if $ac = bd$, $\mathrm{K} \equiv \mathrm{K}'$. Conversely, if tangents at A, C meet at K on DB, $\frac{d^2}{a^2} = \frac{\mathrm{DK}}{\mathrm{BK}} = \frac{c^2}{b^2}$.

9. If AB is a fixed chord of a fixed circle and K a variable point on the circumference, $\triangle$AKB is greatest if arc AK = arc KB. If AQPB is quad. inscribed in fixed circle and if AB is fixed chord, area of AQPB can be increased by moving P if arc BP $\neq$ arc PQ, it can also be increased by moving Q if arc AQ $\neq$ arc QP; $\therefore$ for maximum area, arc BP = arc PQ = arc QA;
$\therefore$ each of these arcs subtends $\angle\,\frac{2\alpha}{3}$ at centre O;
$$\therefore\ \triangle\mathrm{BOP} = \triangle\mathrm{POQ} = \triangle\mathrm{QOA} = \tfrac{1}{2}\mathrm{R}^2\sin\frac{2\alpha}{3};$$
also $\triangle\mathrm{AOB} = \tfrac{1}{2}\mathrm{R}^2\sin(360° - 2\alpha) = -\tfrac{1}{2}\mathrm{R}^2\sin 2\alpha$;
$$\therefore\ \text{area AQPB} = \tfrac{1}{2}\mathrm{R}^2\left(3\sin\frac{2\alpha}{3} - \sin 2\alpha\right) = 2\mathrm{R}^2\sin^3\frac{2\alpha}{3}.$$

10. Sum of roots $=a+b+c+d=2s$. Form equation with roots $s-a,\ s-b,\ s-c,\ s-d$; put $y=s-x$;

$$(s-y)^4-2s(s-y)^3+t(s-y)^2-q(s-y)+2p=0;$$

product of roots $=s^4-2s^4+ts^2-qs+2p$;

$$\therefore\ S=\sqrt{(2p-qs+ts^2-s^4)}.$$

11. AB, DC cut at P; let $PB=e$, $PC=f$. Then as in II. a, No. 16,

$$e=\frac{b(cd+ab)}{d^2-b^2},\quad f=\frac{b(bc+ad)}{d^2-b^2};$$

$$\therefore\ e+f=\frac{b(a+c)(d+b)}{d^2-b^2}=\frac{b(a+c)}{d-b};$$

also $ef(d-b)^2=\dfrac{b^2(cd+ab)(bc+ad)}{(d+b)^2}$; thus

$$\cos^2\tfrac{1}{2}\phi=\frac{(e+f+b)(e+f-b)}{4ef}=\frac{b^2(a+c+d-b)(a+c-d+b)}{4ef(d-b)^2}$$

$$=\frac{(s-b)(s-d)(d+b)^2}{(cd+ab)(bc+ad)}.$$

12. (1) 4 sides and an angle, unique solution. (2) 3 sides, say a, b, c with (i) B, C; unique solution. With (ii) A, B, the vertices A, B, C are known and D lies on a certain line through A. D is given by a circle centre C and may have two possible positions. With (iii) A, C; triangle BCD is known, hence BD, which with a, A can give two positions for D. With (iv) A, D; A, B and the direction of AD are fixed, C lies on a known line parallel to AD distant $c \sin D$ from it, and is at distance b from B; ∴ two positions are possible. (3) 2 sides and 3 angles (i.e. all angles), in general unique solution whether the sides are adjacent or not. But if $A+B=180°$ and if A, B, C, a, c are assigned, the construction is impossible (or indeterminate). (4) 1 side, 4 angles, indeterminate.

13. Let $\angle ABD=\alpha$, $\angle DBC=\beta$; $\therefore\ \angle ABC=\alpha+\beta$.

$$\cos^2(\alpha+\beta)=\cos(\alpha+\beta)\,.\,[2\cos\alpha\cos\beta-\cos(\alpha-\beta)]$$

$$=2\cos\alpha\cos\beta\cos(\alpha+\beta)-\tfrac{1}{2}(\cos 2\alpha+\cos 2\beta);$$

$$\therefore\ \cos^2(\alpha+\beta)+\cos^2\alpha+\cos^2\beta-2\cos(\alpha+\beta)\cos\alpha\cos\beta-1=0.$$

But $\cos(\alpha+\beta)=\dfrac{a^2+b^2-x^2}{2ab}$;

$$\cos\alpha=\frac{a^2+y^2-d^2}{2ay};\quad \cos\beta=\frac{b^2+y^2-c^2}{2by}.$$

Substitute and reduce. **Or,** as follows:

Let the coordinates of A, B, C, D be (0, 0), (x_1y_1), (x_2y_2), (x_3y_3); then $x_1^2+y_1^2=a^2$, $x_2^2+y_2^2=x^2$, $x_3^2+y_3^2=d^2$; also $2(x_1x_2+y_1y_2)=a^2+x^2-(x_1-x_2)^2-(y_1-y_2)^2=a^2+x^2-b^2$, similarly $2(x_2x_3+y_2y_3)=x^2+d^2-c^2$, $2(x_3x_1+y_3y_1)=d^2+a^2-y^2$.

But since
$$\begin{vmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \end{vmatrix}^2=0;$$

$$\therefore\ \begin{vmatrix} x_1^2+y_1^2 & x_1x_2+y_1y_2 & x_1x_3+y_1y_3 \\ x_1x_2+y_1y_2 & x_2^2+y_2^2 & x_2x_3+y_2y_3 \\ x_1x_3+y_1y_3 & x_2x_3+y_2y_3 & x_3^2+y_3^2 \end{vmatrix}=0,$$

or
$$\begin{vmatrix} 2a^2 & a^2+x^2-b^2 & a^2+d^2-y^2 \\ a^2+x^2-b^2 & 2x^2 & x^2+d^2-c^2 \\ a^2+d^2-y^2 & x^2+d^2-c^2 & 2d^2 \end{vmatrix}=0,$$

and this gives the required relation.

14. r is inradius; $\tan\dfrac{A}{2}=\dfrac{r}{\alpha}$, $\tan\dfrac{C}{2}=\dfrac{r}{\gamma}$; $\therefore\ \tan\dfrac{A+C}{2}=\dfrac{r(\alpha+\gamma)}{\alpha\gamma-r^2}$.

Similarly $\tan\dfrac{B+D}{2}=\dfrac{r(\beta+\delta)}{\beta\delta-r^2}$; but $\tan\dfrac{A+C}{2}=-\tan\dfrac{B+D}{2}$;

$$\therefore\ \tan\frac{A+C}{2}=\frac{r(\alpha+\gamma)}{\alpha\gamma-r^2}=\frac{r(\beta+\delta)}{r^2-\beta\delta}=\frac{r(\alpha+\gamma+\beta+\delta)}{\alpha\gamma-\beta\delta}$$

$$=\frac{S}{\alpha\gamma-\beta\delta}=\frac{\sqrt{(abcd)}\,.\,\sin\dfrac{A+C}{2}}{\alpha\gamma-\beta\delta}\ \text{from eqn. (15)};$$

$$\therefore\ \cos\frac{A+C}{2}\,.\,\sqrt{(abcd)}=\alpha\gamma-\beta\delta.$$

15. Circumcentres lie on the perp. bisector of DC and if $\angle DAC=\alpha$, $\angle DBC=\beta$, their distance apart

$$=\tfrac{1}{2}c(\cot\beta-\cot\alpha)=\tfrac{1}{2}c\sin(\alpha-\beta)/(\sin\alpha\sin\beta).$$

Make ABX directly similar to DBC; ∴ BAD is directly similar to BXC, and $\angle AXC=A+C$, and

$$\angle ACX=\angle ACB-\angle XCB=\angle ACB-\angle ADB=\alpha-\beta.$$

Then $\dfrac{c}{a}=\dfrac{c}{AC}\cdot\dfrac{AC}{AX}\cdot\dfrac{AX}{a}=\dfrac{\sin\alpha}{\sin D}\cdot\dfrac{\sin(A+C)}{\sin(\alpha-\beta)}\cdot\dfrac{\sin\beta}{\sin C}$.

16. $AI=r\operatorname{cosec}\tfrac{1}{2}A$; $\therefore\ IA'=\dfrac{R^2-z^2}{r}\sin\tfrac{1}{2}A$,

similarly $IC'=\dfrac{R^2-z^2}{r}\sin\tfrac{1}{2}C=\dfrac{R^2-z^2}{r}\cos\tfrac{1}{2}A$.

A′, C′ are mid-points of arcs BCD, BAD of circumcircle; ∴ A′C′ is a diameter, and the median theorem gives

$$2R^2+2z^2=IA'^2+IC'^2=\frac{(R^2-z^2)^2}{r^2};$$

$$\therefore\ (R-z)^2+(R+z)^2=\frac{(R+z)^2(R-z)^2}{r^2}.$$

17. Suppose in Fig. 20 that $a>b>c>d$; then examine the possibility of the condition, $\angle DAB=\theta=\angle DCB$; this condition is possible if θ can be found such that

$$b^2+c^2-2bc\cos\theta=a^2+d^2-2ad\cos\theta,$$

which gives $\tan^2\frac{1}{2}\theta=\frac{(a-d)^2-(b-c)^2}{(b+c)^2-(a+d)^2}$; the numr. is positive because $a-d>b-c$, the denr., because $b+c>a+d$; thus θ can be found.

Two other crossed cyclic quads. can be deduced by replacing A by its image in the perp. bisector of BD, or D by its image in the perp. bisector of CA.

By No. 6 (i), x and y have two of the values

$$\sqrt{\frac{71\times54}{90}},\quad\sqrt{\frac{90\times54}{71}},\quad\sqrt{\frac{71\times90}{54}}.$$

18. O is centre of circle; $AB=l$, $BC=m$; the arc $ABC=\frac{1}{3}$ circumference; ∴ $\angle ABC=120°$; $\angle AOC=120°$; ∴ $AC=R\sqrt{3}$; but $AC^2=AB^2+BC^2-2AB\cdot BC\cos120°=l^2+m^2+lm$.

19. Use eqn. (20), Ch. I. p. 11; e.g. if M is mid-point of AB, $2\cot(\alpha+\beta)=2\cot OMB=\cot\alpha-\cot\gamma$.

20. $P_n=n\times2r\sin\frac{\pi}{n}$; $P_n'=n\times2r\tan\frac{\pi}{n}$;

$$\therefore\ \frac{P_n'-P_{n+1}'}{P_{n+1}-P_n}=\frac{n\tan\frac{\pi}{n}-(n+1)\tan\frac{\pi}{n+1}}{(n+1)\sin\frac{\pi}{n+1}-n\sin\frac{\pi}{n}}$$

$$=\left(\frac{\tan\alpha}{\alpha}-\frac{\tan\beta}{\beta}\right)\div\left(\frac{\sin\beta}{\beta}-\frac{\sin\alpha}{\alpha}\right)$$

where $\alpha=\frac{\pi}{n}$, $\beta=\frac{\pi}{n+1}$. But if θ is small (*E.T.*, pp. 170, 173), $\sin\theta\simeq\theta-\frac{1}{6}\theta^3$, $\cos\theta\simeq1-\frac{1}{2}\theta^2$; ∴ $\tan\theta=\frac{\sin\theta}{\cos\theta}\simeq\theta+\frac{1}{3}\theta^3$. Replace $\frac{\tan\alpha}{\alpha}$, $\frac{\sin\alpha}{\alpha}$, etc., by these approximations.

21. V is vertex; AB, BC are two sides of base; O is centre of base, AC, OB meet at K, KN is perp. to VB; ∴ $\angle ANC=2\theta$, and $VB=\sqrt{(h^2+a^2)}$, then

$$\cot\theta=\frac{NK}{KA}=\frac{KB\sin OBV}{KA}=\tan\frac{\pi}{n}\cdot\frac{h}{\sqrt{(h^2+a^2)}}.$$

CHAPTER III

EXERCISE III. a. (p. 35.)

19. $\tan\theta=1$.
20. $\tan\theta=-1$.
21. $\cos\theta=\frac{1}{2}$.
22. $\sin\theta=\sin\alpha$.

EXERCISE III. b. (p. 37.)

1. $\cos2\theta=2\cos^2\theta-1=0$.
2. $7\theta=2n\pi\pm5\theta$.
3. $7\theta=2n\pi+3\theta$ or $(2n+1)\pi-3\theta$. The solutions $\frac{n\pi}{2}$, $\frac{(2n+1)\pi}{10}$ are all included in $k\pi$, $\frac{(2k+1)\pi}{10}$.
4. $\sin3\theta=\sin\left(\frac{\pi}{2}-2\theta\right)$; $3\theta=2n\pi+\frac{\pi}{2}-2\theta$ or $(2n+1)\pi-\frac{\pi}{2}+2\theta$. The solutions $\frac{(4n+1)\pi}{10}$, $\frac{(4n+1)\pi}{2}$ are all included in $\frac{(4k+1)\pi}{10}$.
5. $\sin6\theta=\sin(\pi+\theta)$. Use eqn. (1).
6. $\sin5\theta=\sin\left(3\theta-\frac{\pi}{2}\right)$. Use eqn. (1).
7. $\tan5\theta=\tan2\theta$. Use eqn. (3).
8. $\tan3\theta=\tan\left(\frac{\pi}{2}-4\theta\right)$. Use eqn. (3).
9. $2\sin\theta\cos\theta=2\sin^2\theta$; ∴ $\sin\theta=0$ or $\tan\theta=1$.
10. $\cos2\theta=\cos\theta$. Use eqn. (2).
11. $\sin\left(\theta+\frac{\pi}{3}\right)=\sin\frac{\pi}{6}$. Use eqn. (1).
12. $\cos\left(\theta+\frac{\pi}{4}\right)=\cos\frac{\pi}{4}$. Use eqn. (2).

13. $3\sin\theta - 4\sin^3\theta = 3\sin\theta$; $\therefore\ \sin^3\theta = 0$; $\therefore\ \sin\theta = 0$.

14. $\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta = \frac{1}{2}$; take α so that $\cos\alpha = \frac{4}{5}$, $\sin\alpha = \frac{3}{5}$; $\alpha = 36^\circ\ 52'$; $\cos(\theta - \alpha) = \cos 60^\circ$. Use eqn. (2).

15. $4\sin\theta\cos\theta = 1$; $\therefore\ \sin 2\theta = \frac{1}{2} = \sin\dfrac{\pi}{6}$. Use eqn. (1).

16. $13^2 + 84^2 = 85^2$. Take α so that $\cos\alpha = \frac{13}{85}$, $\sin\alpha = \frac{84}{85}$; $\alpha = 81^\circ\ 12'$; $\sin(\theta - \alpha) = \frac{17}{85} = 0{\cdot}2 = \sin 11^\circ\ 32'$. Use eqn. (1).

17. $2\cos 2\theta\cos\theta + \cos 2\theta = 0$; $\therefore\ \cos 2\theta = 0$ or $\cos\theta = -\frac{1}{2}$. Use eqn. (2).

18. $2\cos 4\theta\sin 3\theta = \sin 3\theta$; $\therefore\ \sin 3\theta = 0$ or $\cos 4\theta = \frac{1}{2}$.

19. $\cos 3\theta \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$;

$$\therefore\ \sin 2\theta\sin\theta = 0;\quad \therefore\ \sin 2\theta = 0;\quad \therefore\ \theta = \frac{n\pi}{2}.$$

20. $\tan\theta + \tan 2\theta = \tan(2\theta + \theta) = \dfrac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta}$;

$\therefore\ \tan\theta + \tan 2\theta = 0$; $\therefore\ \tan 2\theta = \tan(-\theta)$; $\therefore\ 2\theta = n\pi - \theta$.

21. $2\cos 2\theta(\cos 4\theta + \cos 2\theta) = 1$;

$$\therefore\ 2\cos 2\theta\cos 4\theta + 2\cos^2 2\theta - 1 = 0;$$
$$\therefore\ 2\cos 2\theta\cos 4\theta + \cos 4\theta = 0;$$
$$\therefore\ \cos 4\theta = 0 \text{ or } \cos 2\theta = -\tfrac{1}{2}.$$

22. $\sin\theta + \cos\theta = 2\sqrt{2}\sin\theta\cos\theta$; $\therefore\ \sin\left(\theta + \dfrac{\pi}{4}\right) = \sin 2\theta$;

$$\therefore\ 2\theta = 2n\pi + \theta + \frac{\pi}{4} \text{ or } (2n+1)\pi - \theta - \frac{\pi}{4}.$$

Both solutions are included in $\dfrac{2n\pi}{3} + \dfrac{\pi}{4}$.

23. $\cos\theta + \sin\theta = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = (\cos\theta + \sin\theta)^2$;

$$\therefore\ \cos\theta + \sin\theta = 0 \text{ or } \cos\theta + \sin\theta = 1;$$
$$\therefore\ \tan\theta = -1 \text{ or } \cos\left(\theta - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}.$$

24. $\dfrac{\sin\theta}{\cos\theta} + \dfrac{1 - \cos 2\theta}{\cos 2\theta} = 0$; $\therefore\ \dfrac{\sin\theta}{\cos\theta} + \dfrac{2\sin^2\theta}{\cos 2\theta} = 0$;

$\therefore\ \sin\theta = 0$ or $\cos 2\theta + 2\sin\theta\cos\theta = 0$, i.e. $\tan 2\theta = -1$.

25. $\cos x\cos\alpha + \sin x\sin\alpha = \frac{1}{2}$; $\therefore\ \cos(x - \alpha) = \cos\dfrac{\pi}{3}$.

26. $\cos 16x + \cos 2x = \cos 8x + \cos 2x$;

$$\therefore\ \cos 16x = \cos 8x;\quad \therefore\ 16x = 2n\pi \pm 8x.$$

Both solutions are included in $\dfrac{n\pi}{12}$.

27. $\dfrac{\cos x}{\sin x} - \dfrac{1}{2\sin x\cos x} = 1$; $\therefore\ 2\cos^2 x - 1 = 2\sin x\cos x$.

$$\therefore\ \cos 2x = \sin 2x;\quad \therefore\ \tan 2x = 1.$$

28. $\cos(2x - \alpha - \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta + 2\sin^2 x$
$= \cos(\alpha + \beta) + \cos(\alpha - \beta) + 1 - \cos 2x$;

$$\therefore\ \cos 2x + \cos(2x - \alpha - \beta) = 1 + \cos(\alpha + \beta);$$
$$\therefore\ 2\cos\left(2x - \frac{\alpha+\beta}{2}\right)\cos\frac{\alpha+\beta}{2} = 2\cos^2\frac{\alpha+\beta}{2};$$
$$\therefore\ \text{if } \cos\frac{\alpha+\beta}{2} \neq 0,\ \cos\left(2x - \frac{\alpha+\beta}{2}\right) = \cos\frac{\alpha+\beta}{2}.$$

29. $\cos^3 x - \sin^3 x \equiv (\cos x - \sin x)(\cos^2 x + \cos x\sin x + \sin^2 x)$
$\equiv (\cos x - \sin x)(1 + \cos x\sin x)$;

$$\therefore\ (\cos x - \sin x)(1 + \cos x\sin x) = 1 + \cos x\sin x;$$
$$\therefore\ 1 + \cos x\sin x = 0 \text{ or } \cos x - \sin x = 1,$$

$1 + \cos x\sin x = 0$ gives $\sin 2x = -2$, which is impossible. For $\cos x - \sin x = 1$, see No. 12.

30. $\operatorname{cosec}\theta - \cot\theta = \dfrac{1 - \cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2}$;

$\therefore$ equation becomes $\tan 2x = \tan 2\alpha$.

31. $\tan(\cot\theta) = \tan\left(\dfrac{\pi}{2} - \tan\theta\right)$; $\therefore\ \cot\theta = n\pi + \dfrac{\pi}{2} - \tan\theta$; but

$$\tan\theta + \cot\theta \equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{2}{\sin 2\theta};\quad \therefore\ \sin 2\theta = \frac{4}{(2n+1)\pi}.$$

32. $\tan^{-1}x = 2\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$;

$$\therefore\ \tan[\tan^{-1}x] = \tan\left[2\tan^{-1}\left(\frac{1-x}{1+x}\right)\right];$$

$$\therefore\ x = \frac{2\left(\frac{1-x}{1+x}\right)}{1 - \left(\frac{1-x}{1+x}\right)^2}, \text{ see } \textit{E.T.}, \text{ p. 218}, = \frac{2(1-x^2)}{4x};$$

$$\therefore\ 2x^2 = 1 - x^2.$$

33. $\cos\left(\theta - \dfrac{\pi}{4}\right) = \dfrac{k}{\sqrt{2}}$;

(i) $\cos\left(\theta - \dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{4}$; (ii) no solution since $\dfrac{2}{\sqrt{2}} > 1$;

(iii) $\cos\dfrac{\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$, see *E.T.*, p. 215;

$$\therefore\ \cos\left(\theta - \frac{\pi}{4}\right) = \cos\frac{\pi}{12}.$$

34. If $\tan\theta$ is + and $\sec\theta$ is −, θ is an angle in third quadrant.

35. From $\sin\theta+\sin 3\theta=\cos\theta$, $2\sin 2\theta\cos\theta=\cos\theta$;

$$\therefore\ \cos\theta=0 \text{ or } \sin 2\theta=\tfrac{1}{2};$$

these give $\theta=n\pi+\frac{\pi}{12}$ or $n\pi+\frac{5\pi}{12}$ or $(2n+1)\frac{\pi}{2}$; but only $\theta=n\pi+\frac{\pi}{12}$ makes $\sin 4\theta=\frac{\sqrt{3}}{2}$.

36. If $n=2k$, $(2n-1)\frac{\pi}{2}+(-1)^n\frac{\pi}{3}=2k\pi-\frac{\pi}{2}+\frac{\pi}{3}=2k\pi-\frac{\pi}{6}$.

If $n=2k+1$, it equals $2k\pi+\frac{\pi}{2}-\frac{\pi}{3}=2k\pi+\frac{\pi}{6}$.

37. Each represents any odd number of right angles.

EXERCISE III. c. (p. 40.)

1. Draw graphs of $y=x^2$ and $y=\cos x$.
2. $2x=2\cos^2 x=1+\cos 2x$; graphs of $y=\cos 2x$ and $y=2x-1$.
3. Sketch graphs of $y=\sin x$, $y=\frac{x}{10}$. They cut at origin and at three points on each side of it.
4. Sketch graphs of $y=\cos x$, $y=1-\frac{2x}{3\pi}$. They cut at (0, 1); $\left(\frac{3\pi}{2}, 0\right)$; $(3\pi, -1)$; and two intermediate points.
5. Draw graphs of $y=\sin x$ and $y=1-\frac{1}{4}x^2$.
6. Sketch graphs of $y=\tan 2x$ and $y=\frac{\pi}{4}-x$; they cut once in each of the intervals $0<x<\frac{\pi}{4}$, $\frac{\pi}{4}<x<\frac{\pi}{2}$, $\frac{3\pi}{4}<x<\pi$.
7. $x+\frac{1}{x}\equiv 2+\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$; $\therefore\ x+\frac{1}{x}\geqslant 2$ if $x>0$; similarly $x+\frac{1}{x}\leqslant -2$ if $x<0$; $\therefore\ x+\frac{1}{x}$ cannot lie between ± 2; $\therefore\ \cos\theta$ cannot lie between $\pm\frac{1}{2}$: $\therefore\ n\pi-\frac{\pi}{3}<\theta<n\pi+\frac{\pi}{3}$.
8. $\sin x\cos x+\sin^2 x=c$; $\therefore\ \sin 2x+1-\cos 2x=2c$;

$$\therefore\ \sin\left(2x-\frac{\pi}{4}\right)=\frac{2c-1}{\sqrt{2}};\quad \therefore\ -1\leqslant\frac{2c-1}{\sqrt{2}}\leqslant 1.$$

9. Rough graphs of $y=c$, $y=\sec x+\operatorname{cosec} x$ show that there is always one solution between $x=\frac{\pi}{2}$, $x=\pi$ and one between

$x=\frac{3\pi}{2}$, $x=2\pi$; there are also two other solutions unless c lies between the minimum and maximum values of $\sec x+\operatorname{cosec} x$. But

$$\frac{d}{dx}(\sec x+\operatorname{cosec} x)=\frac{\sin x}{\cos^2 x}-\frac{\cos x}{\sin^2 x}=\frac{\sin^3 x-\cos^3 x}{\sin^2 x\cos^2 x}$$
$$=2(\sin x-\cos x)(2+\sin 2x)\operatorname{cosec}^2 2x;$$

$\therefore$ a minimum value occurs at $x=\frac{\pi}{4}$ and a maximum value at $x=\frac{5\pi}{4}$; the corresponding values of $\sec x+\operatorname{cosec} x$ are $2\sqrt{2}$ and $-2\sqrt{2}$; $\therefore$ there are two other solutions unless $-2\sqrt{2}<c<2\sqrt{2}$, i.e. $c^2<8$.

10. Take rectangular axes Ox, Oy; C is the point (a, b); take P so that OP = PC = 1. Then θ, ϕ are the angles OP, PC make with Ox. No solution if OC > 2, i.e. $\sqrt{(a^2+b^2)}>2$. If a or b is negative, P lies in another quadrant; use same method.

11. $x+y=n\pi+(-1)^n\frac{\pi}{6}$; $x-y=2k\pi\pm\frac{5\pi}{6}$;

$$\therefore\ 2x=(n+2k)\pi+(-1)^n\cdot\frac{\pi}{6}\pm\frac{5\pi}{6};$$
$$2y=(n-2k)\pi+(-1)^n\cdot\frac{\pi}{6}\mp\frac{5\pi}{6}.$$

For n even, $n=2t$, we have

$$x=(t+k)\pi+\frac{\pi}{2}\ \text{ or }\ (t+k)\pi-\frac{\pi}{3};$$
$$y=(t-k)\pi-\frac{\pi}{3}\ \text{ or }\ (t-k)\pi+\frac{\pi}{2}.$$

The multiples of π in the values of x and y are therefore both even or both odd. Hence two answers as given. Similarly, take $n=2t+1$ and reduce as before.

12. $\cos(x+y)=\frac{1}{4}(\sqrt{6}-\sqrt{2})=\cos\frac{5\pi}{12}$;

$$\cos(x-y)=\tfrac{1}{4}(\sqrt{6}+\sqrt{2})=\cos\frac{\pi}{12};$$
$$\therefore\ x+y=2m\pi\pm\frac{5\pi}{12},\quad x-y=2n\pi\pm\frac{\pi}{12}.$$

13. $\sin x = \tan 3y = \dfrac{\sin 3y}{\cos 3y} = \dfrac{\sin x}{\cos x \cos 3y}$; $\therefore$ $\sin x = 0$, and so $\sin 3y = 0$. The alternative $\cos x \cos 3y = 1$ requires $\cos x = \pm 1$ and $\therefore$ also $\sin x = 0$.

14. $\sin^2 x - \sin x \sin y - \sin x = 0 = \sin x \sin y - \sin^2 y - \sin y$;

$\therefore$ $\sin x(\sin x - \sin y - 1) = 0 = \sin y(\sin x - \sin y - 1)$;

$\therefore$ either $\sin x = 0 = \sin y$ or any solution of $\sin x - \sin y = 1$ may be taken.

15. $(\sqrt{3} - 1)\cos x = (\sqrt{3} + 1)\sin y$; $\therefore$ $\cos x = (2 + \sqrt{3})\sin y$ and $\cos(x + y) = \frac{1}{2}(\cos x + \sqrt{3}\sin y) = (1 + \sqrt{3})\sin y = \cos x - \sin y$;

$\therefore$ $2\sin(x + \frac{1}{2}y)\sin\frac{1}{2}y = \sin y = 2\sin\frac{1}{2}y\cos\frac{1}{2}y$; $\therefore$ $y = 2n\pi$

or $\sin(x + \frac{1}{2}y) = \cos\frac{1}{2}y = \sin(\frac{1}{2}\pi + \frac{1}{2}y)$; $\therefore$ $x = 2m\pi + \dfrac{\pi}{2}$ or

$x + y = 2k\pi - \dfrac{\pi}{2}$. In the last case $\cos(x + y) = 0$;

$\therefore$ $\sin y = 0 = \cos x$,

and these give $x = p\pi + \dfrac{\pi}{2}$, $y = q\pi$ which include the previous answers.

16. See Example 10, p. 39. The geometrical method of III. c, No. 10, shows that there are two solutions in the range 0° to 360°; the two positions of P correspond to an exchange of θ, ϕ. $OC^2 = (\frac{2}{5})^2 + (\frac{3}{4})^2$;

$\therefore$ $OC = \frac{17}{20}$; $\tan xOC = \frac{3}{4} \div \frac{2}{5} = \frac{15}{8}$; $\therefore$ $\angle xOC = 61° 56'$;

also $\angle COP = \cos^{-1}\frac{17}{40} = 64° 51'$; thus, for one P,

$\theta = 61° 56' + 64° 51'$, $\phi = 61° 56' - 64° 51' + 360°$.

17. $5\sin x = 1 + 2\sin y$, $5\cos x = 4 + 2\cos y$; square and add. Reducing, we have $\sin y + 4\cos y = 1$;

$\therefore$ $\cos y(1 - \sin y) = 4\cos^2 y = 4(1 - \sin^2 y)$;

$\therefore$ $\sin y = 1$ or $\cos y = 4 + 4\sin y = 4 + 4(1 - 4\cos y)$;

$\therefore$ $\cos y = \frac{8}{17}$ and $\sin y = -\frac{15}{17}$.

If $\sin y = 1$, then $\cos y = 0$; $\therefore$ $\sin x = \frac{3}{5}$ and $\cos x = \frac{4}{5}$.

If $\sin y = -\frac{15}{17}$ and $\cos y = \frac{8}{17}$, then $\sin x = -\frac{13}{85}$ and $\cos x = \frac{84}{85}$.

18. $x + 3y = 2n\pi \pm \left(\dfrac{\pi}{2} - 2x - 2y\right)$; $2x + 2y = 2k\pi \pm \left(\dfrac{\pi}{2} - 3x - y\right)$.

Since $x - y \neq 2r\pi + \dfrac{\pi}{2}$, these reduce to the single condition $3x + 5y = 2n\pi + \dfrac{\pi}{2}$, $5x + 3y = 2k\pi + \dfrac{\pi}{2}$; solve for x, y.

19. For values between 0 and π, $x = 2y$ or $2y - \pi$, $y = 2z$ or $2z - \pi$, and $z = 2x$ or $2x - \pi$. $x = 2y$, $y = 2z$, $z = 2x - \pi$ gives the first solution and $x = 2y$, $y = 2z - \pi$, $z = 2x - \pi$ the fourth; the others follow by cyclic changes.

20. $\sin x = \sin(-2y)$; $\therefore$ $x = 2\pi - 2y$ or $2y - \pi$. Solutions are given by $(x = 2\pi - 2y$, $y = 2\pi - 2z$, $z = 2\pi - 2x)$, $(x = 2\pi - 2y$, $y = 2\pi - 2z$, $z = 2x - \pi)$, and $(x = 2\pi - 2y$, $y = 2z - \pi$, $z = 2x - \pi)$ and cyclic changes of the last two.

EXERCISE III. d. (p. 44.)

1. Find the sign of $\cos\dfrac{\theta}{2}$ or $\sin\dfrac{\theta}{2}$ for the given angles. The answers are

(i) +, +; (ii) +, +; (iii) −, +;
(iv) +, −; (v) −, +; (vi) +, +.

2. (i) $2n\pi - \dfrac{\pi}{2} < \dfrac{\theta}{2} < 2n\pi + \dfrac{\pi}{2}$; $\therefore$ $\frac{1}{2}\theta$ lies in 1st or 4th quadrant;

(ii) $\frac{1}{2}\theta$ lies in 2nd or 3rd quadrant.

3. $\sin\dfrac{\theta}{2}$ is + if $2n\pi < \frac{1}{2}\theta < (2n + 1)\pi$.

4. Use $\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2} = \sqrt{2}\sin\left(\dfrac{\theta}{2} + 45°\right)$ and

$$\sin\frac{\theta}{2} - \cos\frac{\theta}{2} = \sqrt{2}\sin\left(\frac{\theta}{2} - 45°\right).$$

5. As in No. 4, if $\theta = \dfrac{11\pi}{6}$, $\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2} = -\sqrt{(1 + \sin\theta)}$; $\sin\dfrac{\theta}{2} - \cos\dfrac{\theta}{2} = +\sqrt{(1 - \sin\theta)}$. Other results in same way.

6. $\dfrac{\theta}{2}$ is near 140°; $\therefore$ $\sin\left(\dfrac{\theta}{2} + 45°\right)$ is −, $\sin\left(\dfrac{\theta}{2} - 45°\right)$ is +.

7. $\sin(\theta + 45°)$ is −, $\sin(\theta - 45°)$ is +.

8. $\dfrac{\pi}{4} < \theta < \dfrac{3\pi}{4}$; $\therefore$ $\sin\left(\theta + \dfrac{\pi}{4}\right)$ is +, $\sin\left(\theta - \dfrac{\pi}{4}\right)$ is +.

9. $2n\pi - \dfrac{\pi}{4} < \frac{1}{2}\theta < 2n\pi + \dfrac{3\pi}{4}$;

$\therefore$ $2n\pi < \dfrac{\theta}{2} + \dfrac{\pi}{4} < (2n + 1)\pi$; $\therefore$ $\sin\left(\dfrac{\theta}{2} + \dfrac{\pi}{4}\right)$ is +.

In second case, $(2n + 1)\pi < \dfrac{\theta}{2} + \dfrac{\pi}{4} < (2n + 2)\pi$.

10. $\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$ is + if $2n\pi < \frac{\theta}{2}-\frac{\pi}{4} < (2n+1)\pi$, i.e. if

$$4n\pi+\frac{\pi}{2} < \theta < (4n+2)\pi+\frac{\pi}{2};$$

$$\therefore\ \sin\frac{\theta}{2}-\cos\frac{\theta}{2} = +\sqrt{(1-\sin\theta)} \text{ if}$$

$$4n\pi+\frac{\pi}{2} < \theta < (4n+2)\pi+\frac{\pi}{2}.$$

11. $\sin\frac{\theta}{2}+\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}-\cos\frac{\theta}{2}$ are each +; ∴ by Nos. 9, 10, θ is such that

$$4n\pi-\frac{\pi}{2} < \theta < 4n\pi+\frac{3\pi}{2} \text{ and } 4n\pi+\frac{\pi}{2} < \theta < 4n\pi+\frac{5\pi}{2}.$$

To satisfy both relations, $4n\pi+\frac{\pi}{2} < \theta < 4n\pi+\frac{3\pi}{2}$.

12. $\sin\frac{\theta}{2}+\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}-\cos\frac{\theta}{2}$ are each +; same as No. 11.

13. $\sin\theta+\cos\theta$ and $\sin\theta-\cos\theta$ are each −; ∴ by Nos. 9, 10, $4n\pi+\frac{3\pi}{2} < 2\theta < 4n\pi+\frac{7\pi}{2}$ and

$$(4n+2)\pi+\frac{\pi}{2} < 2\theta < (4n+4)\pi+\frac{\pi}{2};$$

$$\therefore\ 4n\pi+\frac{5\pi}{2} < 2\theta < 4n\pi+\frac{7\pi}{2}. \quad \text{Divide by 2.}$$

14. $\sin\theta+\cos\theta$ is − and $\sin\theta-\cos\theta$ is +. Method of No. 13.

17. $\cos\frac{\pi}{8} = +\sqrt{\left\{\frac{1}{2}\left(1+\cos\frac{\pi}{4}\right)\right\}} = +\sqrt{\{\frac{1}{4}(2+\sqrt{2})\}};$

$$\sin\frac{\pi}{16} = +\sqrt{\left\{\frac{1}{2}\left(1-\cos\frac{\pi}{8}\right)\right\}}.$$

18. $(2n-1)\pi < \frac{1}{2}\theta < 2n\pi$; ∴ $\sin\frac{\theta}{2}$ is −;

$$\therefore\ \sin\frac{\theta}{2} = -\sqrt{\{\tfrac{1}{2}(1-\cos\theta)\}} = -\sqrt{\{\tfrac{1}{2}(1-\tfrac{1}{3})\}}.$$

19. $2n\pi-\frac{\pi}{2} < \frac{1}{2}\theta < 2n\pi+\frac{\pi}{2}$;

$$\therefore\ \cos\frac{\theta}{2} = +\sqrt{\{\tfrac{1}{2}(1+\cos\theta)\}} = +\sqrt{\{\tfrac{1}{2}(1-\tfrac{7}{25})\}}.$$

20. $k \equiv \tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$; ∴ $k\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2} - k = 0$;

$$\therefore\ \tan\frac{\theta}{2} = \frac{1}{k}\{-1 \pm \sqrt{(1+k^2)}\}.$$

(i) If $\frac{\pi}{2} < \theta < \pi$, $k \equiv \tan\theta$ is −; also $\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$;

$$\therefore\ \tan\frac{\theta}{2} \text{ is } +;\quad \therefore\ \tan\frac{\theta}{2} = \frac{1}{k}\{-1-\sqrt{(1+k^2)}\}.$$

In (ii), k is + and $\tan\frac{\theta}{2}$ is −; in (iii), k is − and $\tan\frac{\theta}{2}$ is −.

21. $\tan x+\cot x = \frac{\sin x}{\cos x}+\frac{\cos x}{\sin x} = \frac{\sin^2 x+\cos^2 x}{\sin x\cos x} = \frac{1}{\frac{1}{2}\sin 2x}$.

$$\therefore\ \tan\frac{\theta}{2}+\frac{1}{\tan\frac{\theta}{2}} = \frac{2}{\sin\theta};$$

$$\therefore\ \tan^2\frac{\theta}{2}\,.\,\sin\theta - 2\tan\frac{\theta}{2} + \sin\theta = 0.$$

22. $\tan\frac{\theta}{2} = \pm\sqrt{\left(\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right)} = \pm\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)}$;

$$\tan\frac{\theta}{2} \text{ is + if } n\pi < \frac{\theta}{2} < n\pi+\frac{\pi}{2}.$$

23. $\theta = \alpha$ or $\pi-\alpha$ or $2\pi+\alpha$ or $3\pi-\alpha$ or $4\pi+\alpha$, etc.;
$\frac{\theta}{3} = \frac{\alpha}{3}$ or $\frac{\pi-\alpha}{3}$, etc. Use $\sin\frac{\pi-\alpha}{3} = \sin\frac{2\pi+\alpha}{3}$, etc.

24. (i) $\theta = 60°$ or $300°$ or $420°$, etc. $\frac{\theta}{3} = 20°$ or $100°$ or $140°$, etc.;
(ii) $\theta = 210°$ or $150°$ or $510°$, etc.

25. $\sin\frac{\theta}{2}$ is + if $2n\pi < \frac{\theta}{2} < (2n+1)\pi$, i.e. $2n < \frac{\theta}{2\pi} < 2n+1$, i.e. if the integer just below $\frac{\theta}{2\pi}$ is even.

26. Method of p. 44.
(i) Put $x=4y$, then $4y^3-3y = -\frac{1}{2}$; ∴ $\cos 3\theta = -\frac{1}{2}$ where $y=\cos\theta$; $3\theta = 120°$ or $240°$ or $480°$.
(ii) Put $x=4y$, then $4y^3-3y=\frac{1}{4}$; ∴ $\cos 3\theta = \frac{1}{4}$ where $y=\cos\theta$; $3\theta = 75°\ 31'$ or $360° - 75°\ 31'$, etc.

27. (i) As in No. 26; put $x=6y$, then $\cos 3\theta = \frac{1}{2}$ where $y = \cos\theta$,
$3\theta = 60^\circ$ or 300° or 420°.

(ii) Put $x = 6y = 6\cos\theta$, then $\cos 3\theta = -\frac{20}{27}$;

$\therefore\ 3\theta = 180^\circ - 42^\circ\ 12'$ or $180^\circ + 42^\circ\ 12'$, etc.

28. Method of p. 44; put $x+1=y$; equation becomes
$y^3 - 12y + 8 = 0$; then as in No. 26 (i).

29. $\cos 5\theta = \cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta$

$= (4c^3 - 3c)(2c^2 - 1) - (3s - 4s^3) \cdot 2sc = 16c^5 - 20c^3 + 5c,$

putting $s^2 = 1 - c^2$, where $c = \cos\theta$, $s = \sin\theta$.

In given equation, put $x = 2c$; then

$$32c^5 - 40c^3 + 10c + 1 = 0;$$

$\therefore\ \cos 5\theta = -\frac{1}{2} = \cos 120^\circ$; $5\theta = 120^\circ$ or 240°, etc.

EXERCISE III. e. (p. 48.)

1. (i) $\cos^{-1} x = \theta$, $x = \cos\theta$,

$\sin\theta = \pm\sqrt{(1-x^2)}$, $\theta = \pm\sin^{-1}\sqrt{(1-x^2)}$.

(ii) $\cos^{-1} x = \theta$, $x = \cos\theta$,

$$\tan\theta = \pm\frac{1}{x}\sqrt{(1-x^2)},\ \operatorname{cosec}\theta = \frac{\pm 1}{\sqrt{(1-x^2)}}.$$

2. $\tan^{-1} x = \theta$, $x = \tan\theta$,

$$\sin\theta = \frac{\pm x}{\sqrt{(1+x^2)}},\ \cos\theta = \frac{\pm 1}{\sqrt{(1+x^2)}},\ \cot\theta = \frac{1}{x}.$$

3. $\triangle$ABC, $\angle$C $= 90^\circ$, BC $= 4$, AB $= 5$, then $\angle$A $= \operatorname{cosec}^{-1}(1\frac{1}{4})$.
AC $= 3$; $\therefore\ \angle$A $= \cos^{-1}(\frac{3}{5}) = \tan^{-1}(\frac{4}{3})$.

4. $\sin(\cos^{-1}x) = \sin\{\sin^{-1}[\pm\sqrt{(1-x^2)}]\} = \pm\sqrt{(1-x^2)}$.

5. (i) $\cos(\sin^{-1}x) = \cos\{\cos^{-1}[\pm\sqrt{(1-x^2)}]\}$;

(ii) $\tan(\sin^{-1}x) = \tan\left[\tan^{-1}\dfrac{\pm x}{\sqrt{(1-x^2)}}\right]$.

6. If α is a value of $\cos^{-1}x$, the general value of $2\cos^{-1}x$ is $2(2k\pi \pm \alpha)$; also $x = \cos\alpha$; $\therefore\ 2x^2 - 1 = \cos 2\alpha$; $\therefore$ the general value of $\cos^{-1}(2x^2-1) = \cos^{-1}(\cos 2\alpha)$ is $2m\pi \pm 2\alpha$; $\therefore\ 2\cos^{-1}x = 2n\pi \pm \cos^{-1}(2x^2-1)$.

Simple values are $2\cos^{-1}x - 2\cos^{-1}x = 0$ and
$$2\cos^{-1}x + 2\cos^{-1}x = 4\cos^{-1}x.$$

7. $\sin(2\sin^{-1}x) = 2\sin(\sin^{-1}x)\cos(\sin^{-1}x) = \pm 2x\sqrt{(1-x^2)}$.

8. $\cos^{-1}x = \theta$; $\therefore\ x = \cos\theta$; $\therefore\ \tan^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{1-x}{1+x}$.

9. Use eqn. (9).

10. Use eqn. (9); $2\tan^{-1}m = \tan^{-1}\dfrac{2m}{1-m^2}$.

11. $\tan^{-1}x = \theta$; $\therefore\ x = \tan\theta$;

$$\therefore\ \frac{1-x^2}{1+x^2} = \cos^2\theta - \sin^2\theta = \cos 2\theta;$$

$$\therefore\ 2\theta = 2n\pi \pm \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right).$$

12. $\cos(2\sin^{-1}x) = 1 - 2\sin^2(\sin^{-1}x) = 1 - 2x^2$.

13. $$\tan^{-1}\frac{x}{x+1} = 2\tan^{-1}\frac{1}{x} = \tan^{-1}\frac{\frac{2}{x}}{1-\frac{1}{x^2}} = \tan^{-1}\frac{2x}{x^2-1};$$

$$\therefore\ \frac{x}{x+1} = \frac{2x}{x^2-1}.$$

14. $\sin(3\sin^{-1}x) = 3\sin(\sin^{-1}x) - 4\sin^3(\sin^{-1}x) = 3x - 4x^3$.

15. $\operatorname{cosec}^{-1}\sqrt{5} = \tan^{-1}\frac{1}{2}$; $\cot^{-1}3 = \tan^{-1}\frac{1}{3}$; use eqn. (9).

16. $$\tan^{-1}\left[\tan\left(\frac{\pi}{2} - x\right)\right] + \cot^{-1}\left[\cot\left(\frac{\pi}{2} - x\right)\right]$$
$$= m\pi + \frac{\pi}{2} - x + n\pi + \frac{\pi}{2} - x.$$

17. $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$; $\therefore\ \cos 2\left\{\tan^{-1}\dfrac{x+y}{1-xy}\right\}$

$$= \left\{1 - \left(\frac{x+y}{1-xy}\right)^2\right\} \div \left\{1 + \left(\frac{x+y}{1-xy}\right)^2\right\} = \frac{(1-xy)^2 - (x+y)^2}{(1-xy)^2 + (x+y)^2}.$$

18. As in No. 8,

$$\cos^{-1}\frac{b + a\cos x}{a + b\cos x} = 2\tan^{-1}\sqrt{\left\{\frac{a + b\cos x - b - a\cos x}{a + b\cos x + b + a\cos x}\right\}}$$
$$= 2\tan^{-1}\sqrt{\left\{\frac{(a-b)(1-\cos x)}{(a+b)(1+\cos x)}\right\}} = 2\tan^{-1}\left\{\tan\frac{x}{2}\cdot\sqrt{\left(\frac{a-b}{a+b}\right)}\right\}.$$

19. $\tan^{-1}x = \dfrac{\pi}{2} - \cot^{-1}x$, etc.;

$$\therefore\ \text{left side} = \tan\left(\frac{3\pi}{2} - \cot^{-1}x - \cot^{-1}y - \cot^{-1}z\right).$$

20. $\sin^{-1}p = \cos^{-1}q = \sin^{-1}[\pm\sqrt{(1-q^2)}]$; $\therefore\ p = \pm\sqrt{(1-q^2)}$.

21. $\tan(\tan^{-1}p + \tan^{-1}q) = \tan\left(\dfrac{\pi}{2} - \tan^{-1}r\right)$;

$$\therefore\ \frac{p+q}{1-pq} = \cot(\tan^{-1}r) = \cot\left(\cot^{-1}\frac{1}{r}\right) = \frac{1}{r}.$$

22. $\cos(\cos^{-1}a+\cos^{-1}b)=\cos(\pi-\cos^{-1}c)=-\cos(\cos^{-1}c)=-c$;
$\therefore\ ab-\sqrt{\{(1-a^2)(1-b^2)\}}=-c$;
$\therefore\ (1-a^2)(1-b^2)=(ab+c)^2$.

23. $\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{1}{3}$, using eqn. (8).

24. $\frac{x+(1-x)}{1-x(1-x)}=\frac{9}{7}$; $\therefore\ 9x^2-9x+2=0$.

25. $\tan^{-1}x+2\tan^{-1}x=\frac{\pi}{3}$; $\therefore\ \tan^{-1}x=\frac{\pi}{9}$.

26. $\cos^{-1}(2x^2-1)=2\cos^{-1}x$; $\therefore\ \cos^{-1}x=\frac{\pi}{12}$.

27. $\tan^{-1}\frac{3x-x^3}{1-3x^2}=3\tan^{-1}x$; $\therefore\ \tan^{-1}x=\frac{\pi}{6}$.

EXERCISE III. f. (p. 49.)

1. $\tan 2x=-\tan x$; $\therefore\ 2x=n\pi-x$.

2. $\cos 2x=\cos\left(\frac{\pi}{2}+3x\right)$;

$$\therefore\ \frac{\pi}{2}+3x=2n\pi\pm 2x;\quad \therefore\ x \text{ or } 5x=2n\pi-\frac{\pi}{2},$$

and both are included in $x=(4n-1)\frac{\pi}{10}$.

3. $2\sin 2x\cos x=2\cos x$; $\therefore\ \cos x=0$ or $\sin 2x=1$.

4. $\cos 3x\cos x=1-\sin^2x=\cos^2x$; $\therefore\ \cos x=0$ or $\cos 3x=\cos x$;

$$\therefore\ x=k\pi+\frac{\pi}{2} \text{ or } 3x=2n\pi\pm x;\quad \therefore\ x=n\pi \text{ or } \frac{n\pi}{2};$$

$\therefore$ all solutions are included in $x=\frac{n\pi}{2}$.

5. By *E.T.*, p. 220, $\tan\{x+(x+a)+(x+\beta)\}=0$;
$\therefore\ 3x+a+\beta=n\pi$.

6. Eqn. is equivalent to a quadratic for $\tan x$ which can be factorised; but, as it is satisfied when $x=\beta$ and when $x+a+\beta=\frac{1}{2}\pi$, the quadratic can only have the solutions $\tan\beta$ and $\tan(\frac{1}{2}\pi-a-\beta)$, and the general solutions of the given equation are

$$x=\beta+n\pi,$$
$$x=\tfrac{1}{2}\pi-a-\beta+n\pi.$$

7. $\cos x\cos c=\frac{1}{2}[\cos(2x-a-b)+\cos(a-b)]$
$-\frac{1}{2}[(\cos(a-b)-\cos(a+b)]$
$=\frac{1}{2}[\cos(2x-a-b)+\cos(a+b)]$
$=\cos x\cos(x-a-b)$;
$\therefore\ \cos x=0$ or $\cos(x-a-b)=\cos c$.

8. Put $\theta+\frac{5\pi}{8}=\phi$;

$\therefore\ \sin(3\phi-2\pi)=2\sin\phi$; $\therefore\ \sin 3\phi=2\sin\phi$;
$\therefore\ 3\sin\phi-4\sin^3\phi=2\sin\phi$; $\therefore\ \sin\phi=0$ or $\pm\frac{1}{2}$.

9. $70^2+24^2=74^2$; take a so that $\cos a=\frac{70}{74}$, $\sin a=\frac{24}{74}$;
$\therefore\ \cos a\cos\theta-\sin a\sin\theta=\frac{1}{2}$; $\therefore\ \cos(a+\theta)=\cos 60°$;
$a=18°\ 56'$.

10. $\sin\theta+\sin 4\theta+\sin 2\theta+\sin 3\theta=2\sin\frac{5\theta}{2}\cos\frac{3\theta}{2}+2\sin\frac{5\theta}{2}\cos\frac{\theta}{2}$

$$=2\sin\frac{5\theta}{2}\left(\cos\frac{3\theta}{2}+\cos\frac{\theta}{2}\right)=4\sin\frac{5\theta}{2}\cos\theta\cos\frac{\theta}{2};$$

$$\therefore\ \sin\frac{5\theta}{2}=0 \quad\text{or}\quad \cos\theta=0 \quad\text{or}\quad \cos\frac{\theta}{2}=0.$$

11. $\sin\theta=\frac{2t}{1+t^2}$, $\cos\theta=\frac{1-t^2}{1+t^2}$ where $t=\tan\frac{\theta}{2}$, substituting we have $t^2(\sec a+\sec\beta)+2t\tan a+\sec a-\sec\beta=0$;
$\therefore\ t^2(\sec a+\sec\beta)^2+2t\tan a(\sec a+\sec\beta)$
$=-\sec^2a+\sec^2\beta=\tan^2\beta-\tan^2a$;
$\therefore\ t(\sec a+\sec\beta)+\tan a=\pm\tan\beta$;

$$\therefore\ t=-\frac{\tan a\pm\tan\beta}{\sec a+\sec\beta}=-\frac{\sin(a\pm\beta)}{\cos\beta+\cos a}$$
$$=-\frac{2\sin\frac{1}{2}(a\pm\beta)\cos\frac{1}{2}(a\pm\beta)}{2\cos\frac{1}{2}(a+\beta)\cos\frac{1}{2}(a-\beta)}.$$

12. $2\sin\theta-\tan\theta=\sin\theta(2-\sec\theta)$. Consider values of θ from 0 to 2π, $\sin\theta>0$ if $0<\theta<\pi$; $2-\sec\theta>0$ if

$$0<\theta<\frac{\pi}{3} \text{ or } \frac{\pi}{2}<\theta<\frac{3\pi}{2} \text{ or } \frac{5\pi}{3}<\theta<2\pi.$$

Both factors are + if $0<\theta<\frac{\pi}{3}$ or if $\frac{\pi}{2}<\theta<\pi$.

Similarly, both factors are − if $\frac{3\pi}{2}<\theta<\frac{5\pi}{3}$.

13. $\cos x \sin y + \sin x \cos y = 2 \cot a \sin x \sin y$;

$\therefore \sin(x+y) = \cot a\,[\cos(x-y) - \cos(x+y)]$;

$\therefore \sin 2a + \cot a \cos 2a = \cot a \cos(x-y)$;

$\therefore \cos(2a-a) = \cos a \cos(x-y)$;

$\therefore \cos(x-y) = 1$ if $\cos a \neq 0$;

$\therefore x - y = 2n\pi$; but $x+y = 2a$.

If $\cos a = 0$, $a = n\pi + \frac{\pi}{2}$; then $x + y = (2n+1)\pi$ and $\cot x + \cot y = 0$; this is indeterminate.

14. OX, OY are two rectangular axes; C is the point whose polar coordinates are (c^2, α); OPCQ is a quad. such that $OP = OQ = a^2$, $CP = CQ = b^2$; then θ, ϕ are the angles which OP, PC (or OQ, QC) make with OX. Let $\angle POC = \beta$, $\angle PCO = \gamma$.

15. $\cos^2 x \cos^2 y + \sin^2 x \sin^2 y = 1 = (\sin^2 x + \cos^2 x)(\sin^2 y + \cos^2 y)$;

$\therefore \sin^2 x \cos^2 y + \cos^2 x \sin^2 y = 0$;

$\therefore \sin x \cos y = 0 = \cos x \sin y$;

$\therefore \sin x = 0$ and $\sin y = 0$ or $\cos y = 0$ and $\cos x = 0$.

16. $(2\cos^2 x - 1) + (2\cos^2 y - 1) = 2\cos^2 z - 1$;

$\therefore \cos^2 x + \cos^2 y - \frac{1}{2} = \cos^2 z = (\cos x + \cos y)^2$;

$\therefore \cos x \cos y = -\frac{1}{4}$.

$(4\cos^3 x - 3\cos x) + (4\cos^3 y - 3\cos y) = 4\cos^3 z - 3\cos z$;

$\therefore \cos^3 x + \cos^3 y = \cos^3 z = (\cos x + \cos y)^3$

$= \cos^3 x + \cos^3 y + 3\cos x \cos y(\cos x + \cos y)$;

$\therefore \cos x + \cos y = 0$; since $\cos x \cos y = -\frac{1}{4}$;

$\therefore \cos x = -\cos y = \pm\frac{1}{2}$ and $\cos z = 0$.

17. Put $t = \tan\theta$, $t_1 = \tan\alpha$, etc., and divide by

$$\cos^4\theta \cos\alpha \cos\beta \cos\gamma;$$

then $(1 + tt_1)(1 + tt_2)(1 + tt_3) - t(t - t_1)(t - t_2)(t - t_3)$

$= \sec^4\theta = (1 + t^2)^2$;

expand and factorise,

$$t(t^2+1)(2t - t_1 - t_2 - t_3 - t_1 t_2 t_3) = 0.$$

18. Put $\tan\frac{x}{2} = t$, then $t\left(\frac{2t}{1-t^2}\right)^3 = 1$; $\therefore 8t^4 = (1-t^2)^3$; this reduces to $(t^2+1)(t^4 + 4t^2 - 1) = 0$; $\therefore t^4 + 4t^2 = 1$; $\therefore t^2 = \sqrt{5} - 2$ since t^2 must be positive;

$$\therefore \tan^2 x = \frac{4t^2}{(1-t^2)^2} = \frac{4(\sqrt{5}-2)}{(3-\sqrt{5})^2} = \tfrac{1}{2}(1+\sqrt{5});$$

$$\therefore \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{1-\sqrt{5}}{3+\sqrt{5}} = 2 - \sqrt{5}.$$

19. $\sin\theta = c \pm \sqrt{[(c-2)(c-3)]}$. Put $c = x$ and $\sin\theta = y$ and consider the graph of $y = x \pm \sqrt{[(x-2)(x-3)]}$. This is the same as $y^2 - 2xy + 5x - 6 = 0$ or $(y - 2\frac{1}{2})(y - 2x + 2\frac{1}{2}) = -\frac{1}{4}$. It is a hyperbola, asymptotes $y = 2\frac{1}{2}$ and $y - 2x + 2\frac{1}{2} = 0$, one branch lies in the same angle as the origin.

The hyperbola cuts $y = 1$ at $x = \frac{5}{3}$ and cuts $y = -1$ at $x = \frac{5}{7}$ and cuts each of the lines parallel to Ox between $y = 1$ and $y = -1$ at one point only; the values of x of the points of intersection are between $\frac{5}{3}$ and $\frac{5}{7}$ since for $-1 < y < +1$, x increases steadily as y increases because

$$\frac{dx}{dy} = \frac{2(2-y)(3-y)}{(5-2y)^2} > 0.$$

$\therefore$ there is one value of $\sin\theta$, and only one, if $\frac{5}{3} > c > \frac{5}{7}$.

20. $\cos x = m \pm \sqrt{(1 - 2m - 3m^2)} = m \pm \sqrt{[(1-3m)(1+m)]}$; $\cos x$ is imaginary if $m > \frac{1}{3}$ or if $m < -1$.

(i) $\cos x = m + \sqrt{(1 - 2m - 3m^2)}$. If $m \leqslant \frac{1}{3}$,

$\sqrt{(1 - 2m - 3m^2)} + m \leqslant 1$ if $1 - 2m - 3m^2 \leqslant (1-m)^2$,

i.e. if $0 \leqslant 4m^2$, which is always true. If $-1 \leqslant m < \frac{1}{3}$, $m + \sqrt{(1 - 2m - 3m^2)} \geqslant -1$;

$$\therefore \cos x = m + \sqrt{(1 - 2m - 3m^2)}$$

gives one possible value of $\cos x$ if $-1 \leqslant m \leqslant \frac{1}{3}$.

(ii) $\cos x = m - \sqrt{(1 - 2m - 3m^2)}$.

If $m \leqslant \frac{1}{3}$, $m - \sqrt{(1 - 2m - 3m^2)} \leqslant 1$. If $-1 \leqslant m \leqslant \frac{1}{3}$,

$m - \sqrt{(1 - 2m - 3m^2)} > -1$ if $m + 1 > \sqrt{(1 - 2m - 3m^2)}$,

i.e. if $m^2 + 2m + 1 > 1 - 2m - 3m^2$, since $m + 1 > 0$,

i.e. if $m(m+1) > 0$, i.e. if $m > 0$ or $m < -1$.

$\therefore \cos x = m - \sqrt{(1 - 2m - 3m^2)}$ gives one possible value of $\cos x$ if $0 \leqslant m \leqslant \frac{1}{3}$.

Or, use the method of No. 19. The graph of

$$y = x \pm \sqrt{[(1-3x)(1+x)]}$$

is an ellipse touching $x = -1$ at $(-1, -1)$ and touching $x = \frac{1}{3}$ at $(\frac{1}{3}, \frac{1}{3})$ and touching $y = 1$ at $(0, 1)$ and meeting $y = -1$ again at $(0, -1)$.

21. The solution for $\cos\theta$ amounts to finding x from the simultaneous equations $bhx + aky = ab$, $x^2 + y^2 = 1$. If such a value exists it satisfies $x^2 + y^2 = 1$ and $\therefore x^2 < 1$.

22. $\sin\frac{\theta}{2} + \cos\frac{\theta}{2} = \sqrt{2}\sin\left(\frac{\theta}{2} + 45^\circ\right)$ and this is $-$ since

$$\frac{\theta}{2} + 45^\circ \simeq 255^\circ; \quad \therefore \sin\frac{\theta}{2} + \cos\frac{\theta}{2} = -\sqrt{(1 + \sin\theta)};$$

$\sin\frac{\theta}{2}-\cos\frac{\theta}{2}=\sqrt{2}\sin\left(\frac{\theta}{2}-45^\circ\right)$ and this is + since

$$\frac{\theta}{2}-45^\circ \simeq 165^\circ; \quad \therefore \sin\frac{\theta}{2}-\cos\frac{\theta}{2}=+\sqrt{(1-\sin\theta)}.$$

Result is valid if

$$180^\circ<\frac{\theta}{2}+45^\circ<360^\circ \text{ and } 0^\circ<\frac{\theta}{2}-45^\circ<180^\circ;$$

these require $135^\circ<\frac{\theta}{2}<225^\circ$.

23. One value is $\dfrac{1-\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{1+\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}=\dfrac{2\sin^2\frac{\theta}{4}+2\sin\frac{\theta}{4}\cos\frac{\theta}{4}}{2\cos^2\frac{\theta}{4}+2\sin\frac{\theta}{4}\cos\frac{\theta}{4}}=\tan\frac{\theta}{4}$.

The given expression is unaltered when $2\pi+\theta$ is written instead of θ; $\therefore$ one other value must be $\tan\frac{2\pi+\theta}{4}=-\cot\frac{\theta}{4}$.

Also the given expression is unaltered when $\pi-\theta$ is written instead of θ; $\therefore$ the other two values must be $\tan\frac{\pi-\theta}{4}$ and $-\cot\frac{\pi-\theta}{4}$.

24. $\cos\frac{\theta}{2}$ is − if $2n\pi+\frac{\pi}{2}<\frac{\theta}{2}<2n\pi+\frac{3\pi}{2}$, i.e. if

$(2n+1)\pi<\frac{\theta+\pi}{2}<(2n+2)\pi$, i.e. if $2n+1<\frac{\theta+\pi}{2\pi}<2n+2$, i.e. if the integer just below $\frac{\theta+\pi}{2\pi}$ is odd.

25. $\sin 2px=q=\sin\alpha$, say; then $2px=2n\pi+\alpha$ or $(2k+1)\pi-\alpha$; $\therefore x=\frac{n\pi}{p}+\frac{\alpha}{2p}$ or $\frac{k\pi}{p}+\frac{\pi-\alpha}{2p}$, where n and k have any integral value from 0 to $2p-1$. The two types of solutions give distinct values for $\sin x$ unless

$$\frac{n\pi}{p}+\frac{\alpha}{2p}+\frac{k\pi}{p}+\frac{\pi-\alpha}{2p}\equiv\frac{(n+k)2\pi+\pi}{2p}$$

is an odd multiple of π, which is impossible since $\frac{(n+k)2+1}{2p}$ is not an integer. Therefore there are $4p$ values of $\sin x$, $2p$ of each type; $\sin(2p+1)x=q=\sin\alpha$;

$$x=\frac{2n\pi}{2p+1}+\frac{\alpha}{2p+1} \text{ or } \frac{(2k+1)\pi}{2p+1}-\frac{\alpha}{2p+1};$$

but $\sin\left[\frac{(2k+1)\pi}{2p+1}-\frac{\alpha}{2p+1}\right]=\sin\left[\frac{(2p-2k)\pi}{2p+1}+\frac{\alpha}{2p+1}\right]$; therefore the values of $\sin x$, when n runs from 0 to $2p$, form the same group as are obtained when k runs from 0 to $2p$. Therefore there are only $2p+1$ distinct values of $\sin x$.

26. As in III. d, No. 29, $\cos 5\theta=16c^5-20c^3+5c$ where $c=\cos\theta$; put $x=2k\,.\,c$; the equation reduces to $\cos 5\theta=\cos\alpha$.

27. Use eqn. (8); $(\tan^{-1}p-\tan^{-1}q)+(\tan^{-1}q-\tan^{-1}r)$.

28. Use eqn. (8); $\tan^{-1}\frac{1}{p}-\tan^{-1}\frac{1}{p+q}=$ etc.

29. In No. 28, put $p=1$, $q=1$; $\tan^{-1}1=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$. Then put $p=2$, $q=1$. Then put $p=3$, $q=1$.

30. $\frac{\pi}{4}=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$; now use the values obtained in No. 29.

31. In No. 28, put $p=3$, $q=2$, thus

$$2(\tan^{-1}\tfrac{1}{5}+\tan^{-1}\tfrac{1}{8})+\cot^{-1}7=2\tan^{-1}\tfrac{1}{3}+\tan^{-1}\tfrac{1}{7}$$
$$=\tan^{-1}\tfrac{1}{3}+\tan^{-1}\tfrac{1}{2} \text{ from No. 29.}$$

32. If $\cos\alpha=\dfrac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}$,

$$\tan^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{1+\cos\alpha}=\frac{1-\cos\theta-\cos\phi+\cos\theta\cos\phi}{1+\cos\theta+\cos\phi+\cos\theta\cos\phi}$$
$$=\frac{(1-\cos\theta)(1-\cos\phi)}{(1+\cos\theta)(1+\cos\phi)}=\tan^2\frac{\theta}{2}\cdot\tan^2\frac{\phi}{2}.$$

33. $\cos(\cos^{-1}x+\cos^{-1}2x)=\cos\frac{\pi}{3}=\frac{1}{2}$;

$$\therefore x\,.\,2x-\sqrt{[(1-x^2)(1-4x^2)]}=\tfrac{1}{2};$$
$$\therefore -2\sqrt{[(1-x^2)(1-4x^2)]}=1-4x^2; \quad \therefore 4x^2=1.$$

34. $\sin[\sqrt{(\pi^2-4x^2)}]=\cos x=\sin\left(\frac{\pi}{2}-x\right)$;

$$\therefore \sqrt{(\pi^2-4x^2)}=\frac{\pi-2x}{2} \text{ or } \frac{\pi+2x}{2};$$
$$\therefore 4(\pi^2-4x^2)=(\pi-2x)^2 \text{ or } (\pi+2x)^2;$$
$$\therefore 2x=\pm\pi, \text{ or } 4(\pi+2x)=\pi-2x, \text{ or } 4(\pi-2x)=\pi+2x.$$

CHAPTER IV

EXERCISE IV. a. (p. 54.)

2 to 6. Express that the area "under the curve" lies between the areas of the "upper" and "lower" rectangles.

7. The area measured by hyp (t_1) is a part of that measured by hyp (t_2).

8. Take the upper rectangles instead of the lower rectangles in Fig. 32. Then

$$\text{hyp}\,(2^k) < (2-1)\,.\,1 + (4-2)\,.\,\tfrac{1}{2} + (8-4)\,.\,\tfrac{1}{4} + \dots$$
$$+ (2^k - 2^{k-1}) \cdot \frac{1}{2^{k-1}} = 1 + 1 + 1 + \dots \text{ to } k \text{ terms.}$$

9. (i) Areas of similar triangles are as the squares of corresponding sides, $\text{sq}(nt) = n^2\,.\,\text{sq}(t)$;

(ii) If PN, SM, S′M′ are the ordinates $x = t$, $x = t + t'$, $x = t - t'$,

$$\text{sq}(t+t') - \text{sq}(t-t') = \text{S'M'MS} = \tfrac{1}{2}(\text{S'M'} + \text{SM})\,.\,\text{M'M}$$
$$= \tfrac{1}{2}\,.\,2\text{PN}\,.\,2t' = 4tt', \text{ since PN} = 2t.$$

10. (i) As in No. 2, take ordinates $x = 1$, $x = 1\tfrac{1}{2}$;

(ii) Take ordinates $x = 1$, $x = 1\tfrac{1}{4}$;

(iii) Take ordinates $x = 1$, $x = 1\tfrac{1}{2}$, $x = 2$, $x = 2\tfrac{1}{2}$; rectangle method gives $\tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} < \text{hyp}\,(2\tfrac{1}{2}) < \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}$; but a trapezium (as in Fig. 38) bounded by $x = 1$, $x = \tfrac{3}{2}$, i.e. area $\tfrac{1}{2}(1 + \tfrac{2}{3}) \cdot \tfrac{1}{2} = \tfrac{5}{12}$, can be substituted for the rectangle of area $\tfrac{1}{2}$; $\therefore$ hyp $(2\tfrac{1}{2}) < \tfrac{5}{12} + \tfrac{1}{3} + \tfrac{1}{4} = 1$;

(iv) The tangent at $(2, \tfrac{1}{2})$ to $y = \dfrac{1}{x}$ forms with the ordinates $x = 1$, $x = 3$ and the x-axis a trapezium, base 2, average height $\tfrac{1}{2}$, i.e. area 1.

11. $\text{PN} = \dfrac{1}{t}$, $\text{CN} = t - 1$; upper and lower rectangles have areas $t - 1$ and $\dfrac{t-1}{t}$.

12. QM is ordinate $x = s$; $\text{MC} = 1 - s$, $\text{QM} = \dfrac{1}{s}$; areas of upper and lower rectangles are $\dfrac{1-s}{s}$ and $1 - s$; hyp (s) is negative;

$\therefore\ \dfrac{1-s}{s} > -\text{hyp}\,(s) > 1 - s.$

13. The rectangles have areas $\dfrac{h}{t}, \dfrac{h}{t+h}$; $\therefore$ hyp $(t+h)$ − hyp (t) is between these limits.

14. $\text{PN} = \dfrac{1}{p}$, $\text{QM} = \dfrac{1}{q}$;

$$\therefore\ \text{PNMQ} = \tfrac{1}{2}(q-p)\left(\frac{1}{p} + \frac{1}{q}\right) = \frac{q^2 - p^2}{2pq};$$

similarly $-\text{P'N'M'Q'} = \tfrac{1}{2}\left(\dfrac{1}{p} - \dfrac{1}{q}\right)(p+q)$, the area being reckoned negative because when

$$q > p,\ \frac{1}{p} > \frac{1}{q};\ \therefore\ \text{PNMQ} = -\text{P'N'M'Q'}.$$

If the area represented by hyp (t) is divided up by a large number of ordinates such as PN, QM, etc., then the area hyp $\left(\dfrac{1}{t}\right)$ can be divided up by corresponding ordinates P′N′, Q′M′, etc., such that each trapezium PNMQ = − the corresponding trapezium P′N′M′Q′. By considering the limit of this process, it can be proved that hyp $(t) = -$hyp $\left(\dfrac{1}{t}\right)$.

15. The formula for PNMQ in No. 14 is unchanged when λp, λq are put for p, q. As in No. 14, divide up the area hyp (t), from $x = 1$ to $x = t$, by a large number of ordinates such as PN, QM, etc., then area hyp (λt) − hyp (λ), from $x = \lambda$ to $x = \lambda t$, can be divided up by corresponding ordinates, P′N′, Q′M′, etc., such that each trapezium PNMQ = the corresponding trapezium P′N′M′Q′. Hence in the limit, hyp (λt) − hyp (λ) = hyp (t).

16. The lower rectangles give $\tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{10} = \tfrac{4}{5}$; the upper rectangles give $1 + \tfrac{1}{2} + \tfrac{1}{5} = \tfrac{17}{10}$.

17. (i) $\dfrac{1}{1+x^2} < 1$, $\therefore$ integral $< \displaystyle\int_0^1 dx = 1$;

(ii) $\dfrac{1}{1+x^2} < \dfrac{1}{x^2}$; $\therefore$ integral $< \displaystyle\int_1^t \frac{1}{x^2}\,dx$, for $t > 1$, $= 1 - \dfrac{1}{t}$.

$$\text{Hence } \int_0^t = \int_0^1 + \int_1^t < 1 + \left(1 - \frac{1}{t}\right) < 2.$$

18. Take the ordinates $x = 1$, $x = 2$, $x = 3$, ..., $x = n + 1$; the sum of the areas of the lower rectangles is $1^2 + 2^2 + 3^2 + \dots + n^2$; this is less than the area under the curve from $x = 1$ to $x = n + 1$, i.e. $\displaystyle\int_1^{n+1} x^2 dx = \tfrac{1}{3}\{(n+1)^3 - 1\}$. Take the ordinates $x = 0$, $x = 1$, $x = 2$, ..., $x = n$; the sum of the areas of the

upper rectangles is $1^2+2^2+3^2+\dots+n^2$; this is greater than the area under the curve from $x=0$ to $x=n$, i.e. greater than $\int_0^n x^2\,dx=\frac{1}{3}n^3$.

19. Use the method of No. 18 with the curve $y=\sqrt{x}$; then, as before, $\int_1^{n+1}\sqrt{x}\,dx>\sqrt{1}+\sqrt{2}+\dots+\sqrt{n}>\int_0^n\sqrt{x}\,dx.$

20. Use the method of No. 18 with the curve $y=\sin(\theta x)$, θ being a constant; then, as before, if $0<\theta\leqslant\pi/(2n+1)$,*

$$\int_1^{n+1}\sin(\theta x)\,dx>\sin\theta+\sin 2\theta+\dots+\sin n\theta>\int_0^n\sin(\theta x)\,dx;$$

$$\therefore\ \frac{\cos\theta-\cos(n+1)\theta}{\theta}>\sin\theta+\dots+\sin n\theta>\frac{1-\cos n\theta}{\theta}.$$

EXERCISE IV. b. (p. 58.)

1. (i) and (ii) are special cases of (iii);

(iii) $\frac{d}{dx}\text{hyp}(ax)=\frac{d}{d(ax)}\text{hyp}(ax)\cdot\frac{d(ax)}{dx}=\frac{1}{ax}\cdot a=\frac{1}{x}$;

(iv) $\frac{1}{x}-\frac{1}{x}=0$; $\therefore$ hyp (ax) − hyp (x) is constant, i.e. independent of x, and $\therefore$ =its value when $x=1$, i.e. hyp (a).

2. (i), (ii), (iii) are special cases of (iv);

(iv) $\frac{1}{x^n}\cdot nx^{n-1}=\frac{n}{x}$; $\therefore$ hyp $(x^n)=n$ hyp $x+C$, and $x=1$ gives $C=0$.

3. (i) $\frac{1}{(ax+b)^n}\cdot n(ax+b)^{n-1}\cdot a$;

(ii) $\frac{x+2}{x+1}\cdot\frac{d}{dx}\left(\frac{x+1}{x+2}\right)=\frac{x+2}{x+1}\cdot\frac{(x+2)-(x+1)}{(x+2)^2}.$

4. (i) $\frac{1}{\sin x}\cdot\cos x$; (ii) $\frac{1}{\tan x}\cdot\sec^2 x=\frac{1}{\sin x\cos x}$;

(iii) $\frac{1}{\cot x}\cdot(-\text{cosec}^2 x)=\frac{-1}{\sin x\cos x}.$

5. (viii) $\frac{x}{x-3}\equiv 1+\frac{3}{x-3}$. 7. $\text{cosec}\,2x=\frac{1}{2\sin x\cos x}=\frac{1}{2}\cdot\frac{\sec^2 x}{\tan x}.$

9. $\frac{1}{x}$ is positive because hyp (x) is defined only for positive values of x; $\therefore$ hyp (x) increases as x increases.

* If $\pi/(2n+1)<\theta<\pi/2n$, the last rect., area $\sin n\theta$, is not wholly below curve, but area *above* $<\frac{1}{2}|\sin n\theta-\sin(n+1)\theta|$ whereas there is a surplus $>\frac{1}{2}(\sin 2\theta-\sin\theta)=\sin\frac{1}{2}\theta\cos\frac{3}{2}\theta>\sin\frac{1}{2}\theta\,|\cos(n+\frac{1}{2})\theta|=\frac{1}{2}|\sin n\theta-\sin(n+1)\theta|$ from first rect.

10. $\frac{d}{dx}[x-1-\text{hyp}(x)]=1-\frac{1}{x}=\frac{x-1}{x}$; this is + if $x>1$ and − if $0<x<1$; $\therefore$ $x-1-$hyp (x) decreases as x increases from 0 to 1, and increases as x increases from 1 to ∞; but $x-1-$hyp $(x)=0$ if $x=1$; $\therefore$ it is + for all positive values of x except $x=1$.

11. If x lies between t and $t+h$,

$$\frac{1}{1+x^2}\text{ lies between }\frac{1}{1+t^2}\text{ and }\frac{1}{1+(t+h)^2};$$

$$\therefore\ \int_t^{t+h}\frac{1}{1+x^2}dx\text{ lies between }\frac{h}{1+t^2}\text{ and }\frac{h}{1+(t+h)^2}.$$

Also $\frac{d}{dt}\int_0^t\frac{1}{1+x^2}dx=\lim_{h\to 0}\frac{1}{h}\left\{\int_0^{t+h}-\int_0^t\right\}=\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\frac{1}{1+x^2}dx$;

but $\frac{1}{h}\int_t^{t+h}\frac{1}{1+x^2}dx$ lies between $\frac{1}{1+t^2}$ and $\frac{1}{1+(t+h)^2}$;

$\therefore$ the limit exists and equals $\frac{1}{1+t^2}$.

12. $\tan^{-1}\frac{1}{t}=\int_0^{\frac{1}{t}}\frac{1}{1+x^2}dx=\int_\infty^t\frac{z^2}{z^2+1}\cdot\frac{-1}{z^2}dz$

$=+\int_t^\infty\frac{1}{1+z^2}dz=\int_0^\infty\frac{1}{1+z^2}dz-\int_0^t\frac{1}{1+z^2}dz\triangleq\text{constant}-\tan^{-1}t.$

[The existence of $\int^\infty\frac{1}{1+z^2}dz$ was established in IVa, No. 17.]

EXERCISE IV. c. (p. 61.)

1. hyp (4) =2 hyp (2);
hyp (6) =hyp (2) + hyp (3); hyp (8) =3 hyp (2);
hyp (9) =2 hyp (3); hyp $(\frac{1}{2})=-$hyp (2); hyp $(\frac{1}{4})=-2$ hyp (2);
hyp $(1\frac{1}{2})$ =hyp (3) − hyp (2); hyp $(2\frac{1}{4})$ =2 hyp (3) − 2 hyp (2);
hyp $(2\frac{2}{3})$ =3 hyp (2) − hyp (3).

2. (i) hyp (3) − hyp (2);

(ii) Put $y=\frac{x}{10}$; integral $=\int_2^3\frac{1}{y}dy$ =hyp (3) − hyp (2);

(iii) $\Big[\text{hyp}(x+1)\Big]_1^3$ =hyp (4) − hyp (2) =hyp (2);

(iv) $\Big[\frac{1}{2}\text{hyp}(2x+1)\Big]_0^{1\frac{1}{2}}=\frac{1}{2}$ hyp (4) =hyp (2).

3. $\int_a^{ab}\frac{1}{x}dx=\int_1^b\frac{1}{ay}\cdot a\,dy=\int_1^b\frac{1}{y}dy.$

4. $\int_1^{\frac{a}{b}} \frac{1}{x}\,dx = \int_1^a \frac{1}{x}\,dx + \int_a^{\frac{a}{b}} \frac{1}{x}\,dx =$, putting $x = \frac{a}{y}$, $\int_1^a \frac{1}{x}\,dx - \int_1^b \frac{1}{y}\,dy.$

5 and 6. $\int_1^{2^n} \frac{1}{x}\,dx = \int_1^2 \frac{n}{y}\,dy = n\,\text{hyp}\,(2)$; but $\frac{1}{2} < \text{hyp}\,(2) < 1$, see IV. a, No. 2.

7. Put $x = ty$. The area under the curve, $y = \frac{1}{x}$, between $x = ta$ and $x = tb$ is independent of t.

8. $\text{hyp}\,(x) > \int_1^2 \frac{1}{x}\,dx + \int_2^3 \frac{1}{x}\,dx + \ldots + \int_{[x]-1}^{[x]} \frac{1}{x}\,dx > \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{[x]}$,

since $\int_n^{n+1} \frac{1}{x}\,dx > \int_n^{n+1} \frac{1}{n+1}\,dx = \frac{1}{n+1}$.

Also $\frac{1}{3} + \frac{1}{4} > \frac{2}{4}$, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8}$, $\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16} > \frac{8}{16}$, etc.;

thus $\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{[x]} >$ any assigned number k, if x is large enough.

9. (i) $\text{hyp}\,(b^q) = q\,\text{hyp}\,(b)$; $\therefore \frac{p}{q}\,\text{hyp}\,(b^q) = p\,\text{hyp}\,(b) = \text{hyp}\,(b^p)$;

$$\text{i.e. } \frac{p}{q}\,\text{hyp}\,(a) = \text{hyp}\,(a^{\frac{p}{q}});$$

(ii) If $c = \frac{1}{a}$, $\text{hyp}\,(c) = -\text{hyp}\,(a)$;

$$\therefore \text{hyp}\left(a^{-\frac{p}{q}}\right) = \text{hyp}\left(c^{\frac{p}{q}}\right) = \frac{p}{q}\,\text{hyp}\,(c) = -\frac{p}{q}\,\text{hyp}\,(a).$$

10. $\int_{2^{r-1}}^{2^r} \frac{1}{x}\,dx =$, putting $x = 2^{r-1}y$, $\int_1^2 \frac{1}{y}\,dy$, and this lies between $\frac{1}{2}$ and 1.

11. If CN $= k$, ON $= 1 + k$, PN $= \frac{1}{1+k}$ and $\text{hyp}\,(1+k)$

$$= \text{area ACNP} < \text{trapezium ACNP} = \tfrac{1}{2}\left(1 + \frac{1}{1+k}\right).k$$

$$= k - \tfrac{1}{2}k\left(1 - \frac{1}{1+k}\right).$$

12. In Fig. 40, p. 59, if KC $= k$, OK $= 1 - k$,

P′K $= \frac{1}{1-k}$ and $-\text{hyp}\,(1-k) =$ area P′KCA $<$ trapezium P′KCA

$$= \tfrac{1}{2}\left(\frac{1}{1-k} + 1\right).k = k + \tfrac{1}{2}k\left(\frac{1}{1-k} - 1\right) = k + \frac{k^2}{2(1-k)}.$$

EXERCISE IV. d. (p. 66.)

1. x and y are interchanged. One curve is the image of the other in the line $y = x$.

2. As x varies from $-\infty$ to $+\infty$, 2^x steadily increases from 0 to $+\infty$ and crosses the y-axis at $y = 1$. The same is true of e^{2x}. For approximate shape, see Fig. 44.

3. (i) $y = \log x$ is the image of $y = e^x$ in the line $y = x$, see Fig. 43;

(ii) $y = \log(2x) = \log 2 + \log x$: move the graph in Fig. 43 a distance $\log 2$ parallel to Oy;

(iii) $y = \log(x^2) = 2\log x$, graph as in Fig. 43, doubling each ordinate;

(iv) $y = \log\left(\frac{1}{x}\right) = -\log x$. Take the image in the x-axis of the graph in Fig. 43.

4. (i) $\log x = 1 + \log a = \log e + \log a = \log(ae)$;

(ii) $\log x = \log e - \log b = \log\frac{e}{b}$;

(iii) $\log(\log x) = 0$ if $\log x = 1$, i.e. if $x = e$;

(iv) $\log(\log x) = 1$ if $\log x = e$, i.e. if $x = e^e$.

5. $e^{\log u} = u$, see p. 63;

$$\therefore e^{2\log x} = e^{\log x^2} = x^2 \text{ and } e^{x\log 2} = e^{\log 2^x} = 2^x;$$

also $\log(e^2x) = \log(e^2) + \log x = 2\log e + \log x$, but $\log e = 1$.

6. $\frac{d}{dx}e^x = e^x > 0$; $\therefore$ e^x steadily increases as x increases.

7. (i) The graph of $y = e^{-x}$ is the image of $y = e^x$ (see Fig. 44) in Oy;

(ii) As x increases from 0 to ∞, e^{x^2} steadily increases from 1 to ∞; also the slope of $y = e^{x^2}$ at (0, 1) is 0 and the curve is symmetrical about Oy.

(iii) As x increases from 0 to ∞, e^{-x^2} steadily decreases from 1 to 0; also the slope of $y = e^{-x^2}$ at (0, 1) is 0 and the curve is symmetrical about Oy;

(iv) As x increases from $-\infty$ to 0, $e^{\frac{1}{x}}$ steadily decreases from 1 to 0; as x increases from 0 to ∞, $e^{\frac{1}{x}}$ steadily decreases from $+\infty$ to 1; $e^{\frac{1}{x}}$ is not defined at $x = 0$, but if x is small the slope of $y=e^{\frac{1}{x}}$ is approximately $-\frac{\pi}{2}$; the graph is asymptotic to $y=1$ at each end.

(v) The tangent at (0, 1) to $y=e^x$ is $y=x+1$; hence $y=x+a$ cuts $y=e^x$ at 2 points if $a>1$, but does not cut it at all if $a<1$ (see Fig. 44).

8. $\frac{d}{dx}\{x\log x-x\}=\log x+x\cdot\frac{1}{x}-1$; integral $=\Big[x\log x-x\Big]_1^t$.

9. $\frac{d}{dx}\{(x-1)e^x\}=e^x+(x-1)\,.\,e^x$; integral $=\Big[(x-1)e^x\Big]_0^t$.

In Nos. 10-25, the following theorems are required:

(i) $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$; (ii) $\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$;

(iii) $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$.

10. $\log x\,.\,2x+x^2\,.\,\frac{1}{x}$.

11. $\frac{x\,.\,\frac{1}{x}-\log x}{x^2}$.

12. $\log\left(\frac{1}{x}\right)=-\log x$.

13. $\frac{d(e^{ax})}{d(ax)}\cdot\frac{d(ax)}{dx}=e^{ax}\,.\,a$.

14. As in No. 13.

15. $e^{\log x}=x$.

16. See Example 1, p. 65.

17. $\log(\sec x)=-\log(\cos x)$, see No. 16.

18. As in No. 13.

19. $\frac{d(e^{\tan^2 x})}{d(\tan^2 x)}\cdot\frac{d(\tan^2 x)}{d(\tan x)}\cdot\frac{d(\tan x)}{dx}$.

20. $\exp(x\sec x)\cdot\frac{d}{dx}(x\sec x)$.

21. $\frac{d\cos(e^x)}{d(e^x)}\cdot\frac{d(e^x)}{dx}$.

22. $\frac{d[\operatorname{cosec}(\log x)]}{d(\log x)}\cdot\frac{d(\log x)}{dx}$.

23. $\log(a+bx)^n=n\log(a+bx)$.

24. $\frac{1}{e^x+e^{-x}}\cdot(e^x-e^{-x})$.

25. (i) $\cot\frac{x}{2}\,.\,\sec^2\frac{x}{2}\cdot\frac{1}{2}=\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{1}{\sin x}$;

(ii) As in (i), $\frac{1}{\sin 2\left(\frac{\pi}{4}+\frac{x}{2}\right)}$;

(iii) $\frac{1}{x+\sqrt{(a^2+x^2)}}\cdot\left\{1+\frac{x}{\sqrt{(a^2+x^2)}}\right\}$

$$=\frac{1}{x+\sqrt{(a^2+x^2)}}\cdot\frac{\sqrt{(a^2+x^2)}+x}{\sqrt{(a^2+x^2)}}.$$

26. By inspection, or put $3x=y$.

27. $\frac{2x+4}{2x+3}=1+\frac{1}{2x+3}$.

28. $\frac{x-1}{2x+3}=\frac{1}{2}\left(1-\frac{5}{2x+3}\right)$.

29. By inspection, or put $3x=y$.

30. $\int e^{ax}\,dx=\frac{1}{a}\,.\,e^{ax}$.

31. Put $2+3\cos x=y$, then

$-3\sin x\,.\,dx=dy$ and integral $=-\frac{1}{3}\int\frac{1}{y}\,dy$.

32. Put $x^2+3x+4=y$, then $(2x+3)dx=dy$ and integral $=\int\frac{1}{y}\,dy$.

33. Put $\log x=y$, then $\frac{1}{x}dx=dy$ and integral $=\int y\,dy$.

34. Put $\tan\frac{x}{2}=t$, then $\frac{1}{2}\sec^2\frac{x}{2}\cdot dx=dt$ and integral $=\int\frac{2}{t}\,dt$.

35. Put $x^2=y$, then $2x\,.\,dx=dy$ and integral $=\int\frac{1}{2}e^y\,dy$.

36. $\int\frac{\cos 3x}{\sin 3x}\,dx=\int\frac{\frac{1}{3}d(\sin 3x)}{\sin 3x}$.

37. $\int\frac{dx}{1+e^x}=\int\frac{e^{-x}dx}{e^{-x}+1}=\int\frac{-d(e^{-x}+1)}{e^{-x}+1}=-\log(e^{-x}+1)$.

38. These are the reverse results of No. 25. The results in (i) and (ii) may be obtained direct by putting $\tan\frac{x}{2}=t$, then $\sec^2\frac{x}{2}\cdot\frac{1}{2}dx=dt$ and so $dx=\frac{2dt}{1+t^2}$.

39. $\frac{3x+1}{x^2-1}=\frac{1}{x+1}+\frac{2}{x-1}$.

40. $\frac{3x+7}{(x+1)(x+2)(x+3)}=\frac{2}{x+1}-\frac{1}{x+2}-\frac{1}{x+3}$.

41. $\frac{x-1}{x(x-2)^2}=-\frac{\frac{1}{4}}{x}+\frac{\frac{1}{4}}{x-2}+\frac{\frac{1}{2}}{(x-2)^2}$.

42. Integral $=\int\left(\frac{1}{x-3}-\frac{x}{x^2+1}\right)dx=\log(x-3)-\int\frac{\frac{1}{2}d(x^2+1)}{x^2+1}$.

43. $\frac{x^3}{x-1}=\frac{(x^3-1)+1}{x-1}=x^2+x+1+\frac{1}{x-1}$.

44. $\frac{x^2}{(x-a)(x-b)}=1+\frac{1}{a-b}\left(\frac{a^2}{x-a}-\frac{b^2}{x-b}\right)$.

45. $\int\log x\,dx=\int\log x\,.\,1\,.\,dx=(\log x)\,.\,x-\int x\cdot\frac{1}{x}dx$.

46. $\int x\log x\,dx=\frac{x^2}{2}\cdot\log x-\int\frac{x^2}{2}\cdot\frac{1}{x}dx$.

47. $\int x^n\log x\,dx=\frac{x^{n+1}}{n+1}\log x-\int\frac{x^{n+1}}{n+1}\cdot\frac{1}{x}dx$.

48. $\int xe^x\,dx=x\,.\,e^x-\int e^x\cdot\frac{dx}{dx}dx=xe^x-\int e^x\,dx$.

49. $\int x^2e^x dx=x^2\,.\,e^x-\int e^x\,.\,2x\,dx$, and then use No. 48.

50. $\log(\sqrt{x})=\frac{1}{2}\log x$; then as in No. 45.

51. Put $\mathsf{P}=\int e^{ax}\sin bx\,dx$, $\mathsf{Q}=\int e^{ax}\cos bx\,dx$;

$$\frac{d}{dx}(e^{ax}\sin bx)=e^{ax}(a\sin bx+b\cos bx);$$
$$\therefore\ e^{ax}\sin bx=a\mathsf{P}+b\mathsf{Q};$$
$$\frac{d}{dx}(e^{ax}\cos bx)=e^{ax}(a\cos bx-b\sin bx);$$
$$\therefore\ e^{ax}\cos bx=a\mathsf{Q}-b\mathsf{P};\ \text{solve for P and Q.}$$

52. $\frac{d}{dx}(x^2\log x)=2x\log x+x=x(2\log x+1)=0$ when $\log x=-\frac{1}{2}$, that is $x=\frac{1}{\sqrt{e}}$, and is − or + according as $x<$ or $>\frac{1}{\sqrt{e}}$; hence a minimum for $x=\frac{1}{\sqrt{e}}$.

53. $\frac{d}{dx}\left(\frac{\log x}{x}\right)=\frac{1-\log x}{x^2}=0$ when $x=e$ and is + or − according as $x<$ or $>e$; hence a maximum for $x=e$; the max. value $=\frac{\log e}{e}=\frac{1}{e}$.

$\frac{\log x}{x}$ is undefined if $x<0$; when x increases from 0 to e, $\frac{\log x}{x}$ steadily increases from $-\infty$ to $\frac{1}{e}$; when x increases from e to ∞, $\frac{\log x}{x}$ steadily decreases from $\frac{1}{e}$ to 0, see

Example 4, p. 68. Draw the graph of $y=\frac{\log x}{x}$; the line $y=\mathsf{A}$ cuts it at 1 point if $\mathsf{A}\leqslant 0$, at 2 points if $0<\mathsf{A}<\frac{1}{e}$, at 1 point if $\mathsf{A}=\frac{1}{e}$, at no points if $\mathsf{A}>\frac{1}{e}$.

EXERCISE IV. e. (p. 70.)

1. $\log t<t-1$, put $\frac{t_2}{t_1}$ for t.

2. $\log t>1-\frac{1}{t}$, put $\frac{a+b}{a}$ for t.

3. In eqn. (18), put $1+x$ for t.

4. In eqn. (18), put $\frac{1}{1-x}$ for t.

5. By No. 3, $x>\log(1+x)$ if $x>-1$, $x\neq 0$; $\therefore e^x>1+x$. If $x=0$, $e^x=1=1+x$. If $x<-1$, $e^x>0>1+x$.

6. Write $-x$ for x in No. 5; then $e^{-x}\geqslant 1-x$; $\therefore$ for $x<1$, $e^x\leqslant\frac{1}{1-x}$, and for $x>1$, $e^x>0>\frac{1}{1-x}$.

7. If $p>0$, $\log(p^p)=p\log p>p\left(1-\frac{1}{p}\right)$, by eqn. (18), for $p\neq 1$,

$$=p-1;\ \therefore\ p^p>e^{p-1}=\frac{1}{e}\cdot e^p;\ \therefore\ \left(\frac{e}{p}\right)^p \not< e,\ \text{unless } p=1.$$

8. As in IV. d, No. 53, $\frac{d}{dx}\left(\frac{\log x}{x}\right)$ is − if $x>e$; $\therefore\frac{\log x}{x}$ steadily decreases as x increases from e to ∞.

Or $\frac{\log(x+h)}{x+h}-\frac{\log x}{x}$, where $h>0$, $x>0$, has same sign as $x\log(x+h)-(x+h)\log x=x\{\log(x+h)-\log x\}-h\log x$

$$=x\log\left(1+\frac{h}{x}\right)-h\log x<h-h\log x,\ \text{by eqn. (18)},$$
$$<0\ \text{if } x>e.$$

9. $(n+1)\log n-n\log(n+1)=\log n-n\{\log(n+1)-\log n\}$

$$=\log n-n\log\left(1+\frac{1}{n}\right)>\log n-1,\ \text{by eqn. (18)},\ >0\ \text{if } n>e;$$
$$\therefore\ \log n^{n+1}>\log(n+1)^n.$$

10. In Example 3, p. 68, write $1+x$ for x.

11. In Example 4 (i), p. 68, write x^p for x; then

$$\frac{\log(x^p)}{x^p}=\frac{p\log x}{x^p}\to 0\ \text{when } x^p\to\infty;$$

but if $p>0$, $x\to\infty$ when $x^p\to\infty$.

12. In Example 4 (ii), p. 68, write x^p for x; then
$$x^p \log(x^p) = px^p \log x \to 0 \text{ when } x^p \to 0$$
through + values; but if $p > 0$, $x \to 0$ when $x^p \to 0$.

13. As in Example 6, p. 69; here $u_{n+1} - u_n = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right) > 0$,

by eqn. (18). Also, by eqn. (18), $\log \frac{n}{n-1} > \frac{1}{n}$;
$$\therefore \log n = \sum_2^n \log \frac{r}{r-1} > \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n};$$
$$\therefore u_n = 1 + \left(\tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} - \log n\right) - \frac{1}{n} < 1 - \frac{1}{n} < 1.$$

14. When $n \to \infty$,
$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n-1} - \log n \equiv \left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} - \log n\right\} - \frac{1}{n}$$
$\to \gamma$ since $\frac{1}{n} \to 0$, see also Example 6, pp. 69, 70;

$\therefore$ by No. 13, $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} - \log n < \gamma$ for all values of n; take $n = 2$, then $1 - \log 2 < \gamma$.

15. Expression $= \left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n} - \log(2n)\right\} + \log 2$
$\to \gamma + \log 2$ when $n \to \infty$.

16. Expression $= \left\{1 + \frac{1}{2} + \ldots + \frac{1}{2n} - \log(2n)\right\}$
$$- \left\{1 + \tfrac{1}{2} + \ldots + \frac{1}{n} - \log n\right\} + \log(2n) - \log n,$$
but $\log(2n) - \log n = \log\left(\frac{2n}{n}\right) = \log 2$;
$\therefore$ expression $\to \gamma - \gamma + \log 2 = \log 2$.

17. Expression $= \left\{1 + \frac{1}{2} + \ldots + \frac{1}{2n} - \log(2n)\right\}$
$$- 2\left\{\tfrac{1}{2} + \tfrac{1}{4} + \ldots + \frac{1}{2n} - \tfrac{1}{2}\log n\right\} + \log 2n - \log n$$
$$= \left\{1 + \tfrac{1}{2} + \ldots + \frac{1}{2n} - \log(2n)\right\} - \left\{1 + \tfrac{1}{2} + \ldots + \frac{1}{n} - \log n\right\} + \log 2,$$
as in No. 16; this tends to $\gamma - \gamma + \log 2$.

18. Expression $= \left\{1 + \frac{1}{2} + \ldots + \frac{1}{2n} - \log(2n)\right\}$
$$- \left\{\tfrac{1}{2} + \tfrac{1}{4} + \ldots + \frac{1}{2n} - \tfrac{1}{2}\log n\right\} + \log 2n - \log n + \frac{1}{2n+1}$$

$$= \left\{1 + \tfrac{1}{2} + \ldots + \frac{1}{2n} - \log(2n)\right\} - \tfrac{1}{2}\left\{1 + \tfrac{1}{2} + \ldots + \frac{1}{n} - \log n\right\}$$
$$+ \log 2 + \frac{1}{2n+1}, \text{ as in No. 16};$$

but when $n \to \infty$, $\frac{1}{2n+1} \to 0$; $\therefore$ limit $= \gamma - \frac{1}{2}\gamma + \log 2$.

19. If $u_n = 1 + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3n+1} - \frac{1}{3}\log n$,
$$u_{n+1} - u_n = \frac{1}{3n+4} - \tfrac{1}{3}\log(n+1) + \tfrac{1}{3}\log n$$
$$= \frac{1}{3n+4} - \tfrac{1}{3}\log\frac{n+1}{n} < \frac{1}{3n+4} - \tfrac{1}{3}\left(1 - \frac{n}{n+1}\right),$$
by eqn. (18), $= \frac{1}{3n+4} - \frac{1}{3n+3} < 0$; $\therefore$ when n increases, u_n steadily decreases; but
$$u_n > \tfrac{1}{3} + \tfrac{1}{6} + \ldots + \frac{1}{3n+3} - \tfrac{1}{3}\log n$$
$$> \tfrac{1}{3}\left\{1 + \tfrac{1}{2} + \ldots + \frac{1}{n} - \log n\right\} > 0,$$
by Example 6, p. 69. Hence, as on p. 70, u_n tends to a limit.

20. $\log \frac{n+p}{n-1} = \log(n+p) - \log(n-1)$
$$= \sum_n^{n+p}\left\{\log(m) - \log(m-1)\right\}$$
$$= \sum_n^{n+p} \log\frac{m}{m-1} > \sum_n^{n+p}\left(1 - \frac{m-1}{m}\right), \text{ by eqn. (18)},$$
$$= \sum_n^{n+p} \frac{1}{m} = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{n+p}.$$
Similarly,
$$\log\frac{n+p+1}{n} = \sum_{n+1}^{n+p+1} \log\frac{m}{m-1} < \sum_{n+1}^{n+p+1}\left(\frac{m}{m-1} - 1\right),$$
by eqn. 18, $= \sum_{n+1}^{n+p+1} \frac{1}{m-1} = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{n+p}$. Put $p = nq - n$, then $\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{nq}$ lies between $\log\frac{nq}{n-1}$ and $\log\frac{nq+1}{n}$, i.e. between $\log\left(q + \frac{q}{n-1}\right)$ and $\log\left(q + \frac{1}{n}\right)$; each of these tends to $\log q$.

EXERCISE IV. f. (p. 72.)

1. $\dfrac{1}{x+\sqrt{(x^2-a^2)}}\cdot\left\{1+\dfrac{x}{\sqrt{(x^2-a^2)}}\right\}$

$$=\frac{1}{x+\sqrt{(x^2-a^2)}}\cdot\frac{\sqrt{(x^2-a^2)}+x}{\sqrt{(x^2-a^2)}}.$$

2. By No. 1, $\displaystyle\int\frac{1}{\sqrt{(x^2-a^2)}}dx=\log[x+\sqrt{(x^2-a^2)}]$;

$\therefore$ integral $=\frac{1}{2}\displaystyle\int_{1\cdot5}^{2\cdot5}\frac{1}{\sqrt{(x^2-\frac{9}{4})}}dx=\frac{1}{2}\{\log[2\cdot5+\sqrt{(2\cdot5^2-\frac{9}{4})}]$

$-\log[1\cdot5+\sqrt{(1\cdot5^2-\frac{9}{4})}]\}=\frac{1}{2}\{\log4\cdot5-\log1\cdot5\}=\frac{1}{2}\log\dfrac{4\cdot5}{1\cdot5}$.

3. By partial fractions,

$$\text{integral}=\int\left\{1+\frac{3}{x-1}+\frac{3}{(x-1)^2}+\frac{2}{(x-1)^3}\right\}dx.$$

4. Integration by parts gives

$$\int x^{n+1}e^x\,.\,dx=x^{n+1}\,.\,e^x-\int e^x(n+1)x^n\,dx$$
$$=x^{n+1}e^x-(n+1)\int x^ne^x\,dx.$$

5. If $y=x^{\frac{1}{x}}$, $\log y=\dfrac{1}{x}\log x$; $\therefore$ y is a maximum when $x=e$, see solution of IV. d, No. 53.

6. $\dfrac{dy}{dx}=-\dfrac{a\sin(\log x)}{x}+\dfrac{b\cos(\log x)}{x}$;

$$\therefore\ x\frac{dy}{dx}=-a\sin(\log x)+b\cos(\log x);$$
$$\therefore\ \frac{d}{dx}\left(x\frac{dy}{dx}\right)=-\frac{1}{x}[a\cos(\log x)+b\sin(\log x)];$$
$$\therefore\ x\frac{d^2y}{dx^2}+\frac{dy}{dx}=-\frac{y}{x};\quad\therefore\ x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y.$$

7. If $y=e^{kx}$, $\dfrac{dy}{dx}=k\,.\,e^{kx}=ky$; $\dfrac{d^2y}{dx^2}=k\dfrac{dy}{dx}=k^2y$;

$$\therefore\ y(k^2-5k+6)=0;\quad\therefore\ (k-2)(k-3)=0.$$

8. $y\,.\,e^{mx}=a\sin(nx+\alpha)$; $\therefore$ differentiating,

$\dfrac{dy}{dx}\cdot e^{mx}+my\,.\,e^{mx}=an\cos(nx+\alpha)$; differentiate again;

$$\therefore\ \left(\frac{d^2y}{dx^2}\cdot e^{mx}+m\frac{dy}{dx}\,.\,e^{mx}\right)+\left(m\frac{dy}{dx}\,.\,e^{mx}+m^2y\,.\,e^{mx}\right)$$
$$=-an^2\sin(nx+\alpha)=-n^2ye^{mx};$$
$$\therefore\ \frac{d^2y}{dx^2}+2m\frac{dy}{dx}+m^2y=-n^2y.$$

9. $\dfrac{dy}{dx}=e^x(\sin x+\cos x)=e^x\,.\,\sqrt{2}\sin\left(x+\dfrac{\pi}{4}\right)$;

$$\therefore\ \text{similarly, }\frac{d^2y}{dx^2}=e^x\,.\,(\sqrt{2})^2\sin\left(x+\frac{2\pi}{4}\right),\text{ etc.};$$

and $\dfrac{d^4y}{dx^4}=e^x\,.\,(\sqrt{2})^4\,.\,\sin\left(x+\dfrac{4\pi}{4}\right)=-4e^x\sin x=-4y$.

10. $\log y=x\log x$; $\therefore$ $\dfrac{1}{y}\dfrac{dy}{dx}=1+\log x=\log e+\log x=\log(ex)$.

11. Integral $=\displaystyle\int\tfrac{1}{2}e^{ax}(1-\cos2bx)\,dx=\tfrac{1}{2}\left\{\int e^{ax}dx-\int e^{ax}\cos2bx\,dx\right\}$

$$=\tfrac{1}{2}\left\{\frac{e^{ax}}{a}-\frac{1}{a^2+4b^2}\cdot e^{ax}\,.\,(a\cos2bx+2b\sin2bx)\right\}$$

by IV. d, No. 51.

12. For graph of $\log x$, see Fig. 43, p. 63; $\log x<0$ if $x<1$; $\therefore$ $\log(\log x)$ is undefined if $x<1$; when $x\to1$, $(x>1)$, $\log x\to0$; $\therefore$ $\log(\log x)\to-\infty$; when x increases from 1 to e, $\log(\log x)$ steadily increases from $-\infty$ to 0; when x increases from e to ∞, $\log(\log x)$ steadily increases from 0 to ∞, but less rapidly than $\log x$; in fact $\dfrac{\log(\log x)}{\log x}\to0$ when $x\to\infty$, by Example 4 (i), p. 68.

13. By eqn. (18), $\log(\sqrt{t})<\sqrt{t}-1$, $t>0$, $t\neq1$; $\therefore$ $\frac{1}{2}\log t+1<\sqrt{t}$; but $\log(e^2t)=2\log e+\log t=2+\log t$.

If $t=1$, $\log(e^2t)=2\sqrt{t}$.

14. Put $\dfrac{a}{b}=1+x$, then $\log(1+x)\simeq\dfrac{a^2-b^2}{2ab}=\dfrac{(1+x)^2-1}{2(1+x)}$

$$=\frac{(2x+x^2)(1-x+x^2-x^3)}{2(1-x^4)}\simeq\frac{2x-x^2+x^3}{2}=x-\tfrac{1}{2}x^2+\tfrac{1}{2}x^3.$$

15. By Example 4 (i), p. 68,

$$\log(x^{\frac{1}{x}})=\frac{1}{x}\log x\to0\text{ when }x\to\infty;\quad\therefore\ x^{\frac{1}{x}}\to1.$$

16. By Example 4 (ii), p. 68,

$$\log(x^x)=x\log x\to0\text{ when }x\to0\text{ for }x>0;\quad\therefore\ x^x\to1.$$

17. (i) $\log t^2>1-\dfrac{1}{t^2}$; $\therefore$ $2\log t>1-\dfrac{1}{t^2}$;

(ii) $\log\sqrt{t}>1-\dfrac{1}{\sqrt{t}}$; $\therefore$ $\frac{1}{2}\log t>1-\dfrac{1}{\sqrt{t}}$;

(iii) $\frac{1}{2}\left(1-\frac{1}{t^2}\right)=\frac{1}{2}\left(1+\frac{1}{t}\right)\left(1-\frac{1}{t}\right)<\left(1-\frac{1}{t}\right)$ since $t>1$; writing $\sqrt{t}$ for t, this becomes

$$\tfrac{1}{2}\left(1-\frac{1}{t}\right)<\left(1-\frac{1}{\sqrt{t}}\right); \quad \therefore \log t>2\left(1-\frac{1}{\sqrt{t}}\right)$$

gives more information than

$$\log t>1-\frac{1}{t} \text{ or } \log t>\tfrac{1}{2}\left(1-\frac{1}{t^2}\right).$$

18. In $\log t<t-1$, put $t=\sqrt{x}$; then $\frac{1}{2}\log x=\log\sqrt{x}<\sqrt{x}-1$.

19. $t-\frac{1}{2}t^2=\int_0^t(1-x)\,dx<\int_0^t\left(1-\frac{x}{1+x}\right)dx$, for $t>0$,

$$=\log(1+t)<\int_0^t\left(1-\frac{x}{1+t}\right)dx=t-\frac{\frac{1}{2}t^2}{1+t}$$

$$=t-\frac{t^2}{2(1+t)}=\frac{2t+t^2}{2(1+t)}.$$

20. In No. 19, put $t=x-1$. **Or** $\frac{2}{x}<1+\frac{1}{x^2}$, for $x>1$; $\therefore$ for $t>1$,

$$2\log t=\int_1^t\frac{2}{x}dx<\int_1^t\left(1+\frac{1}{x^2}\right)dx=\left[x-\frac{1}{x}\right]_1^t=t-\frac{1}{t}.$$

21. $t-\frac{t^2}{2}+\frac{t^3}{3(1+t)}=\int_0^t\left(1-x+\frac{x^2}{1+t}\right)dx<\int_0^t\left(1-x+\frac{x^2}{1+x}\right)dx$, for $t>0$, $=\int_0^t\frac{1}{1+x}dx=\log(1+t)<\int_0^t(1-x+x^2)\,dx$, for $t>0$, $=t-\frac{1}{2}t^2+\frac{1}{3}t^3$.

22. $p>q$; $\therefore pe^{-x}>qe^{-x}$;

$$\therefore p(e^x+e^{-x})>pe^x+qe^{-x}; \quad \therefore p>\frac{pe^x+qe^{-x}}{e^x+e^{-x}}$$

since $e^x+e^{-x}>0$; similarly, $pe^x+qe^{-x}>q(e^x+e^{-x})$.

23. (i) $\int x^2e^x\,dx=x^2e^x-\int 2xe^x\,dx$

$$=x^2e^x-2\left\{xe^x-\int e^x\,dx\right\}=x^2e^x-2xe^x+2e^x;$$

(ii) By IV e, No. 5, $1+x^2<e^{x^2}<e^x$ since, for $0<x<1$, $x^2<x$:

$$\therefore \int_0^1 x^2(1+x^2)\,dx<\int_0^1 x^2e^{x^2}dx<\int_0^1 x^2e^x\,dx$$

$$=\left[(x^2-2x+2)e^x\right]_0^1=e-2; \text{ also } \int_0^1 x^2(1+x^2)\,dx$$

$$=\left[\frac{x^3}{3}+\frac{x^5}{5}\right]_0^1=\tfrac{1}{3}+\tfrac{1}{5}=\tfrac{8}{15}.$$

24. By eqn. (18), $x>\log(1+x)>\frac{x}{1+x}$;

$$\therefore \int_0^t x^2\,dx>\int_0^t x\log(1+x)\,dx>\int_0^t\frac{x^2}{1+x}dx, \quad \text{for } t>0,$$

$$>\int_0^t\frac{x^2-1}{x+1}dx=\int_0^t(x-1)\,dx=\tfrac{1}{2}t^2-t.$$

25. For $x>1$, $p>0$, we have $x^p>1$; $\therefore \frac{x^p}{x}>\frac{1}{x}$; $\therefore \frac{1}{x}<\frac{1}{x^{1-p}}$;

$$\therefore \int_1^t\frac{1}{x}dx<\int_1^t\frac{1}{x^{1-p}}dx=\int_1^t x^{p-1}dx=\frac{t^p-1}{p}<\frac{t^p}{p};$$

$\therefore \log t<\frac{t^p}{p}$. Now given any positive number n, it is possible to choose p, so that $0<p<n$, then

$$\frac{\log t}{t^n}<\frac{t^p}{pt^n}=\frac{1}{p}\cdot\frac{1}{t^{n-p}},$$

where $n-p>0$, so that $\frac{1}{t^{n-p}}\to 0$ when $t\to\infty$; $\therefore \frac{\log t}{t^n}$, being positive, tends to 0 when $t\to\infty$.

If $t=e^y$, $\log t=y$, and $y\to\infty$ when $t\to\infty$. Hence $\frac{y}{e^{ny}}\to 0$ when $y\to\infty$; $\therefore$ if $q>0$, $\left(\frac{y}{e^{ny}}\right)^q\to 0$ when $y\to\infty$, thus $\frac{y^q}{e^{nqy}}=\frac{y^q}{e^{ry}}$, putting $nq=r$, tends to 0 when $y\to\infty$, for $r>0$; this last result is obviously still true for $q<0$.

26. For $0<\theta<\frac{1}{2}\pi$, $\sin\theta<\theta<\tan\theta$; $\therefore 1<\theta\operatorname{cosec}\theta<\sec\theta$; $\therefore 0<\log(\theta\operatorname{cosec}\theta)^n<\log\sec^n\theta$; $\therefore$ by Example 7, p 70, putting $\theta=\frac{x}{n}$, $\lim_{n\to\infty}\log\left(\frac{x/n}{\sin x/n}\right)^n=0$, and $\lim_{n\to\infty}\left(\frac{x/n}{\sin x/n}\right)^n=1$.

EXERCISE IV. g. (p. 74.)

1. $\frac{1}{\sqrt{(1+x^2)}}$ steadily decreases as x increases from 0 to 2 and takes the values 1, $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{5}}$ for $x=0, 1, 2$;

$$\therefore \int_0^2\frac{1}{\sqrt{(1+x^2)}}dx=\int_0^1\frac{1}{\sqrt{(1+x^2)}}dx+\int_1^2\frac{1}{\sqrt{(1+x^2)}}dx$$

$$<\int_0^1 1\,dx+\int_1^2\frac{1}{\sqrt{2}}dx=1+\frac{1}{\sqrt{2}}<1{\cdot}71;$$

and integral $>\int_0^1\frac{1}{\sqrt{2}}dx+\int_1^2\frac{1}{\sqrt{5}}dx=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{5}}>1{\cdot}15$.

2. $\sum_{r=1}^{n} r^p < \sum_{r=1}^{n}\int_r^{r+1} x^p dx = \int_1^{n+1} x^p dx = \frac{(n+1)^{p+1}-1}{p+1}$; and similarly

series $> \sum_{r=1}^{n}\int_{r-1}^{r} x^p dx = \int_0^n x^p dx = \frac{n^{p+1}}{p+1}$.

3. $\tan^{-1} n = \int_0^n \frac{1}{1+x^2}\,dx$

$$= \sum_1^n \int_{r-1}^{r} \frac{1}{1+x^2}\,dx > \sum_1^n \frac{1}{1+r^2} > \sum_1^n \int_r^{r+1} \frac{1}{1+x^2}\,dx$$

$$= \int_1^{n+1} \frac{1}{1+x^2}\,dx = \tan^{-1}(n+1) - \tan^{-1} 1 = \tan^{-1}\frac{n}{n+2}.$$

4. For $x > 1$, $(x-1)\log x > 0$; $\therefore \int_1^x (u-1)\log u\,du > 0$; but

integral $= \left[\log u(\tfrac{1}{2}u^2 - u)\right]_1^x - \int_1^x (\tfrac{1}{2}u^2 - u)\frac{1}{u}\,du$

$$= (\tfrac{1}{2}x^2 - x)\log x - \left[\tfrac{1}{2}\frac{u^2}{2} - u\right]_1^x$$

$$= (\tfrac{1}{2}x^2 - x)\log x - \tfrac{1}{4}x^2 + x + \tfrac{1}{4} - 1$$

$$= \tfrac{1}{4}\{2x(x-2)\log x - (x^2 - 4x + 3)\};$$

$\therefore\ x^2 - 4x + 3 - 2x(x-2)\log x < 0.$

5. If $y = \frac{1}{x}$, $\int_0^{\frac{1}{t}} \frac{1}{\sqrt{(1+x^4)}}\,dx = \int_\infty^t \frac{y^2}{\sqrt{(y^4+1)}}\cdot\frac{-dy}{y^2} = \int_t^\infty \frac{1}{\sqrt{(1+y^4)}}\,dy$;

$\therefore$ the sum of the integrals $= \int_0^\infty \frac{1}{\sqrt{(1+y^4)}}\,dy$, which is independent of t.

6. $\frac{a}{c} - \frac{b}{d} = \frac{ad - bc}{cd}$; but ad, bc are negative and $|bc| > |ad|$;

$\therefore\ bc < ad$; also $cd > 0$; $\therefore\ \frac{ad-bc}{cd} > 0$.

7. From eqn. (18), putting $t = \frac{1}{1-z}$, $z < \log\frac{1}{1-z} < \frac{z}{1-z}$ or $z < -\log(1-z) < \frac{z}{1-z}$; $\therefore$ since $0 < x < 1$ and $0 < y < 1$, we have $0 < -\log(1-x) < \frac{x}{1-x}$ and $-\log(1-y) > y > 0$;

$\therefore\ \frac{-\log(1-x)}{-\log(1-y)} < \frac{x}{1-x}\cdot\frac{1}{y}$; similarly, $-\log(1-x) > x > 0$

and $0 < -\log(1-y) < \frac{y}{1-y}$; $\therefore\ \frac{-\log(1-x)}{-\log(1-y)} > x\cdot\frac{1-y}{y}$.

8. $(1 + \tfrac{1}{2}s)^2 = 1 + s + \tfrac{1}{4}s^2 > 1 + s$;

$$\therefore\ \log(1+t) = \int_0^t \frac{1}{1+s}\,ds > \int_0^t \frac{1}{(1+\frac{1}{2}s)^2}\,ds$$

$$= \left[-\frac{2}{1+\frac{1}{2}s}\right]_0^t = -\frac{4}{2+t} + 2.$$

9. Put $t = x - 1$ in No. 8.

10. $\left(1 - \frac{1}{s}\right)^2 > 0$, for $s \neq 1$; $\therefore\ \tfrac{1}{2}\left(1 + \frac{1}{s^2}\right) > \frac{1}{s}$; $\therefore$ for $s > 1$,

$$\log x = \int_1^x \frac{1}{s}\,ds < \int_1^x \tfrac{1}{2}\left(1 + \frac{1}{s^2}\right)ds = \left[\tfrac{1}{2}\left(s - \frac{1}{s}\right)\right]_1^x = \tfrac{1}{2}\left(x - \frac{1}{x}\right).$$

11. Put $x = t + 1$ in No. 10.

12. Write $\sqrt{x}$ for x in Nos. 9, 10, and note that $\log\sqrt{x} = \tfrac{1}{2}\log x$.

13. Writing $\frac{2+x}{2-x}$ for x in No. 9, we have $\log\frac{2+x}{2-x} > \frac{2\left(\frac{2+x}{2-x} - 1\right)}{\frac{2+x}{2-x} + 1}$,

provided $\frac{2+x}{2-x} > 1$, i.e. $\frac{2+x}{2-x} - 1 > 0$, i.e. $\frac{2x}{2-x} > 0$,

i.e. $0 < x < 2$. Hence $\log\frac{2+x}{2-x} > x$, i.e. $\frac{2+x}{2-x} > e^x$.

14. In eqn. (18), write $\sqrt[n]{t}$ for t; then $\log(t^{\frac{1}{n}}) < \sqrt[n]{t} - 1$;

$$\therefore\ \frac{1}{n}\log t < \sqrt[n]{t} - 1.$$

15. $\frac{d}{dt}\left(\frac{\log t}{t-1}\right) = \frac{(t-1)\cdot\frac{1}{t} - \log t}{(t-1)^2} = \frac{1}{(t-1)^2}\left(1 - \frac{1}{t} - \log t\right) < 0$, for $t \neq 1$, by eqn. (18); $\therefore\ \frac{\log t}{t-1}$ steadily decreases as t increases from 1 upwards.

In Fig. 31, p. 52, if OC $= 1$, ON $= t$, area ACNP $= \log t$ and CN $= t - 1$; $\therefore\ \frac{\log t}{t-1} =$ average height of hyperbola above the asymptote Ox, and this steadily decreases as t increases.

16. If $t = 1 + \frac{1}{x}$, when t decreases steadily down towards 1, x increases steadily through positive values; also

$$\frac{\log t}{t-1} = x\log\left(1 + \frac{1}{x}\right) = \log\left(1 + \frac{1}{x}\right)^x.$$

17. By eqn. (18), $\log t < t - 1,\ t \neq 1$; put $t = 1 + \dfrac{1}{x}$; then

$$\log\left(1+\frac{1}{x}\right) < \frac{1}{x}, \text{ where } x > 0;$$

$$\therefore\ \log\left\{\left(1+\frac{1}{x}\right)^x\right\} = x\log\left(1+\frac{1}{x}\right) < 1.$$

18. By eqn. (18), putting $t = 1 + x$, $\dfrac{x}{1+x} < \log(1+x) < x = \dfrac{x}{1+0}$ for $x > -1$; $\therefore \log(1+x)$ lies between the extreme values of $\dfrac{x}{1+\theta x}$ which correspond to $\theta = 1$ and $\theta = 0$.

19. (i) As in No. 18, $\dfrac{x}{1+x} < \log(1+x) < x$;

$$\therefore\ \frac{1}{1+x} < \frac{1}{x}\log(1+x) < 1, \text{ for } x > 0;$$

$$\therefore\ \frac{1}{x}\log(1+x) = \log\{(1+x)^{\frac{1}{x}}\} \to 1 \text{ when } x \to 0,\ x > 0;$$

$$\therefore\ (1+x)^{\frac{1}{x}} \to e;$$

(ii) Write $\dfrac{1}{x}$ for x in (i).

20. As in No. 18, $\dfrac{x}{1+x} < \log(1+x) < x$;

$$\therefore\ \int_0^1 \frac{x}{(1+x)(1+x^2)}dx < \int_0^1 \frac{\log(1+x)}{1+x^2}dx < \int_0^1 \frac{x}{1+x^2}dx$$

$$= \left[\tfrac{1}{2}\log(1+x^2)\right]_0^1 = \tfrac{1}{2}\log 2; \text{ also } \frac{x}{(1+x)(1+x^2)}$$

$$= \tfrac{1}{2}\left\{\frac{1}{1+x^2} + \frac{x}{1+x^2} - \frac{1}{1+x}\right\}; \quad \therefore\ \int_0^1 \frac{x}{(1+x)(1+x^2)}dx$$

$$= \left[\tfrac{1}{2}\{\tan^{-1}x + \tfrac{1}{2}\log(1+x^2) - \log(1+x)\}\right]_0^1$$

$$= \tfrac{1}{2}\{\tan^{-1}1 + \tfrac{1}{2}\log 2 - \log 2\} = \tfrac{1}{2}\left(\frac{\pi}{4} - \tfrac{1}{2}\log 2\right).$$

21. (i) $\displaystyle\int_1^\infty \frac{\log x}{1+x^2}dx = \int_1^0 \frac{-\log y}{1+\frac{1}{y^2}}\cdot\left(-\frac{1}{y^2}\right)dy$, putting $x = \dfrac{1}{y}$,

$$= -\int_0^1 \frac{\log y}{1+y^2}dy = -\int_0^1 \frac{\log x}{1+x^2}dx;$$

(ii) $\displaystyle\int_0^\infty \frac{\log x}{1+x^2}dx = 0 = \int_0^\infty \frac{\log(cy)}{1+c^2y^2}\cdot c\,dy = \int_0^\infty \frac{\log c + \log y}{1+c^2y^2}\cdot c\,dy$;

$$\therefore\ \int_0^\infty \frac{\log y}{1+c^2y^2}dy = -\int_0^\infty \frac{\log c}{1+c^2y^2}dy = -\log c\,.\left[\frac{1}{c}\tan^{-1}cy\right]_0^\infty$$

$$= -\log c\,.\frac{1}{c}\cdot\frac{\pi}{2} = -\frac{\pi\log c}{2c},$$

22. $\displaystyle\Gamma(n) \equiv \int_0^\infty x^{n-1}e^{-x}dx = \left[x^{n-1}.(-e^{-x})\right]_0^\infty + \int_0^\infty e^{-x}(n-1)x^{n-2}dx$

$$= (n-1)\int_0^\infty x^{n-2}e^{-x}dx,$$

since $x^{n-1}e^{-x} \to 0$ when $x \to \infty$ by IV. f, No. 25, and $x^{n-1}e^{-x} \to 0$ when $x \to 0$ for $n > 1$; $\therefore \Gamma(n) = (n-1)\Gamma(n-1)$ for $n > 1$.

Hence writing $n+1$ for n, $\Gamma(n+1) = n\Gamma(n)$ for $n > 0$.

But $\Gamma(1) = \displaystyle\int_0^\infty e^{-x}dx = \left[-e^{-x}\right]_0^\infty = 1$; $\therefore\ \Gamma(2) = 1$;

$\therefore\ \Gamma(3) = 2\Gamma(2) = 2\,!$; $\therefore\ \Gamma(4) = 3\Gamma(3) = 3\,!$ etc.;

thus if $m-1$ is a positive integer $\Gamma(m) = (m-1)!$

23. $\displaystyle f(n+2) \equiv \int_0^\infty \tfrac{1}{2}x^{n+1}\,.\,2xe^{-x^2}dx = \left[\tfrac{1}{2}x^{n+1}(-e^{-x^2})\right]_0^\infty$

$$+ \int_0^\infty e^{-x^2}\tfrac{1}{2}(n+1)x^n\,dx = \tfrac{1}{2}(n+1)\int_0^\infty e^{-x^2}\,.\,x^n\,dx = \tfrac{1}{2}(n+1)f(n)$$

since $x^{n+1}e^{-x^2} \to 0$, when $x \to \infty$ (see IV, f, No. 25), and $x^{n+1}e^{-x^2} \to 0$ when $x \to 0$ for $n > -1$.

24. (i) Put $x = 1 - y$, $\mathrm{B}(m, n) = \displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}dx$

$$= \int_1^0 (1-y)^{m-1}\,.\,y^{n-1}\,.\,(-1)\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}dy = \mathrm{B}(n, m);$$

(ii) $\mathrm{B}(m+1, n) = \displaystyle\int_0^1 x^m(1-x)^{n-1}dx$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\{1-(1-x)\}\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx - \int_0^1 x^{m-1}(1-x)^n\,dx$$

$$= \mathrm{B}(m, n) - \left[(1-x)^n\cdot\frac{x^m}{m}\right]_0^1 + \int_0^1 \frac{x^m}{m}d(1-x)^n$$

$$=\mathrm{B}(m, n)+\int_0^1 \frac{1}{m}\cdot x^m(-n)(1-x)^{n-1}\,dx$$

$$=\mathrm{B}(m, n)-\frac{n}{m}\mathrm{B}(m+1, n);$$

$$\text{hence } \mathrm{B}(m+1, n)\,.\left\{1+\frac{n}{m}\right\}=\mathrm{B}(m, n);$$

(iii) Put $x=\sin^2\theta$, then $dx=2\sin\theta\cos\theta\,.\,d\theta$ and $(1-x)^{n-1}$ $=(\cos^2\theta)^{n-1}=\cos^{2n-2}\theta$; and $\mathrm{B}(m, n)$

$$=\int_0^{\frac{\pi}{2}}\sin^{2m-2}\theta\,.\,\cos^{2n-2}\theta\,.\,2\sin\theta\cos\theta\,.\,d\theta;$$

(iv) By (ii), $\mathrm{B}(m+1, n+1)=\dfrac{m}{m+n+1}\mathrm{B}(m, n+1)$

$$=\frac{m}{m+n+1}\mathrm{B}(n+1, m), \text{ by (i)},$$

$$=\frac{m}{m+n+1}\cdot\frac{n}{m+n}\mathrm{B}(m, n), \text{ using (ii)}.$$

25. $\tan^{-1}x-\dfrac{\pi}{4}=\displaystyle\int_1^x\frac{1}{1+y^2}\,dy<\int_1^x\frac{1}{2y}\,dy$, for $x>1$,
since $1+y^2-2y=(1-y)^2>0$, i.e. $1+y^2>2y$,
i.e. $\dfrac{1}{2y}>\dfrac{1}{1+y^2}$; and $\displaystyle\int_1^x\frac{1}{2y}\,dy=\tfrac{1}{2}\log x$.

26. $\dfrac{d}{d\theta}\dfrac{\sin\theta}{\theta}=\dfrac{\theta\cos\theta-\sin\theta}{\theta^2}=\dfrac{\cos\theta\,(\theta-\tan\theta)}{\theta^2}<0$ for $0<\theta<\dfrac{\pi}{2}$;

$\therefore$ $\dfrac{\sin\theta}{\theta}$ steadily decreases as θ increases from 0 to $\dfrac{\pi}{2}$;

$\therefore$ $1>\dfrac{\sin\theta}{\theta}>\dfrac{1}{\frac{1}{2}\pi}$ for $0<\theta<\dfrac{\pi}{2}$; $\therefore$ $\operatorname{cosec}\theta<\dfrac{\pi}{2\theta}<\dfrac{2}{\theta}$. Also $\displaystyle\int_\epsilon^1\log\frac{1}{\theta}\,d\theta=-\int_\epsilon^1\log\theta\,d\theta=-\Big[\theta\log\theta-\theta\Big]_\epsilon^1=\epsilon\log\epsilon-\epsilon+1$; but by Ex. 4 (ii), p. 68, $\epsilon\log\epsilon\to0$ when $\epsilon\to0$, $\epsilon>0$; $\therefore$ the integral $\to1$.

Also $\displaystyle\int_\epsilon^{\frac{\pi}{2}}\log\operatorname{cosec}\theta\,d\theta<\int_\epsilon^{\frac{\pi}{2}}\log\left(\frac{2}{\theta}\right)d\theta$

$$=\int_\epsilon^{\frac{\pi}{2}}(\log 2-\log\theta)\,d\theta=\Big[\theta\log 2-\theta\log\theta+\theta\Big]_\epsilon^{\frac{\pi}{2}}$$

$$=\Big[\theta\log\frac{2e}{\theta}\Big]_\epsilon^{\frac{\pi}{2}}=\frac{\pi}{2}\log\frac{4e}{\pi}-\epsilon\log\frac{2e}{\epsilon};$$

but, as before, $\epsilon\log\dfrac{2e}{\epsilon}=\epsilon(\log 2e-\log\epsilon)\to0$ when $\epsilon\to0$;

$\therefore$ $\displaystyle\int_\epsilon^{\frac{\pi}{2}}\log\operatorname{cosec}\theta\,d\theta$, being positive and steadily increasing as $\epsilon\to0$, remains less than a fixed number, $\dfrac{\pi}{2}\log\dfrac{4e}{\pi}$, and so tends to a limit.

27. The existence of the integral has been established in No. 26.

Put $\theta=\dfrac{\pi}{2}-\phi$,

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\log(\sin\theta)\,d\theta=-\int_{\frac{\pi}{2}}^0\log(\cos\phi)\,d\phi=\int_0^{\frac{\pi}{2}}\log(\cos\phi)\,d\phi;$$

$$\therefore\ 2\mathrm{I}=\int_0^{\frac{\pi}{2}}\{\log(\cos\theta)+\log(\sin\theta)\}\,d\theta=\int_0^{\frac{\pi}{2}}\log\left(\frac{\sin 2\theta}{2}\right)d\theta$$

$$=\int_0^{\frac{\pi}{2}}\{\log(\sin 2\theta)-\log 2\}\,d\theta.$$

Put $2\theta=\psi$, then

$$\int_0^{\frac{\pi}{2}}\log(\sin 2\theta)\,d\theta=\int_0^{\pi}\log(\sin\psi)\,.\,\tfrac{1}{2}d\psi$$

$$=2\int_0^{\frac{\pi}{2}}\log(\sin\psi)\,.\,\tfrac{1}{2}d\psi,$$

since $\log(\sin\psi)$ is symmetrical about $\psi=\dfrac{\pi}{2}$,

$$=\int_0^{\frac{\pi}{2}}\log(\sin\psi)\,d\psi=\mathrm{I};\quad\therefore\ 2\mathrm{I}=\mathrm{I}-\int_0^{\frac{\pi}{2}}\log 2\,d\theta$$

$$=\mathrm{I}-\frac{\pi}{2}\log 2;\quad\therefore\ \mathrm{I}=-\frac{\pi}{2}\log 2.$$

28. $\mathrm{I}=\displaystyle\int_0^1\frac{\log(1+x)}{1+x^2}\,dx=$, putting $x=\tan\theta$,

$$\int_0^{\frac{\pi}{4}}\frac{\log(1+\tan\theta)}{\sec^2\theta}\,.\,\sec^2\theta\,d\theta=\int_0^{\frac{\pi}{4}}\log(1+\tan\theta)\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\log\left\{1+\tan\left(\frac{\pi}{4}-\phi\right)\right\}d\phi, \text{ putting } \theta=\frac{\pi}{4}-\phi,$$

$$=\int_0^{\frac{\pi}{4}}\log\left\{1+\frac{1-\tan\phi}{1+\tan\phi}\right\}d\phi=\int_0^{\frac{\pi}{4}}\log\frac{2}{1+\tan\phi}\,d\phi$$

$$=\int_0^{\frac{\pi}{4}}\log 2\,.\,d\phi-\int_0^{\frac{\pi}{4}}\log(1+\tan\phi)\,d\phi=\frac{\pi}{4}\log 2-\mathrm{I};$$

$$\therefore\ 2\mathrm{I}=\frac{\pi}{4}\log 2.$$

29. By (18), Ch. IV, if $a_r \neq \mathrm{A}$, $\log(a_r/\mathrm{A}) < a_r/\mathrm{A} - 1$;

$\therefore$ if $a_1, a_2, \ldots, a_n$ are not all equal, $\Sigma\log(a_r/\mathrm{A}) < n - n, = 0$;

$\therefore \log(a_1 a_2 \ldots a_n/\mathrm{A}^n) < 0$; $\therefore a_1 a_2 \ldots a_n < \mathrm{A}^n$.

CHAPTER V

EXERCISE V. a. (p. 82.)

1. In eqn. (5), put $x=1$. **2.** In eqn. (6), put $x=1$.

3. In eqn. (5), put $x=2$.

4. $\dfrac{2n}{(2n+1)!}=\dfrac{1}{(2n)!}-\dfrac{1}{(2n+1)!}$;

$$\text{series}=\left(\frac{1}{2!}-\frac{1}{4!}+\frac{1}{6!}-\ldots\right)-\left(\frac{1}{3!}-\frac{1}{5!}+\ldots\right)$$

$$=(1-\cos 1)-(1-\sin 1).$$

5. In eqn. (6), put $x=\dfrac{\pi}{2}$, series $=1-\cos\dfrac{\pi}{2}$.

6. In eqn. (5), put $x=2$, 1st series $=1+\sin 2=(\sin 1+\cos 1)^2$.

2nd series $=(1+1)-\left(\dfrac{1}{2!}+\dfrac{1}{3!}\right)+\left(\dfrac{1}{4!}+\dfrac{1}{5!}\right)-\ldots=\sin 1+\cos 1.$

7. $\cos 1 \simeq 1-\dfrac{1}{2!}+\dfrac{1}{4!}-\dfrac{1}{6!}+\dfrac{1}{8!}$. **8.** $\sin 3^\circ=\sin\dfrac{\pi}{60}\simeq\dfrac{\pi}{60}-\dfrac{\pi^3}{60^3\,.\,3!}$.

9. $\tan x-\sin x=\tan x(1-\cos x)\simeq x\cdot\dfrac{x^2}{2}=\tfrac{1}{2}x^3$.

10. $\sin^2\theta=\tfrac{1}{2}(1-\cos 2\theta)$; put $x=2\theta$ in eqn. (6).

11. $\dfrac{x}{\sin x}\simeq\dfrac{1}{1-\dfrac{x^2}{6}+\dfrac{x^4}{120}}=(1-y)^{-1}\simeq 1+y+y^2$ where $y=\dfrac{x^2}{6}-\dfrac{x^4}{120}$.

12. (i) $\sin\left(\dfrac{\pi}{4}+x\right)\cos x=\dfrac{1}{\sqrt{2}}(\sin x+\cos x)\cos x$

$$=\tfrac{1}{4}\sqrt{2}(\sin 2x+1+\cos 2x);$$

use eqns. (5), (6);

(ii) Put $\dfrac{\pi}{4}+x=y$,

$$\text{expression}=\sin y\cos\left(y-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\sin y(\cos y+\sin y)$$

$$=\tfrac{1}{4}\sqrt{2}(\sin 2y+1-\cos 2y),\ \text{use eqns. (5), (6).}$$

13. $\cos^3 x=\tfrac{1}{4}(3\cos x+\cos 3x)$; use eqn. (6).

14. $\dfrac{3\sin\theta}{2+\cos\theta}\simeq 3\sin\theta\div\left(3-\dfrac{\theta^2}{2}+\dfrac{\theta^4}{24}\right)$

$$\simeq\left(\theta-\frac{\theta^3}{6}+\frac{\theta^5}{120}\right)\cdot\left(1-\frac{\theta^2}{6}+\frac{\theta^4}{72}\right)^{-1}$$

$$\simeq\left(\theta-\frac{\theta^3}{6}+\frac{\theta^5}{120}\right)\cdot\left[1+\left(\frac{\theta^2}{6}-\frac{\theta^4}{72}\right)+\left(\frac{\theta^2}{6}-\frac{\theta^4}{72}\right)^2\right]$$

$$\simeq\left(\theta-\frac{\theta^3}{6}+\frac{\theta^5}{120}\right)\left(1+\frac{\theta^2}{6}+\frac{\theta^4}{72}\right).$$

15. By method of p. 82, $\tan x\simeq x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\dfrac{17x^7}{315}$;

hence $\tan x-24\tan\dfrac{x}{2}\simeq-11x-\dfrac{2x^3}{3}+\dfrac{x^5}{30}+kx^7$ where $k>0$;

by eqn. (5), $4\sin x-15x\simeq-11x-\dfrac{2x^3}{3}+\dfrac{x^5}{30}-\dfrac{4x^7}{7!}$.

16. $\operatorname{cosec}^2 x-\dfrac{1}{x^2}=\left(\dfrac{x}{\sin x}\right)^2\dfrac{1}{x^2}\left(1-\dfrac{\sin x}{x}\right)\left(1+\dfrac{\sin x}{x}\right)$.

From p. 80, $\dfrac{1}{3!}-\dfrac{x^2}{5!}<\dfrac{1}{x^2}\left(1-\dfrac{\sin x}{x}\right)<\dfrac{1}{3!}$;

$\therefore \dfrac{1}{x^2}\left(1-\dfrac{\sin x}{x}\right)\to\dfrac{1}{6}$. Also $\dfrac{\sin x}{x}\to 1$. Use $\operatorname{cosec}^2 x=1+\cot^2 x$.

17. $\sin(\theta+\alpha)=\sin\theta\cos\alpha+\cos\theta\sin\alpha\simeq\sin\theta+\alpha\cos\theta$;

$$\text{expression}\simeq\frac{\sin\theta+\alpha}{\sin\theta+\alpha\cos\theta}=1+\frac{\alpha(1-\cos\theta)}{\sin\theta+\alpha\cos\theta}$$

$$\simeq 1+\alpha\cdot\frac{1-\cos\theta}{\sin\theta}=1+\alpha\tan\frac{\theta}{2}.$$

18. First approximation is $1-\tfrac{1}{2}\theta^2=2\theta$; $\therefore\ \theta=-2+\sqrt{6}\simeq 0{\cdot}45$; put $\theta=0{\cdot}45+\alpha$; $\cos(\cdot 45)\cos\alpha-\sin(\cdot 45)\sin\alpha=0{\cdot}9+2\alpha$; neglecting α^2,

$$\therefore\ \alpha[2+\sin(\cdot 45)]=\cos(\cdot 45)-0{\cdot}9;\ \ \cdot 45^c=25^\circ\,47';$$

$$\therefore\ \alpha=\frac{\cdot 9004-0{\cdot}9}{2+\cdot 4349}=\cdot 0002;\ \ \therefore\ \theta=\cdot 4502.$$

19. Put $\theta=\frac{3\pi}{2}-\alpha=4\cdot7-\beta$; $\sin(4\cdot7-\beta)=(4\cdot7-\beta)\,.\cos(4\cdot7-\beta)$;

$\therefore\ \sin(4\cdot7)-\beta\cos(4\cdot7)=(4\cdot7-\beta)\,.\,[\cos(4\cdot7)+\beta\sin(4\cdot7)]$,

neglecting β^2, $=4\cdot7\cos(4\cdot7)+4\cdot7\beta\sin(4\cdot7)-\beta\cos(4\cdot7)$;

$\therefore\ \beta=\frac{1}{4\cdot7}-\cot(4\cdot7)=0\cdot213-\cot 269^\circ\,17'\simeq0\cdot2$;

$$\therefore\ \theta\simeq4\cdot7-0\cdot2.$$

If we put $\theta=4\cdot5-\gamma$, we find in same way $\gamma\simeq0\cdot007$.

20. $\frac{\tan(\theta-\phi)}{\tan\phi}=1+\lambda$; $\therefore\ \frac{\tan(\theta-\phi)-\tan\phi}{\tan(\theta-\phi)+\tan\phi}=\frac{\lambda}{2+\lambda}$;

$\therefore\ \frac{\sin(\theta-2\phi)}{\sin\theta}=\frac{\lambda}{2+\lambda}\simeq\frac{1}{2}\lambda$, neglecting λ^2;

$\therefore\ \theta-2\phi\simeq\frac{1}{2}\lambda\sin\theta$;

$\therefore\ \tan\phi\simeq\tan(\frac{1}{2}\theta-\frac{1}{4}\lambda\sin\theta)\simeq\frac{\tan\frac{1}{2}\theta-\frac{1}{4}\lambda\sin\theta}{1+\tan\frac{1}{2}\theta\,.\,\frac{1}{4}\lambda\sin\theta}$

$\simeq(\tan\frac{1}{2}\theta-\frac{1}{4}\lambda\sin\theta)(1-\tan\frac{1}{2}\theta\,.\,\frac{1}{4}\lambda\sin\theta)$

$\simeq\tan\frac{1}{2}\theta-\frac{1}{4}\lambda\sin\theta\,(1+\tan^2\frac{1}{2}\theta)$

$=\tan\frac{1}{2}\theta-\frac{1}{2}\lambda\sin\frac{1}{2}\theta\,.\,\cos\frac{1}{2}\theta\,.\,\sec^2\frac{1}{2}\theta$

$=\tan\frac{1}{2}\theta-\frac{1}{2}\lambda\tan\frac{1}{2}\theta.$

21. (i) $2(1-\cos x)-x\sin x=2\sin x\left(\frac{1-\cos x}{\sin x}-\frac{x}{2}\right)$

$$=2\sin x\left(\tan\frac{x}{2}-\frac{x}{2}\right);$$

but for $0<\frac{x}{2}<\frac{\pi}{2}$, $\tan\frac{x}{2}>\frac{x}{2}$ and $\sin x>0$.

(ii) If $y=x(2+\cos x)-3\sin x$,

$\frac{dy}{dx}=2-2\cos x-x\sin x>0$ by (i) for $0<x<\pi$;

$\therefore$ y increases as x increases; but $y=0$ when $x=0$;

$\therefore\ y>0$ for $0<x<\pi$.

Or By (i) $\int_0^x\{2(1-\cos x)-x\sin x\}dx>0$; evaluate the integral.

22. If $t=\tan\frac{x}{2}$, $\tan x\,.\,\sin x=\frac{2t}{1-t^2}\cdot\frac{2t}{1+t^2}$

$=\frac{4t^2}{1-t^4}>4t^2=4\tan^2\frac{x}{2}>4\left(\frac{x}{2}\right)^2$;

$\therefore\ \tan x\,.\,\sin x>x^2$; but $\sin x>0$ and $x>0$

$\therefore\ \frac{\tan x}{x}>\frac{x}{\sin x}.$

23. $\theta=\sin\theta+\frac{\theta^3}{3!}-\frac{\theta^5}{5!}+\dots$ Substitute successive approximations in turn in this expression.

1st approx., $\theta=\sin\theta$;

2nd approx., $\theta=\sin\theta+\frac{\sin^3\theta}{3!}$;

3rd approx., $\theta=\sin\theta+\frac{1}{3!}(\sin\theta+\frac{1}{6}\sin^3\theta)^3-\frac{\sin^5\theta}{5!}$

$=\sin\theta+\frac{1}{6}(\sin^3\theta+\frac{1}{2}\sin^5\theta)-\frac{\sin^5\theta}{120}.$

24. Same method as No. 23. $\phi=nt+e\sin\phi$.

1st approx., $\phi=nt$;

2nd approx., $\phi=nt+e\sin nt$;

3rd approx., $\phi=nt+e\sin(nt+e\sin nt)$

$=nt+e\sin nt\cos(e\sin nt)+e\cos nt\sin(e\sin nt)$

$\simeq nt+e\sin nt+e\cos nt\,.\,e\sin nt.$

25. $e^x-1>0$ if $x>0$; $\therefore\ \int_0^x(e^t-1)dt>0$ if $x>0$; $\therefore\ e^x-x-1>0$;

$\therefore\ \int_0^x(e^t-t-1)dt>0$ if $x>0$; $\therefore\ e^x-\frac{x^2}{2}-x-1>0$; etc.

26. $f_1(x)=1-e^{-x}>0$ if $x>0$; $f_2(x)=\int_0^x(1-e^{-t})\,dt=x-1+e^{-x}>0$;

$f_3(x)=\int_0^x(t-1+e^{-t})dt=\frac{x^2}{2!}-x+1-e^{-x}>0$; etc.; $\therefore$ as on p. 80,

$$s_{2r}\equiv1-\frac{x}{1!}+\frac{x^2}{2!}-\dots-\frac{x^{2r-1}}{(2r-1)!}<e^{-x}$$

$$<1-\frac{x}{1!}+\frac{x^2}{2!}-\dots+\frac{x^{2r}}{(2r)!}\equiv s_{2r+1};$$

$\therefore\ 0<(s_{2r+1}-e^{-x})<(s_{2r+1}-s_{2r})\to0$ when $r\to\infty$; $\therefore$ by the argument on p. 80, s_{2r} and s_{2r+1} each tend to e^{-x} when $r\to\infty$.

EXERCISE V. b. (p. 86.)

1. (i) In eqn. (11), put $x=1$;

(ii) In eqn. (11), put $x=\frac{1}{2}$;

(iii) In eqn. (12), put $x=\frac{1}{3}$, sum $=\frac{1}{2}\log 2$.

2. (i) $\log(x+a)-\log a=\log\frac{x+a}{a}=\log\left(1+\frac{x}{a}\right)$; use eqn. (11);

(ii) $\log x-\log y=-\log\frac{y}{x}=-\log\left(1-\frac{x-y}{x}\right)$; in eqn. (11), write $-\frac{x-y}{x}$ for x; true if $-1\leqslant\frac{x-y}{x}<1$, i.e. if $-2\leqslant\frac{x-y}{x}-1<0$;

(iii) Left side $=-\log\left(1-\frac{1}{x^2}\right)=-\log\frac{x^2-1}{x^2}=\log\frac{x^2}{x^2-1}$ if $x^2>1$;

right side $=\log\frac{1+y}{1-y}$, where $y=\frac{1}{2x^2-1}$, $=\log\frac{2x^2}{2x^2-2}$,

if $-1<\frac{1}{2x^2-1}<1$, which is true if $x^2>1$.

3. (ii) $\log(1-x)+\log(1+3x)$; expand each separately;

(iii) $\log(1+5x+6x^2)=\log[(1+2x)(1+3x)]$
$=\log(1+2x)+\log(1+3x)$;

(iv) $\log(1+x)^2=2\log(1+x)$;

(v) $\log(x+2)=\log\frac{x+2}{2}+\log 2=\log 2+\log\left(1+\frac{x}{2}\right)$;

(vi) $\log[(x+1)(x+2)]=\log 2+\log(1+x)+\log\left(1+\frac{x}{2}\right)$;

(vii) $\log(1-2x^2)-\log(1-x)$; the first expansion is in powers of x^2 and $\therefore$ contains a term x^n only if n is even;

(viii) $\log(1+x+x^2)=\log\frac{1-x^3}{1-x}=\log(1-x^3)-\log(1-x)$.

4. (iv) If $\log_{10}N=x$, then $N=10^x$; $\therefore$ $\log_e N=\log_e 10^x=x\log_e 10$; $\therefore$ $x=\log_e N\div\log_e 10$.

5. (i) Expression $=(1-2x)\left[-\Sigma\frac{1}{n}2^nx^n\right]$; $\therefore$ the term in x^n is $-\frac{1}{n}\cdot 2^nx^n+2x\cdot\frac{1}{n-1}\cdot 2^{n-1}x^{n-1}$;

(ii) $(1+6x+9x^2)\,.\,\Sigma[(-1)^{n-1}\cdot\frac{1}{n}\cdot 3^nx^n]$; $\therefore$ term in x^n is

$$(-1)^{n-1}\cdot\frac{1}{n}\cdot 3^nx^n+6x\,.\,(-1)^{n-2}\cdot\frac{1}{n-1}\cdot 3^{n-1}x^{n-1}$$
$$+9x^2\,.\,(-1)^{n-3}\cdot\frac{1}{n-2}\cdot 3^{n-2}x^{n-2}$$
$$=(-1)^{n-1}\,.\,3^n\left\{\frac{1}{n}-\frac{2}{n-1}+\frac{1}{n-2}\right\}x^n;$$

(iii) $(1-x)\Sigma\left[(-1)^{r-1}\cdot\frac{1}{r}\left(\frac{x}{2}\right)^{2r}\right]$
$$=\Sigma\left[(-1)^{r-1}\cdot\frac{1}{r}\left(\frac{x}{2}\right)^{2r}+(-1)^r\cdot\frac{2}{r}\left(\frac{x}{2}\right)^{2r+1}\right];$$

if n is even, put $2r=n$ or $r=\frac{n}{2}$; if n is odd, put $2r+1=n$ or $r=\frac{n-1}{2}$.

6. (i) $\frac{nx^{n+1}}{n+1}=x^{n+1}-\frac{x^{n+1}}{n+1}$. If $|x|<1$, $x^2+x^3+x^4+\ldots=\frac{x^2}{1-x}$ and

$-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots=\log(1-x)+x$;

$$\text{sum}=\log(1-x)+x+\frac{x^2}{1-x};$$

(ii) $\frac{x^{2n}}{2n(2n-1)}=\frac{x^{2n}}{2n-1}-\frac{x^{2n}}{2n}$.

If $|x|<1$,

$\frac{x^2}{1}+\frac{x^4}{3}+\frac{x^6}{5}+\ldots=x\times\frac{1}{2}\log\frac{1+x}{1-x}$ by eqn. (12) and

$-\frac{x^2}{2}-\frac{x^4}{4}-\frac{x^6}{6}-\ldots=\frac{1}{2}\{\log(1+x)+\log(1-x)\}$;

(iii) $\frac{(-x)^n}{n(n+1)}=\frac{(-x)^n}{n}-\frac{(-x)^n}{n+1}$.

If $|x|<1$,

$-x+\frac{x^2}{2}-\frac{x^3}{3}+\ldots=-\log(1+x)$

and $\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}-\ldots=\frac{1}{x}\{x-\log(1+x)\}$ for $x\neq 0$.

If $x=0$, each term $=0$.

7. $\log\frac{x+1}{x-1}=\log\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$; write $\frac{1}{x}$ for x in eqn. (12).

8. Expression $=\log\left[\frac{n^2}{(n+1)(n-1)}\right]=\log\frac{n^2}{n^2-1}$
$$=-\log\frac{n^2-1}{n^2}=-\log\left(1-\frac{1}{n^2}\right).$$

9. Expression $= \log\left[\frac{(x+2)(x-1)^2}{(x-2)(x+1)^2}\right] = \log\frac{x^3-3x+2}{x^3-3x-2} = \log\frac{1+y}{1-y}$,
where $y = \frac{2}{x^3-3x}$; use eqn. (12). Valid if $\left(\frac{2}{x^3-3x}\right)^2 < 1$,
if $(x^3-3x)^2>4$, if $x^6-6x^4+9x^2-4>0$, if
$(x^2-4)(x^2-1)^2>0$, if $x^2>4$.

10. $s_n = (1-\frac12)+(\frac12-\frac13)+\dots+\left(\frac1n-\frac1{n+1}\right)=1-\frac1{n+1}$; when $n\to\infty$,
$s_n\to1$.

11. $s_n=(1-\frac12)-(\frac12-\frac13)+(\frac13-\frac14)-\dots+(-1)^{n+1}\left(\frac1n-\frac1{n+1}\right)$

$$= -1+2\left[1-\tfrac12+\tfrac13-\dots+(-1)^{n+1}\cdot\frac1n\right]-(-1)^{n+1}\cdot\frac1{n+1};$$

when $n\to\infty$, the expression in the bracket $\to\log 2$.

12. $\frac{1}{n(n+1)(n+2)} = \frac{\frac12}{n}-\frac{1}{n+1}+\frac{\frac12}{n+2}$;

$$2s_n = \Sigma(-1)^{n-1}\left[\frac1n-\frac{2}{n+1}+\frac1{n+2}\right]$$
$$=(\tfrac11-\tfrac22+\tfrac13)-(\tfrac12-\tfrac23+\tfrac14)+(\tfrac13-\tfrac24+\tfrac15)-\dots$$
$$=1-\tfrac32+4\left[\tfrac13-\tfrac14+\tfrac15-\dots+(-1)^{n-1}\cdot\frac1n\right]$$
$$+(-1)^n\cdot\left(\frac3{n+1}-\frac1{n+2}\right);$$

when $n\to\infty$, the expression in the bracket $\to\log2-(1-\frac12)$;
$\therefore\ 2s_n\to1-\frac32+4[\log2-\frac12]$.

13. $\frac{1}{(2n-1)2n(2n+1)} = \frac{\frac12}{2n-1}-\frac1{2n}+\frac{\frac12}{2n+1}$; ∴ as in No. 12,

$$s_n = -\tfrac12+\left[1-\tfrac12+\tfrac13-\dots+\frac1{2n-1}-\frac1{2n}\right]+\frac{\frac12}{2n+1}.$$

14. $\frac1{n(2n+1)}=\frac1n-\frac2{2n+1}=\frac2{2n}-\frac2{2n+1}$;

$$\therefore\ \tfrac12 s_n = (\tfrac12-\tfrac13)+(\tfrac14-\tfrac15)+\dots+\left(\frac1{2n}-\frac1{2n+1}\right)$$
$$=1-\left\{1-\tfrac12+\tfrac13-\tfrac14+\dots-\frac1{2n}+\frac1{2n+1}\right\}.$$

15. $\frac{1}{n(n+1)(2n+1)} = \frac1n+\frac1{n+1}-\frac4{2n+1}=2\left(\frac1{2n}-\frac2{2n+1}+\frac1{2n+2}\right)$;
∴ as in No. 12,

$$\tfrac12 s_n = -\tfrac12+2\left\{\tfrac12-\tfrac13+\tfrac14-\dots-\frac1{2n+1}\right\}+\frac1{2n+2};$$

when $n\to\infty$, the expression in the bracket $\to 1-\log 2$.

16. $\log\left|\cot\frac A2\right| = \frac12\log\cot^2\frac A2=\frac12\log\frac{1+\cos A}{1-\cos A}$; put $x=\cos A$ in eqn. (12); put $A=\frac\pi3$ and note that
$\cos A=\frac12$ and $\log\cot\frac A2=\log\sqrt3=\frac12\log3$.

17. By No. 6 (ii), $(1+x)\log(1+x)+(1-x)\log(1-x)=\Sigma\frac{x^{2n}}{n(2n-1)}$;
∴ numerator $=\log(1+x)+\log(1-x)+\Sigma\frac{x^{2n}}{n(2n-1)}$

$$=\log(1-x^2)+\Sigma\frac{x^{2n}}{n(2n-1)} \overset{*}{\simeq} \left(-x^2-\frac{x^4}2\right)+\left(x^2+\frac{x^4}6\right)=-\tfrac13x^4.$$

18. Put $1-x=y$ and suppose y small;

$$\text{expression}=\frac{y+\log(1-y)}{1-\sqrt{(1-y^2)}}=\frac{\{y+\log(1-y)\}\{1+\sqrt{(1-y^2)}\}}{1-(1-y^2)}$$
$$\simeq \frac{{}^*\left\{y+\left(-y-\frac{y^2}2\right)\right\}\cdot2}{y^2}=-1.$$

19. Neglect ϵ^3 and let $x=1+A\epsilon+B\epsilon^2$;
∴ $\log x\simeq(A\epsilon+B\epsilon^2)-\frac12A^2\epsilon^2$;
substituting in the eqn.,
$(1+A\epsilon+B\epsilon^2)(A\epsilon+B\epsilon^2-\frac12A^2\epsilon^2)+A\epsilon+B\epsilon^2=\epsilon$;
equating coefficients of ϵ, ϵ^2, $2A=1$;
$B-\frac12A^2+A^2+B=0$; ∴ $A=\frac12$, $B=-\frac1{16}$.

20. The graphs of $\log x$ and $\frac15\left(2\cdot7-\frac2x\right)\equiv0\cdot54-\frac2{5x}$ show that there are two roots, $x\simeq0\cdot2$ and $x\simeq1\cdot2$. Put $x=0\cdot2+\alpha$ and suppose α is small.
$\log(0\cdot2+\alpha)\equiv\log(0\cdot2)+\log(1+5\alpha)\simeq\log(0\cdot2)+5\alpha$;
∴ $\log(0\cdot2)+5\alpha=0\cdot54-\frac2{1+5\alpha}\simeq0\cdot54-2(1-5\alpha)$;
neglecting α^2.
But $\log(0\cdot2)=-1\cdot61$; hence $\alpha\simeq-0\cdot03$;
∴ $x\simeq0\cdot2-0\cdot03=0\cdot17$.
Next put $x=1\cdot2+\beta$ and suppose β is small; as before,
$\log(1\cdot2+\beta)\equiv\log(1\cdot2)+\log\left(1+\frac{5\beta}6\right)\simeq0\cdot182+\frac{5\beta}6$;
∴ $0\cdot182+\frac{5\beta}6=0\cdot54-\frac2{6+5\beta}\simeq0\cdot54-\frac26\left(1-\frac{5\beta}6\right)$;
hence $\beta\simeq0\cdot04$; ∴ $x\simeq1\cdot24$.
For closer approximations, put $x=0\cdot17+\gamma$ and $x=1\cdot24+\delta$.

* The error, for example, in replacing $\log(1-y)$ by $-y-\frac12y^2$ is, by eqn. (9), $\int_0^{|y|}\frac{x^2}{1-x}dx$, which, for $|y|<\frac12$, $<2\int_0^{|y|}x^2dx=\frac23(|y|)^3$.

EXERCISE V. c. (p. 89.)

1. (i) Put $y = \frac{1}{2}$ in eqn. (14);

(ii) Series $= [1 - \frac{1}{3} + \frac{1}{5} - \ldots] - [y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \ldots]$ where $y = \frac{1}{\sqrt{3}}$,

$$= \tan^{-1}(1) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{4} - \frac{\pi}{6}.$$

2. That value of $\tan^{-1}(\tan x) = x - r\pi$ which lies between $-\frac{\pi}{4}$ and $+\frac{\pi}{4}$.

4. To prove relation, see Ex. III. f, No. 29.

5. Use Ch. III. eqn. (9) and Ch. V, eqn. (14).

6. Series $= 2[(\frac{1}{3}) - \frac{1}{3}(\frac{1}{3})^3 + \frac{1}{5}(\frac{1}{3})^5 - \ldots] + [\frac{1}{7} - \frac{1}{3}(\frac{1}{7})^3 + \ldots]$

$= 2\tan^{-1}(\frac{1}{3}) + \tan^{-1}(\frac{1}{7}) = \frac{\pi}{4}$, see Ex. III. f, No. 29.

7. By eqn. (12), $\frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$, if $x^2 < 1$; by eqn. (14), $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$, if $x^2 < 1$; add and halve.

8. $\tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) = \tan^{-1}\frac{1+\tan\theta}{1-\tan\theta} = \tan^{-1}\left[\tan\left(\theta + \frac{\pi}{4}\right)\right]$

$= n\pi + \frac{\pi}{4} + \theta = n\pi + \frac{\pi}{4} + \tan^{-1}(\tan\theta)$; use eqn. (14).

9. $x = y + \frac{x^3}{3} - \frac{x^5}{5}$; 1st approx., $x = y$; 2nd approx., $x = y + \frac{y^3}{3}$;

3rd approx., $x = y + \frac{1}{3}\left(y + \frac{y^3}{3}\right)^3 - \frac{1}{5}y^5 = y + \frac{1}{3}(y^3 + y^5) - \frac{1}{5}y^5$.

If $y = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta$, $y \simeq \tan^{-1}(\tan\theta) = \theta$; then $\tan\theta = x \simeq y + \frac{1}{3}y^3 + \frac{2}{15}y^5 \simeq \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5$.

10. $x = \epsilon - \tan^{-1}x = \epsilon - \left(x - \frac{x^3}{3} + \ldots\right)$; 1st approx., $2x = \epsilon$, $x = \frac{1}{2}\epsilon$;

2nd approx., $2x = \epsilon + \frac{1}{3}x^3 = \epsilon + \frac{1}{3}(\frac{1}{2}\epsilon)^3$, $x = \frac{1}{2}\epsilon + \frac{1}{48}\epsilon^3$; 3rd approx., $2x = \epsilon + \frac{1}{3}x^3 - \frac{1}{5}x^5 = \epsilon + \frac{1}{3}(\frac{1}{2}\epsilon + \frac{1}{48}\epsilon^3)^3 - \frac{1}{5}(\frac{1}{2}\epsilon)^5$.

EXERCISE V. d. (p. 95.)

1. Put $x = -1$ in eqn. (16).

2. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$; $\frac{1}{e} = e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots$; subtract.

Also addition gives $\frac{1}{2}\left(e + \frac{1}{e}\right) = 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots$.

3. $u_n = \frac{n}{(n-1)!} = \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$;

$s_n = 1 + \left(1 + \frac{1}{1!}\right) + \left(\frac{1}{1!} + \frac{1}{2!}\right) + \ldots + u_n$

$= \left\{1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-2)!}\right\} + \left\{1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!}\right\}$;

$\therefore$ when $n \to \infty$, $s_n \to e + e$.

4. $u_n = \frac{n}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$;

$s_n = \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \ldots + u_n = 1 - \frac{1}{(n+1)!}$;

$\therefore$ when $n \to \infty$, $s_n \to 1$.

5. $u_n = \frac{n^2}{(n+1)!} = \frac{1}{(n-1)!} - \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)$;

$s_n = 1 + \frac{1}{1!} + \ldots + \frac{1}{(n-1)!} - \left(1 - \frac{1}{(n+1)!}\right)$;

$\therefore$ when $n \to \infty$, $s_n \to e - 1$.

6. $u_n = \frac{n^2}{(n-1)!} = \frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}$;

$s_n = 1 + \left(3 + \frac{1}{1!}\right) + \left(1 + \frac{3}{1!} + \frac{1}{2!}\right) + \ldots + u_n$

$= \left[1 + \frac{1}{1!} + \ldots + \frac{1}{(n-3)!}\right] + 3\left[1 + \frac{1}{1!} + \ldots + \frac{1}{(n-2)!}\right] + \left[1 + \frac{1}{1!} + \ldots + \frac{1}{(n-1)!}\right]$;

$\therefore$ when $n \to \infty$, $s_n \to e + 3e + e$.

7. $u_n = \frac{2n}{(2n+1)!} = \frac{1}{(2n)!} - \frac{1}{(2n+1)!}$;

$s_n = \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots + u_n$;

$\therefore$ by No. 1, $s_n \to \frac{1}{e}$.

8. $u_n = \frac{n}{(2n-2)!} = \frac{1}{2}\left[\frac{1}{(2n-3)!} + \frac{2}{(2n-2)!}\right]$;

$2s_n = 2 + \left(\frac{1}{1!} + \frac{2}{2!}\right) + \left(\frac{1}{3!} + \frac{2}{4!}\right) + \ldots + 2u_n$

$= \left[\frac{1}{1!} + \frac{1}{3!} + \ldots + \frac{1}{(2n-3)!}\right] + 2\left[1 + \frac{1}{2!} + \frac{1}{4!} + \ldots + \frac{1}{(2n-2)!}\right]$

$\therefore$ as in No. 2, $2s_n \to \frac{1}{2}\left(e - \frac{1}{e}\right) + \left(e + \frac{1}{e}\right) = \frac{3e}{2} + \frac{1}{2e}$.

9. $u_n = \frac{\frac{1}{2}n(n+1)}{n!} = \frac{1}{2}\left[\frac{1}{(n-2)!} + \frac{2}{(n-1)!}\right];$

$$\therefore\ 2s_n = 2 + \left(1 + \frac{2}{1!}\right) + \left(\frac{1}{1!} + \frac{2}{2!}\right) + \dots + 2u_n$$

$$= \left[1 + \frac{1}{1!} + \dots + \frac{1}{(n-2)!}\right] + 2\left[1 + \frac{1}{1!} + \dots + \frac{1}{(n-1)!}\right] \to e + 2e.$$

10. $u_n = \frac{2^n - 1}{n!} = \frac{1}{n!}\cdot 2^n - \frac{1}{n!};$

$$s_n = \left(\frac{1}{1!}\cdot 2 - \frac{1}{1!}\right) + \left(\frac{1}{2!}\cdot 2^2 - \frac{1}{2!}\right) + \dots + u_n$$

$$= \left(\frac{1}{1!}\cdot 2 + \frac{1}{2!}\cdot 2^2 + \dots + \frac{1}{n!}\cdot 2^n\right)$$

$$-\left(\frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}\right) \to (e^2 - 1) - (e - 1).$$

11. $u_n = \frac{n^3}{(n+1)!} = \frac{1}{(n-2)!} + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right);$

$$\therefore\ s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(n-2)!} + \left(1 - \frac{1}{(n+1)!}\right);$$

$\therefore$ when $n \to \infty$, $s_n \to e + 1$.

12. $u_n = \frac{n^2}{(2n+1)!} = \frac{1}{4}\left[\frac{1}{(2n-1)!} - \frac{1}{(2n)!} + \frac{1}{(2n+1)!}\right];$

$$\therefore\ 4s_n = \left[\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{(2n-1)!} - \frac{1}{(2n)!}\right]$$

$$+ \left[\frac{1}{3!} + \frac{1}{5!} + \dots + \frac{1}{(2n+1)!}\right];$$

$\therefore$ from No. 2, $4s_n \to \left(1 - \frac{1}{e}\right) + \left[\frac{1}{2}\left(e - \frac{1}{e}\right) - 1\right] = \frac{1}{2}\left(e - \frac{3}{e}\right).$

13. $u_n = \frac{5n-2}{(2n)!} = \frac{\frac{5}{2}}{(2n-1)!} - \frac{2}{(2n)!};$

$$\therefore\ s_n = \frac{5}{2}\left[\frac{1}{1!} + \frac{1}{3!} + \dots + \frac{1}{(2n-1)!}\right]$$

$$-2\left[\frac{1}{2!} + \frac{1}{4!} + \dots + \frac{1}{(2n)!}\right];$$

$\therefore$ by No. 2, $s_n \to \frac{5}{2}\left[\frac{1}{2}\left(e - \frac{1}{e}\right)\right]$

$$-2\left[\frac{1}{2}\left(e + \frac{1}{e}\right) - 1\right] = \frac{5}{4}\left(e - \frac{1}{e}\right) - \left(e + \frac{1}{e}\right) + 2.$$

14. $u_n = \frac{n(n+2)}{(n+1)!} = \frac{1}{(n-1)!} + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right);$ $\therefore$ as in No. 11,

$$s_n = \left[1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(n-1)!}\right] + \frac{1}{1!} - \frac{1}{(n+1)!} \to e + 1.$$

15. $u_n = (-1)^{n-1}\cdot\frac{n^3}{n!} = (-1)^{n-1}\left[\frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}\right];$

$$\therefore\ s_n = \left(1 - \frac{1}{1!} + \frac{1}{2!}\dots\right) - 3\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots\right)$$

$$+\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots\right) \to \frac{1}{e} - \frac{3}{e} + \frac{1}{e}.$$

16. $u_n = (-1)^{n-1}\frac{(n+1)^3}{n!}$

$$= (-1)^{n-1}\left[\frac{1}{(n-3)!} + \frac{6}{(n-2)!} + \frac{7}{(n-1)!} + \frac{1}{n!}\right];$$

$$\therefore\ s_n = \left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots\right) - 6\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots\right)$$

$$+ 7\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots\right) + \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots\right);$$

$$\therefore\ s_n \to \frac{1}{e} - 6\left(\frac{1}{e}\right) + 7\left(\frac{1}{e}\right) + \left(1 - \frac{1}{e}\right) = 1 + \frac{1}{e}.$$

17. As in No. 2,

$$1 + \frac{1}{2!} + \frac{1}{4!} + \dots = \frac{1}{2}\left(e + \frac{1}{e}\right) \text{ and } 1 + \frac{1}{3!} + \frac{1}{5!} + \dots = \frac{1}{2}\left(e - \frac{1}{e}\right);$$

$$\text{expression} = \frac{1}{4}\left(e + \frac{1}{e}\right)^2 - \frac{1}{4}\left(e - \frac{1}{e}\right)^2.$$

18. As in No. 17, expression $= \left[\frac{1}{2}\left(e + \frac{1}{e}\right) - 1\right] \div \left[\frac{1}{2}\left(e - \frac{1}{e}\right)\right]$

$$= (e^2 - 2e + 1) \div (e^2 - 1) = \frac{(e-1)^2}{e^2 - 1}.$$

19. $u_r = \frac{1}{(r-3)!} + \frac{2}{(r-2)!} + \frac{1}{(r-1)!} - \left(\frac{1}{r!} - \frac{1}{(r+1)!}\right);$

$\therefore$ as in No. 11, $s_n \to e + 2e + e - 1$.

20. $u_r = \frac{r(r+1)(2r+1)}{6\,.\,(r)!} = \frac{1}{6}\left[\frac{2}{(r-3)!} + \frac{9}{(r-2)!} + \frac{6}{(r-1)!}\right];$

$\therefore$ as in No. 6, $s_n \to \frac{1}{6}(2e + 9e + 6e)$.

21. $u_r = \frac{r^2(r+1)^2}{4\,.\,(r+1)!} = \frac{1}{4}\left[\frac{1}{(r-3)!} + \frac{4}{(r-2)!} + \frac{2}{(r-1)!}\right];$

$\therefore$ as in No. 6, $s_n \to \frac{1}{4}(e + 4e + 2e)$.

22. $s_n = \frac{x}{2!} + \frac{x^2}{3!} + \dots + \frac{x^n}{(n+1)!} = \frac{1}{x}\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{n+1}}{(n+1)!}\right]$

$= \frac{1}{x}\left[1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n+1}}{(n+1)!}\right] - \frac{1}{x}(1+x) \to \frac{1}{x} \cdot e^x - \frac{1}{x}(1+x).$

23. $\frac{x^r}{(r+2) \, . \, r!} = \frac{(r+1)x^r}{(r+2)!} = \frac{x^r}{(r+1)!} - \frac{x^r}{(r+2)!}$,

by No. 22, $\Sigma \frac{x^r}{(r+1)!} = \frac{1}{x}(e^x - 1 - x)$;

also $\frac{x}{3!} + \frac{x^2}{4!} + \dots = \frac{1}{x^2}\left(\frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)$

$= \frac{1}{x^2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - \frac{1}{x^2}\left(1 + x + \frac{x^2}{2}\right)$

and this $\to \frac{1}{x^2} \cdot e^x - \frac{1}{x^2}(1 + x + \frac{1}{2}x^2)$;

$\therefore$ sum $= \frac{1}{x}(e^x - 1 - x) - \frac{1}{x^2}(e^x - 1 - x - \frac{1}{2}x^2)$.

24. Series $= \frac{x}{3!} + \frac{x^2}{5!} + \frac{x^3}{7!} + \dots = \frac{1}{\sqrt{x}}\left[\frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!} + \frac{(\sqrt{x})^7}{7} + \dots\right]$;

but $\frac{1}{2}(e^y - e^{-y}) = y + \frac{y^3}{3!} + \frac{y^5}{5!} + \dots$; put $y = \sqrt{x}$.

25. $u_r = \frac{x^r(r+1)(r+2)}{(r+3)!} = x^r\left[\frac{1}{(r+1)!} - \frac{2}{(r+2)!} + \frac{2}{(r+3)!}\right]$.

From No. 22, $\Sigma \frac{x^r}{(r+1)!} = \frac{1}{x}(e^x - 1 - x)$;

from No. 23, $\Sigma \frac{x^r}{(r+2)!} = \frac{1}{x^2}(e^x - 1 - x - \frac{1}{2}x^2)$;

similarly $\Sigma \frac{x^r}{(r+3)!} = \frac{1}{x^3}(e^x - 1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3)$;

$\therefore$ series $= \frac{1}{x^3}[x^2(e^x - 1 - x) - 2x(e^x - 1 - x - \frac{1}{2}x^2)$
$+ 2(e^x - 1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3)]$.

26. $(2+3x)\left(1 + x + \frac{x^2}{2!} + \dots\right) = 2 + 5x + \dots + x^n\left[\frac{2}{n!} + \frac{3}{(n-1)!}\right] + \dots$.

27. $(1 + 2x + 3x^2)\left(1 - x + \frac{x^2}{2!} - \dots + (-1)^n\frac{x^n}{n!} \dots\right)$

$= 1 + x + \frac{3}{2}x^2 + \dots$;

coefficient of x^n is $(-1)^n\left[\frac{1}{n!} - \frac{2}{(n-1)!} + \frac{3}{(n-2)!}\right]$.

28. Series $= e^{x+1} = e \, . \, e^x = e\left(1 + x + \frac{x^2}{2!} + \dots\right)$.

29. Expression $= e^{2x} + e^{-2x}$; use eqn. (16).

30. $(1 + 2x - 4x^2)\left[1 + (\frac{1}{2}x) + \frac{1}{2!}(\frac{1}{2}x)^2 + \dots + \frac{1}{n!}(\frac{1}{2}x)^n + \dots\right]$

$= 1 + \frac{5x}{2} - \frac{23x^2}{8} \dots$;

coefficient of x^n is

$\frac{1}{n!}(\frac{1}{2})^n + 2 \cdot \frac{1}{(n-1)!}(\frac{1}{2})^{n-1} - 4 \cdot \frac{1}{(n-2)!} \cdot (\frac{1}{2})^{n-2}$.

31. Series $= \left(x^2 - \frac{1}{2!}x^4 + \frac{1}{3!}x^6 - \dots\right) - \left(y^2 - \frac{1}{2!}y^4 + \frac{1}{3!}y^6 - \dots\right)$

$= (1 - e^{-x^2}) - (1 - e^{-y^2})$.

32. Series $= 1 - e^{-\log 2} = 1 - e^{\log \frac{1}{2}} = 1 - \frac{1}{2}$.

33. Coefficient of x^4 in

$\left[2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)\right]^n$ or in $2^n\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!}\right)^n$

or in $2^n\left[1 + n\left(\frac{x^2}{2!} + \frac{x^4}{4!}\right) + \frac{n(n-1)}{1 \, . \, 2} \cdot \left(\frac{x^2}{2!}\right)^2\right]$

is $2^n\left[\frac{n}{4!} + \frac{n(n-1)}{1 \, . \, 2} \cdot \frac{1}{4}\right]$.

34. If x is small, $e^x + \log(1+x) - 1 - 2x \simeq \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) +$

$\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - 1 - 2x$, with error $< \frac{e^x x^4}{4!} + \int_0^{|x|}\frac{t^3}{1+t}\,dt$,

by p. 91 and eqn. (7), $= \frac{1}{2}x^3$, with error $< x^4$ for $|x| < \frac{1}{2}$.

35. Put $x = 1 + y$ so that $y \to 0$; $\log x \simeq y - \frac{1}{2}y^2$, cf. No. 34;

$x^x = e^{x \log x} \simeq e^{(1+y)\left(y - \frac{y^2}{2}\right)} = e^{y + \frac{1}{2}y^2}$

$\simeq 1 + (y + \frac{1}{2}y^2) + \frac{y^2}{2} = 1 + y + y^2$

$\therefore$ Numerator $\simeq \left(y - \frac{y^2}{2}\right) - (1 + y + y^2) + 1 = -\frac{3}{2}y^2$,

and denominator $\simeq \left(y - \frac{y^2}{2}\right) - (1 + y) + 1 = -\frac{y^2}{2}$.

36. $(1+x)^{\frac{1}{1-x}} = e^{\frac{1}{1-x}\log(1+x)} \simeq e^{(1+x+x^2)\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)} \simeq e^{x + \frac{x^2}{2} + \frac{5x^3}{6}}$

$\simeq 1 + \left(x + \frac{x^2}{2} + \frac{5x^3}{6}\right) + \frac{1}{2!}\left(x + \frac{x^2}{2}\right)^2 + \frac{1}{3!}x^3$.

37. (i) By formula (18) in Chap. IV. with $\frac{1}{1-x}$ for t, $x<\log\frac{1}{1-x}$ if $1-x$ is pos. and $\neq 1$; $\therefore e^x<\frac{1}{1-x}$ if $0<x<1$ or if $x<0$.

(ii) See Ex. IV. g, No. 13. If $-2<x<0$, put $x=-y$ then

$$0<y<2;\quad \therefore e^{-x}=e^y<\frac{2+y}{2-y}=\frac{2-x}{2+x};\quad \therefore e^x>\frac{2+x}{2-x}.$$

This is also true if $x<-2$ as the r.h.s. is then negative.

Or as follows:

(i) $\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}\right)<(1+x+x^2+\dots+x^n)-\frac{1}{2}x^2$; expressions in brackets have limits e^x and $\frac{1}{1-x}$ if $0<x<1$;

$\therefore e^x\leqslant\frac{1}{1-x}-\frac{1}{2}x^2<\frac{1}{1-x}$. If $x<0$, let $x=-y$, then

$e^y>1+y>0$; $\therefore e^x<\frac{1}{1+y}=\frac{1}{1-x}$.

(ii) $\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}\right)<\left(1+x+\frac{x^2}{2}+\frac{x^3}{2^2}+\dots+\frac{x^n}{2^{n-1}}\right)-\frac{1}{12}x^3$,

and limit of second bracket is $1+\frac{2x}{2-x}$ if $0<x<2$;

$$\therefore e^x\leqslant\frac{2+x}{2-x}-\frac{1}{12}x^3<\frac{2+x}{2-x}.$$

If $-2<x<0$, let $x=-y$, then $0<y<2$;

$$\therefore e^{-x}=e^y<\frac{2+y}{2-y}=\frac{2-x}{2+x};\quad \therefore e^x>\frac{2+x}{2-x}.$$

This is true also if $x<-2$ as the r.h.s. is then negative.

EXERCISE V. e. (p. 97.)

1. (i) In eqn. (5), put $x=\pi$; series $=\frac{1}{\pi}\sin\pi$;

(ii) Series $=\frac{1}{2}(1-\frac{1}{2}+\frac{1}{3}-\dots)$; put $x=1$ in eqn. (11);

(iii) From eqn. (12), $\frac{1}{2x}\log\frac{1+x}{1-x}=1+\frac{x^2}{3}+\frac{x^4}{5}+\dots$; put $x=\frac{1}{\sqrt{2}}$;

$$\text{sum}=\frac{1}{\sqrt{2}}\log\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}=\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$=\frac{1}{\sqrt{2}}\log\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)}=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2$$

$$=\sqrt{2}\log(\sqrt{2}+1).$$

(iv) From eqn. (14), $\frac{1}{y}\tan^{-1}y=1-\frac{y^2}{3}+\frac{y^4}{5}-\dots$; put $y=\frac{1}{\sqrt{3}}$;

(v) $e+e^{-1}=2\left(1+\frac{1}{2!}+\frac{1}{4!}+\dots\right)$; series $=\frac{1}{2}\left(e+\frac{1}{e}\right)-1$;

(vi) $\sin 1=1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\dots$;

$$\tfrac{1}{2}(e-e^{-1})=1+\frac{1}{3!}+\frac{1}{5!}+\dots;\ \text{add.}$$

2. 1st series $=\frac{1}{2}\left[\frac{2^2}{2!}-\frac{2^4}{4!}+\frac{2^6}{6!}-\dots\right]=\frac{1}{2}(1-\cos 2)=\sin^2 1$;

2nd series $=\sin 1$.

3. $\theta\cot\theta=\cos\theta\cdot\left(\frac{\sin\theta}{\theta}\right)^{-1}\simeq\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\right)\cdot\left(1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}\right)^{-1}$

$$\simeq\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\right)\left[1+\left(\frac{\theta^2}{3!}-\frac{\theta^4}{5!}\right)+\left(\frac{\theta^2}{3!}-\frac{\theta^4}{5!}\right)^2\right];$$

simplify, neglecting θ^6, etc.

4. $8\sin\frac{\theta}{2}-\sin\theta\simeq 8\left[\frac{\theta}{2}-\frac{1}{3!}\cdot\left(\frac{\theta}{2}\right)^3+\frac{1}{5!}\cdot\left(\frac{\theta}{2}\right)^5\right]-\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\right)$; this reduces to $3\left(\theta-\frac{\theta^5}{480}\right)$; equation is

$$\sin\theta\simeq\tfrac{5}{6}\cdot\tfrac{1}{3}\left(8\sin\frac{\theta}{2}-\sin\theta\right);$$

$$\therefore 40\sin\frac{\theta}{2}=23\sin\theta\equiv 46\sin\frac{\theta}{2}\cos\frac{\theta}{2};$$

$$\therefore \sin\frac{\theta}{2}=0\ \text{or}\ \cos\frac{\theta}{2}=\tfrac{20}{23};$$

$$\therefore \theta=0\ \text{or}\ \frac{\theta}{2}=\pm 29^\circ 35',\ \theta=\pm 59^\circ 10'=\pm 1{\cdot}033^c.$$

5. Numerator $=\sin\theta(1-\cos 2\alpha)-\cos\theta\sin 2\alpha+\sin\alpha$

$$\simeq\sin\theta\cdot\frac{(2\alpha)^2}{2!}-\cos\theta\cdot(2\alpha)+\alpha$$

$$=\alpha(1-2\cos\theta)+2\alpha^2\sin\theta.$$

Since α is a factor of numerator, but not of the denominator, we can neglect terms in α^2 in denominator; denominator $=\cos\theta(1-\cos 2\alpha)+\sin\theta\sin 2\alpha+\cos\alpha$

$$\simeq\sin\theta\cdot 2\alpha+1;$$

$$\therefore\ \text{fraction}\simeq\frac{\alpha(1-2\cos\theta)+2\alpha^2\sin\theta}{1+2\alpha\sin\theta}$$

$$\simeq[\alpha(1-2\cos\theta)+2\alpha^2\sin\theta]\cdot[1-2\alpha\sin\theta]$$

$$\simeq\alpha(1-2\cos\theta)+2\alpha^2\sin\theta-2\alpha^2\sin\theta(1-2\cos\theta)$$

$$=\alpha(1-2\cos\theta)+4\alpha^2\sin\theta\cos\theta.$$

6. $\sqrt{(\cos\theta)} = \left[1 - \left(\frac{\theta^2}{2} - \frac{\theta^4}{24}\right)\right]^{\frac{1}{2}}$

$$= 1 - \tfrac{1}{2}\left(\frac{\theta^2}{2} - \frac{\theta^4}{24}\right) + \frac{\frac{1}{2}\,.\,(-\frac{1}{2})}{1\,.\,2}\,.\left(\frac{\theta^2}{2}\ldots\right)^2 \simeq 1 - \frac{\theta^2}{4} + \frac{\theta^4}{48} - \frac{\theta^4}{32}$$

$$= 1 - \frac{\theta^2}{4} - \frac{\theta^4}{96}; \quad \therefore \ (3 + \cos\theta)\,.\,\sqrt{(\cos\theta)}$$

$$\simeq \left(4 - \frac{\theta^2}{2} + \frac{\theta^4}{24}\right)\left(1 - \frac{\theta^2}{4} - \frac{\theta^4}{96}\right) \simeq 4 - \frac{3\theta^2}{2} + \frac{\theta^4}{8}$$

$$= 1 + 3\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}\right) \simeq 1 + 3\cos\theta.$$

7. 1st approx., $\cos(\theta + \alpha) = \cos\alpha - \phi\cos\beta\sin\alpha$;

$\therefore \ \cos\alpha - \cos(\theta + \alpha) = \phi\cos\beta\sin\alpha$;

$\therefore \ 2\sin\frac{\theta}{2}\sin\left(\alpha + \frac{\theta}{2}\right) = \phi\cos\beta\sin\alpha$;

$\therefore \ \theta\sin\alpha = \phi\cos\beta\sin\alpha$;

2nd approx., $\cos(\theta + \alpha) = \cos\alpha\,(1 - \frac{1}{2}\phi^2) - \phi\cos\beta\sin\alpha$;

$$\therefore \ \phi\cos\beta\sin\alpha + \tfrac{1}{2}\phi^2\cos\alpha = 2\sin\frac{\theta}{2}\sin\left(\alpha + \frac{\theta}{2}\right)$$

$$= \theta\left(\sin\alpha + \frac{\theta}{2}\cos\alpha\right);$$

$$\therefore \ \theta\sin\alpha = \phi\cos\beta\sin\alpha + \tfrac{1}{2}\phi^2\cos\alpha - \tfrac{1}{2}\theta^2\cos\alpha$$

$$= \phi\cos\beta\sin\alpha + \tfrac{1}{2}\phi^2\cos\alpha - \tfrac{1}{2}(\phi\cos\beta)^2\cos\alpha$$

$$= \phi\cos\beta\sin\alpha + \tfrac{1}{2}\phi^2\cos\alpha\sin^2\beta.$$

8. For small values of ϵ, $\frac{\pi}{2}\sin^2 x - x + \epsilon$ is $+, -, +, -$ for $x = 0$, $\frac{\pi}{6}, \frac{\pi}{2}, \pi$ and $\therefore$ has roots between those values. This may also be seen by drawing the graphs of $\frac{\pi}{2}\sin^2 x$ and $x - \epsilon$. A rough sketch of the graphs shows that there cannot be more than 3 roots.

9. $16\sin\frac{\pi}{12} = 4{\cdot}14112$; $12\left(\frac{\pi}{12}\right) + 1 = \pi + 1 = 4{\cdot}14159$.

Put $\theta = \frac{\pi}{12} + \alpha$; $16\left(\sin\frac{\pi}{12} + \alpha\cos\frac{\pi}{12}\right)$

$= \pi + 12\alpha + 1$, neglecting α^2;

$$\therefore \ \alpha\left(16\cos\frac{\pi}{12} - 12\right) = \pi + 1 - 16\sin\frac{\pi}{12}$$

$$= 4{\cdot}14159 - 4{\cdot}14112 = 0{\cdot}00047;$$

$\therefore \ 3{\cdot}46\alpha \simeq 0{\cdot}00047$;

$\therefore \ \alpha \simeq 0{\cdot}00014$; $\theta = \frac{3{\cdot}14159}{12} + 0{\cdot}00014.$

10. From Ex. 2, p. 82, $\tan\frac{x}{2} \simeq \frac{x}{2} + \frac{x^3}{24}$;

1st approx., $\frac{x}{2} = a$;

2nd approx., $(1 + x)\frac{x}{2} = a$;

$$\therefore \ \frac{x}{2} = a - \tfrac{1}{2}x^2 = a - \tfrac{1}{2}(2a)^2, \ x = 2a - 4a^2;$$

3rd approx., $(1 + x + \frac{1}{2}x^2)\left(\frac{x}{2} + \frac{x^3}{24}\right) = a$; $\therefore \ \frac{x}{2} + \frac{x^2}{2} + \frac{7x^3}{24} = a$;

$$\therefore \ \frac{x}{2} = a - \tfrac{1}{2}x^2 - \frac{7x^3}{24} = a - \tfrac{1}{2}(2a - 4a^2)^2 - \tfrac{7}{24}(2a)^3$$

$$= a - 2a^2 + \frac{17a^3}{3};$$

4th approx., $\left(1 + x + \frac{1}{2}x^2 + \frac{x^3}{6}\right)\left(\frac{1}{2}x + \frac{x^3}{24}\right) = a$;

$$\therefore \ \frac{x}{2} + \frac{x^2}{2} + \frac{7x^3}{24} + \frac{x^4}{8} = a;$$

$$\therefore \ \frac{x}{2} = a - \tfrac{1}{2}\left(2a - 4a^2 + \frac{34a^3}{3}\right)^2 - \tfrac{7}{24}(2a - 4a^2)^3 - \tfrac{1}{8}(2a)^4.$$

11. By No. 3, $\theta\cot\theta \simeq 1 - \frac{\theta^2}{3} - \frac{\theta^4}{45}$;

$$\therefore \ \theta^2 = 3\epsilon - \frac{\theta^4}{15} = 3\epsilon\left(1 - \frac{\theta^4}{45\epsilon}\right);$$

$$\therefore \ \theta = \sqrt{(3\epsilon)}\,.\left(1 - \frac{\theta^4}{90\epsilon}\right), \text{ neglecting } \theta^6;$$

1st approx., $\theta = \sqrt{(3\epsilon)}$;

2nd approx., $\theta = \sqrt{(3\epsilon)}\,.\left\{1 - \frac{[\sqrt{(3\epsilon)}]^4}{90\epsilon}\right\}$

$$= \sqrt{(3\epsilon)}\,.\left\{1 - \frac{9\epsilon^2}{90\epsilon}\right\} = \sqrt{(3\epsilon)}\,.\left\{1 - \frac{\epsilon}{10}\right\}.$$

12. Series $= 1 + (\frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{6}x^3 + \ldots) + (\frac{1}{3}y^2 + \frac{1}{5}y^4 + \frac{1}{7}y^6 + \ldots)$ where

$$x = \tfrac{1}{4}, y = \tfrac{1}{2}, \ = \tfrac{1}{2}(x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \ldots) + \frac{1}{y}(y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \ldots)$$

$$= -\tfrac{1}{2}\log(1 - x) + \frac{1}{y}\,.\,\tfrac{1}{2}\log\frac{1 + y}{1 - y} = -\tfrac{1}{2}\log(1 - \tfrac{1}{4}) + \log\frac{1\frac{1}{2}}{\frac{1}{2}}$$

$$= \tfrac{1}{2}\log\tfrac{4}{3} + \log 3 = \tfrac{1}{2}\log(\tfrac{4}{3} \times 3^2).$$

13. Write x for $\dfrac{1}{n+1}$ so that $0<x<1$; series $=\Sigma x^r\left(\dfrac{1}{r}-\dfrac{1}{r+1}\right)$

$$=\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)-\left(\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\dots\right)$$

$$=-\log(1-x)-\frac{1}{x}\{-x-\log(1-x)\}$$

$$=1+n\log\left(\frac{n}{n+1}\right),\text{ substituting for } x,$$

$$=1-n\log\left(\frac{n+1}{n}\right)=1-\log\left(1+\frac{1}{n}\right)^n.$$

14. (i) $\log\left(1+\dfrac{1}{x-3}\right)=\log\dfrac{x-2}{x-3}=\log\dfrac{2\left(1-\dfrac{x}{2}\right)}{3\left(1-\dfrac{x}{3}\right)}$

$$=\log\tfrac{2}{3}+\log\left(1-\frac{x}{2}\right)-\log\left(1-\frac{x}{3}\right)$$

$$=\log\tfrac{2}{3}-\left\{\frac{x}{2}+\tfrac{1}{2}\cdot\left(\frac{x}{2}\right)^2+\tfrac{1}{3}\cdot\left(\frac{x}{2}\right)^3+\dots\right\}$$

$$+\left\{\frac{x}{3}+\tfrac{1}{2}\left(\frac{x}{3}\right)^2+\tfrac{1}{3}\cdot\left(\frac{x}{3}\right)^3+\dots\right\},$$

valid if $-1\leqslant\dfrac{x}{2}<1$.

(ii) $\log(1-x^2+x^4)=\log\dfrac{1+x^6}{1+x^2}$

$=\log(1+x^6)-\log(1+x^2)=$, if $x^2<1$,

$(x^6-\tfrac{1}{2}x^{12}+\tfrac{1}{3}x^{18}-\dots)-(x^2-\tfrac{1}{2}x^4+\tfrac{1}{3}x^6-\dots)$.

Coefficient of x^n is zero if n is odd. For n even it is $\left(\dfrac{2}{n}-\dfrac{6}{n}\right)(-1)^{\frac{n}{2}}$ if n is divisible by 3, and otherwise is $\dfrac{2}{n}(-1)^{\frac{n}{2}}$.

15. $\log\dfrac{x+y}{x-y}=\log\dfrac{1+\dfrac{y}{x}}{1-\dfrac{y}{x}}$; use eqn. (12); valid if $-1<\dfrac{y}{x}<1$.

16. $\log\dfrac{(x+h)^2}{x(x+2h)}=\log\dfrac{(x+h)^2}{(x+h)^2-h^2}$

$$=-\log\frac{(x+h)^2-h^2}{(x+h)^2}=-\log\left[1-\left(\frac{h}{x+h}\right)^2\right];$$

valid if $\dfrac{h^2}{(x+h)^2}<1$, i.e. if $x^2+2xh+h^2>h^2$, i.e. if $x(x+2h)>0$.

17. By eqn. (12), $2\left(\tfrac{1}{9}+\dfrac{1}{3\,.\,9^3}+\dots\right)$

$$=\log\frac{1+\frac{1}{9}}{1-\frac{1}{9}}=\log\tfrac{10}{8}=\log 10-3\log 2;$$

$\log_{10}2=\log 2\div\log 10$.

18. $3a+b+c=\log\{(\tfrac{6}{5})^3\,.\,(\tfrac{10}{9})\,.\,(\tfrac{25}{24})\}=\log 2$.

19. By eqn. (12), series $=\tfrac{1}{2}\log\dfrac{1+\cos 2\theta}{1-\cos 2\theta}$, for $-1<\cos 2\theta<1$, $=\tfrac{1}{2}\log\cot^2\theta=\log[+\sqrt{(\cot^2\theta)}]$. For $\dfrac{\pi}{2}<\theta<\pi$, $\cot\theta$ is negative; $\therefore$ $+\sqrt{(\cot^2\theta)}=-\cot\theta$.

20. By eqn. (12), series $=\tfrac{1}{2}\log\dfrac{1+\dfrac{x}{1+x^2}}{1-\dfrac{x}{1+x^2}}$, for $-1<\dfrac{x}{1+x^2}<1$, which holds for all values of x;

$$\therefore\ \text{series}=\tfrac{1}{2}\log\frac{1+x^2+x}{1+x^2-x}=\tfrac{1}{2}\log\left(\frac{1-x^3}{1-x}\times\frac{1+x}{1+x^3}\right)$$

$$=\tfrac{1}{2}\log\frac{1+x}{1-x}-\tfrac{1}{2}\log\frac{1+x^3}{1-x^3}=\Sigma\frac{x^{2r-1}}{2r-1}-\Sigma\frac{x^{6r-3}}{2r-1},\ \text{if } x^2<1.$$

If n is odd, coefficient of x^{3n} is $\dfrac{1}{3n}-\dfrac{1}{n}$.

21. (i) Series $=(1-\tfrac{1}{2})+(\tfrac{1}{3}-\tfrac{1}{4})+(\tfrac{1}{5}-\tfrac{1}{6})+\dots=\log 2$, by eqn. (11).

(ii) Series $=(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{4}-\tfrac{1}{5})+(\tfrac{1}{6}-\tfrac{1}{7})+\dots=1-\log 2$.

(iii) $u_n=\dfrac{1}{(4n-3)(4n-2)(4n-1)}=\dfrac{\frac{1}{2}}{4n-3}-\dfrac{1}{4n-2}+\dfrac{\frac{1}{2}}{4n-1}$;

$$\therefore\ 2s_n=(\tfrac{1}{1}-\tfrac{2}{2}+\tfrac{1}{3})+(\tfrac{1}{5}-\tfrac{2}{6}+\tfrac{1}{7})+(\tfrac{1}{9}-\tfrac{2}{10}+\tfrac{1}{11})+\dots+2u_n$$

$$=\left(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\dots+\frac{1}{4n-1}\right)$$

$$-\left(\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{6}-\dots+\frac{1}{4n-2}\right);$$

$\therefore$ when $n\to\infty$, $2s_n\to\log 2-\tfrac{1}{2}\log 2=\tfrac{1}{2}\log 2$.

(iv) $u_n=\dfrac{2+3n}{n(n+1)(2n+1)}=\dfrac{2}{n}-\dfrac{2}{2n+1}-\dfrac{1}{n+1}$

$$=\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{2}{2n}-\frac{2}{2n+1}\right);$$

$$\therefore\ s_n = \left(1 - \frac{1}{n+1}\right) + 2\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} \ldots - \frac{2}{2n+1}\right)$$
$$\to 1 + 2(1 - \log 2).$$

Also $u_n = \left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{1}{n(2n+1)}$

$$< \left(\frac{1}{n} - \frac{1}{n+1}\right) + \tfrac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right);$$

$\therefore\ u_{n+1} + u_{n+2} + \ldots + u_{n+k} <$

$\left(\frac{1}{n+1} - \frac{1}{n+k+1}\right) + \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+k}\right)$, which, for all positive values of k, $< \frac{1}{n+1} + \frac{1}{2}\cdot\frac{1}{n} < \frac{1}{n}(1 + \frac{1}{2})$.

22. $\frac{6}{(2n-1)(2n)(2n+1)(2n+2)}$

$$= \frac{2}{(2n-1)(2n)(2n+1)} - \frac{2}{(2n)(2n+1)(2n+2)}$$
$$= \frac{1}{(2n-1)(2n)} - \frac{2}{(2n)(2n+1)} + \frac{1}{(2n+1)(2n+2)};$$

$$\therefore\ 6(\text{sum to } n \text{ terms}) = \left\{\frac{1}{1.2} + \frac{1}{3.4} + \ldots + \frac{1}{(2n-1)(2n)}\right\}$$
$$-2\left\{\frac{1}{2.3} + \frac{1}{4.5} + \ldots + \frac{1}{2n(2n+1)}\right\}$$
$$+\left\{\frac{1}{3.4} + \frac{1}{5.6} + \ldots + \frac{1}{(2n+1)(2n+2)}\right\}$$

$$= s_{2n} - 2\left\{\left(\frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{2n(2n+1)}\right) - \left(\frac{1}{1.2} + \frac{1}{3.4} + \ldots + \frac{1}{(2n-1)(2n)}\right)\right\}$$
$$+\left\{s_{2n} - \tfrac{1}{2} + \frac{1}{(2n+1)(2n+2)}\right\}$$
$$= 2s_{2n} - \tfrac{1}{2} + \frac{1}{(2n+1)(2n+2)} - 2\left\{1 - \frac{1}{2n+1} - s_{2n}\right\},$$

where $s_{2n} = \frac{1}{1.2} + \frac{1}{3.4} + \ldots + \frac{1}{(2n-1)(2n)} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \ldots$ to $2n$ terms; $\lim s_{2n} = \log 2$; $\therefore$ sum to infinity is $\frac{2}{3}\log 2 - \frac{5}{12}$.

23. (i) $\log(1+x)^{\frac{1}{x}} = \frac{1}{x}\log(1+x) \simeq 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4}$;

$$\therefore\ \log\{\log(1+x)^{\frac{1}{x}}\} \simeq -\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4}\right) - \tfrac{1}{2}\left(\frac{x}{2} - \frac{x^2}{3}\right)^2 - \tfrac{1}{3}\left(\frac{x}{2}\right)^3$$
$$\simeq -\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} - \tfrac{1}{2}\left(\frac{x^2}{4} - \frac{x^3}{3}\right) - \frac{x^3}{24}.$$

(ii) $\sin x \,.\, \sqrt{(1+x^2)} \simeq \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\cdot\left[1 + \tfrac{1}{2}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})}{1.2}x^4\right]$
$\simeq x + \frac{x^3}{3} - \frac{x^5}{5}$; use eqn. (12) and simplify.

24. $\log\sin\theta - \log\theta = \log\frac{\sin\theta}{\theta} \simeq \log\left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}\right)$

$$\simeq -\left(\frac{\theta^2}{6} - \frac{\theta^4}{120}\right) - \tfrac{1}{2}\left(\frac{\theta^2}{6}\ldots\right)^2 = -\frac{\theta^2}{6} + \frac{\theta^4}{120} - \frac{\theta^4}{72}.$$

25. By Exercise IV. g, Nos. 9, 10, p. 74, $\frac{x}{x-1} - \frac{1}{\log x}$ lies between $\frac{x}{x-1} - \frac{x+1}{2(x-1)}$ and $\frac{x}{x-1} - \frac{2x}{x^2-1}$, i.e. between $\frac{1}{2}$ and $\frac{x}{x+1}$, $\therefore \to \frac{1}{2}$.

26. (i) Series $= \frac{1}{2}\{(1 - \frac{1}{3}) + (\frac{1}{5} - \frac{1}{7}) + (\frac{1}{9} - \frac{1}{11}) + \ldots\} = \frac{1}{2}\tan^{-1}(1)$, by eqn. (14).

(ii) Series $= \frac{1}{2}\left\{\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)\right\}$;
use eqn. (12) and eqn. (14).

27. $\tan^{-1}x - \sin x \simeq \left(x - \frac{x^3}{3}\right) - \left(x - \frac{x^3}{6}\right) = -\frac{x^3}{6}$;

$$x^2 - 2x + 2\log(1+x) \simeq x^2 - 2x + 2\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) = \frac{2x^3}{3};$$

The moduli of the errors in $\tan^{-1}x$, $\sin x$, $\log(1+x)$ are by pp. 88, 80, 84 less than $\frac{1}{5}|x^5|$, $\frac{1}{5!}|x^5|$, $\frac{1}{4}x^4$, respectively.

28. Expression $\simeq ax\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$

$$+\tfrac{1}{2}b\left(\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720}\right) - x\left(x - \frac{x^3}{3} + \frac{x^5}{5}\right)$$

coefficient of $x^2 = a + \frac{b}{4} - 1 = 0$;

coefficient of $x^4 = -\frac{a}{6}-\frac{b}{48}+\frac{1}{3}=0$;

solve $4a+b=4$; $8a+b=16$; these give $a=3$, $b=-8$;

coefficient of $x^6=\frac{a}{120}+\frac{b}{1440}-\frac{1}{5}=\frac{1}{40}-\frac{1}{180}-\frac{1}{5}=-\frac{13}{72}$.

29. If $\frac{1}{2}\cos^{-1}k=\phi$, then $\cos 2\phi=k$;

$$\therefore\ \tan^2\phi=\frac{1-k}{1+k};\quad \therefore\ \phi=\tan^{-1}\sqrt{\left(\frac{1-k}{1+k}\right)};$$

$$\therefore\ \text{given expression}=\tan^{-1}\sqrt{\left\{\frac{1+\cos\alpha\cos\theta-\cos\alpha-\cos\theta}{1+\cos\alpha\cos\theta+\cos\alpha+\cos\theta}\right\}}$$

$$=\tan^{-1}\sqrt{\left\{\frac{(1-\cos\alpha)(1-\cos\theta)}{(1+\cos\alpha)(1+\cos\theta)}\right\}}=\tan^{-1}\sqrt{\left(\tan^2\frac{\alpha}{2}\tan^2\frac{\theta}{2}\right)}$$

$$=\pm\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\theta}{2}\right).$$

30. $u_r=\frac{1}{2}\left[\frac{1}{(2r-2)!}+\frac{1}{(2r-1)!}\right]$;

$$2s_n=\left(1+\frac{1}{1!}\right)+\left(\frac{1}{2!}+\frac{1}{3!}\right)+\ldots+2u_n;$$

$$\therefore\ 2s_n\to e \text{ when } n\to\infty.$$

31. $u_r=\frac{\frac{1}{2}r(r+1)}{(r+1)!}=\frac{\frac{1}{2}}{(r-1)!}$; $s_n=\frac{1}{2}\left[1+\frac{1}{1!}+\frac{1}{2!}+\ldots\frac{1}{(n-1)!}\right]\to\frac{1}{2}e$.

32. $u_r=\frac{2}{(r-2)!}+\frac{3}{(r-1)!}-\frac{1}{r!}$;

$$\therefore\ s_n=\left(3-\frac{1}{1!}\right)+\left(2+\frac{3}{1!}-\frac{1}{2!}\right)+\left(\frac{2}{1!}+\frac{3}{2!}-\frac{1}{3!}\right)+\ldots+u_n$$

$$=2\left[1+\frac{1}{1!}+\ldots+\frac{1}{(n-2)!}\right]+3\left[1+\frac{1}{1!}+\ldots+\frac{1}{(n-1)!}\right]$$

$$-\left[\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right];$$

$\therefore$ when $n\to\infty$, $s_n\to 2e+3e-(e-1)$.

33. $u_r=\frac{(r+1)(r+1)}{(r+2)!}=\frac{1}{r!}-\left(\frac{1}{(r+1)!}-\frac{1}{(r+2)!}\right)$;

$$\therefore\ s_n=\left(\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right)-\frac{1}{2!}+\frac{1}{(n+2)!}.$$

(Cf. V. d, No. 11); $\therefore$ when $n\to\infty$, $s_n\to(e-1)-\frac{1}{2}$.

34. $u_r=\frac{1}{2}\left[\frac{5}{(2r)!}-\frac{3}{(2r+1)!}\right]$;

$$\therefore\ 2s_n=5\left[\frac{1}{2!}+\frac{1}{4!}+\ldots+\frac{1}{(2n)!}\right]-3\left[\frac{1}{3!}+\frac{1}{5!}+\ldots+\frac{1}{(2n+1)!}\right];$$

$$\therefore\ \text{when } n\to\infty,\ 2s_n\to 5\left[\tfrac{1}{2}\left(e+\frac{1}{e}\right)-1\right]$$

$$-3\left[\tfrac{1}{2}\left(e-\frac{1}{e}\right)-1\right]=e+\frac{4}{e}-2.$$

35. $u_r=\frac{8}{(r-2)!}+\frac{12}{(r-1)!}+\frac{2}{r!}-\frac{1}{(r+1)!}$;

$$\therefore\ s_n=\left[12+\frac{2}{1!}-\frac{1}{2!}\right]+\left[8+\frac{12}{1!}+\frac{2}{2!}-\frac{1}{3!}\right]+\ldots+u_n$$

$$=8\left[1+\frac{1}{1!}+\ldots+\frac{1}{(n-2)!}\right]+12\left[1+\frac{1}{1!}+\ldots+\frac{1}{(n-1)!}\right]$$

$$+2\left[\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right]-\left[\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{(n+1)!}\right];$$

$\therefore$ when $n\to\infty$, $s_n\to 8e+12e+2(e-1)-(e-2)$.

36. Series $=e^{a+bx}=e^a\,.\,e^{bx}=e^a\left(1+bx+\frac{b^2x^2}{2!}+\ldots\right)$.

37. Expression $=e^{-x}[e^{4x}-4e^{2x}+6-4e^{-2x}+e^{-4x}]$
$=e^{-x}(e^x-e^{-x})^4\simeq(1-x)\,.\,(2x)^4$.

38. Expression $=e^{(1+x)\log(1+x)}$;

$$(1+x)\log(1+x)\simeq(1+x)\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)\simeq x+\frac{x^2}{2}-\frac{x^3}{6};$$

$$\therefore\ \text{expression}\simeq 1+\left(x+\frac{x^2}{2}-\frac{x^3}{6}\right)+\frac{1}{2!}\left(x+\frac{x^2}{2}\right)^2+\frac{1}{3!}(x)^3$$

$$\simeq 1+x+\frac{x^2}{2}-\frac{x^3}{6}+\tfrac{1}{2}(x^2+x^3)+\tfrac{1}{6}x^3.$$

39. $\left(1+\frac{1}{x}\right)^x=e^{x\log\left(1+\frac{1}{x}\right)}\simeq e^{x\left(\frac{1}{x}-\frac{1}{2x^2}\right)}$

$$=e^{1-\frac{1}{2x}}=e\,.\,e^{-\frac{1}{2x}}\simeq e\left(1-\frac{1}{2x}\right).$$

40. $\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}=e^{(x+\frac{1}{2})\log\left(1+\frac{1}{x}\right)}\simeq e^{(x+\frac{1}{2})\left(\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}\right)}$

$$\simeq e^{1+\frac{1}{12x^2}}=e\,.\,e^{\frac{1}{12x^2}}\simeq e\left(1+\frac{1}{12x^2}\right).$$

41. $x=a^{\frac{n}{n+p}}=a\,.\,a^{-\frac{p}{n+p}}=a\,.\,e^{-\frac{p}{n+p}\log a}$, since $a>0$,

$\simeq a\left[1-\frac{p}{n+p}\log a\right]$, neglecting $\frac{p^2}{n^2}$, $\simeq a\left[1-\frac{p}{n}\log a\right]$,

since $\frac{p}{n+p}=\frac{p}{n}\,.\left(1+\frac{p}{n}\right)^{-1}\simeq\frac{p}{n}\left(1-\frac{p}{n}\right)=\frac{p}{n}-\frac{p^2}{n^2}$.

42. $u = k + a\sin u$; 1st approx., $u = k$; 2nd approx., $u = k + a\sin k$;
3rd approx., $u = k + a\sin(k + a\sin k) = k + a\sin k\cos(a\sin k)$
$+ a\cos k\sin(a\sin k) = k + a\sin k + a^2\sin k\cos k$;
$\therefore\ \cos u = \cos k\,.\,\cos(a\sin k + a^2\sin k\cos k)$.
$- \sin k\,.\,\sin(a\sin k + a^2\sin k\cos k)$
$= \cos k\,.(1 - \frac{1}{2}a^2\sin^2 k) - \sin k\,.(a\sin k + a^2\sin k\cos k)$
$= \cos k - a\sin^2 k - \frac{3}{2}a^2\sin k\cos k$.

EXERCISE V. f. (p. 100.)

1. $2\theta(9 + 6\cos\theta) - (28\sin\theta + \sin 2\theta)$

$$\simeq 2\theta\left(9 + 6 - \frac{6\theta^2}{2} + \frac{6\theta^4}{24} - \frac{6\theta^6}{720}\right) - 28\left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \frac{\theta^7}{5040}\right)$$

$$- \left(2\theta - \frac{8\theta^3}{6} + \frac{32\theta^5}{120} - \frac{128\theta^7}{5040}\right) = -\theta^7\left(\tfrac{1}{60} - \tfrac{1}{180} - \tfrac{16}{630}\right)$$

$$= \frac{\theta^7}{70};\ \ 9 + 6\cos\theta \simeq 15;\ \ \therefore\ \text{expression} \simeq \frac{\theta^7}{70}\cdot\tfrac{1}{15}.$$

2. $x(a + b\cos x) - (a + b)\sin x = \int_0^x \{a(1 - \cos x) - bx\sin x\}\,dx$.

$$\text{But}\ \frac{1 - \cos x}{x\sin x} = \frac{1}{x}\cdot\tan\frac{x}{2} = \tfrac{1}{2}\left\{\tan\frac{x}{2} \div \frac{x}{2}\right\} > \tfrac{1}{2},\ \text{for}$$

$$0 < \frac{x}{2} < \frac{\pi}{2},\quad > \frac{b}{a};$$

∵ since x, $\sin x$, a are all positive, $a(1 - \cos x) > bx\sin x$;
∴ the integrand is positive; ∴ integral is positive, see p. 79; ∴ $x(a + b\cos x) > (a + b)\sin x$; but $a + b\cos x > 0$; ∴ result follows.

3. $\sin\frac{\theta}{2} < \frac{\theta}{2} < \tan\frac{\theta}{2}$;

$$\therefore\ \theta\sin\theta = 2\theta\sin\frac{\theta}{2}\cos\frac{\theta}{2} < 4\sin^2\frac{\theta}{2} < 4\left(\frac{\theta}{2}\right)^2 = \theta^2 < \tan^2\theta;$$

$$\therefore\ \int_0^\theta \theta\sin\theta\,d\theta < \int_0^\theta 4\sin^2\frac{\theta}{2}\,d\theta < \int_0^\theta \tan^2\theta\,d\theta$$

$$= \int_0^\theta (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta,\ \text{for}\ \ 0 < \theta < \frac{\pi}{2};$$

$$\therefore\ (\sin\theta - \theta\cos\theta) < \int_0^\theta 2(1 - \cos\theta)\,d\theta = 2(\theta - \sin\theta) < \tan\theta - \theta;$$

∴ $3\sin\theta < \theta(2 + \cos\theta)$, i.e. $\theta(2\,\text{cosec}\,\theta + \cot\theta) > 3$; also $3\theta < 2\sin\theta + \tan\theta$. $\theta < \frac{\pi}{2}$ is needed for the existence of $\int_0^\theta \tan^2\theta\,d\theta$.

4. Radius r, $\theta = \frac{\pi}{n}$; $x = 2r\sin\theta$, $y = 2r\tan\theta$;

$$\frac{n}{3}(2x + y) = \frac{2nr}{3}(2\sin\theta + \tan\theta) \simeq \frac{2nr}{3}\left\{2\left(\theta - \frac{\theta^3}{3!}\right) + \left(\theta + \frac{\theta^3}{3}\right)\right\}$$

$$= 2nr\theta = 2\pi r,\ \text{neglecting}\ \theta^5.$$

5. Radius r, $\theta = \frac{\pi}{n}$; $\mathsf{A} = \frac{1}{2}nr^2\sin 2\theta$, $\mathsf{B} = nr^2\tan\theta$,

$$\tfrac{1}{3}(\mathsf{A} + 2\mathsf{B}) = \frac{nr^2}{3}\left(\tfrac{1}{2}\sin 2\theta + 2\tan\theta\right)$$

$$\simeq \frac{nr^2}{3}\left\{\tfrac{1}{2}\left(2\theta - \frac{8\theta^3}{3!}\right) + 2\left(\theta + \frac{\theta^3}{3}\right)\right\}$$

$$= nr^2\theta = \pi r^2,\ \text{neglecting}\ \theta^5.$$

6. Let $n\theta = \pi = m\phi$; side $= \frac{l}{n}$, inradius $= \frac{l}{2n}\cot\theta$;

$$\therefore\ \text{area} = n\cdot\frac{l}{2n}\cdot\frac{l}{2n}\cot\theta \simeq \frac{l^2}{4n}\left(\frac{1}{\theta} - \frac{\theta}{3}\right),$$

$$\text{from V. e, No. 3,} = \frac{l^2}{4}\left(\frac{1}{\pi} - \frac{\theta}{3n}\right);$$

$$\therefore\ \text{difference of areas} \simeq \frac{l^2}{4}\left(\frac{\phi}{3m} \sim \frac{\theta}{3n}\right) = \frac{l^2\pi}{12}\left(\frac{1}{m^2} \sim \frac{1}{n^2}\right).$$

7. $2\log 3 + 2\log 5 - 5\log 2 - \log 7$

$$= \log\frac{9 \times 25}{32 \times 7} = \log\tfrac{225}{224} = \log\frac{1 + \frac{1}{449}}{1 - \frac{1}{449}} = 2(x + \tfrac{1}{3}x^3 + \ldots),$$

where $x = \frac{1}{449}$; this differs from $\frac{2}{449}$ by an amount $< 2(\frac{1}{3}x^3 + \frac{1}{3}x^5 + \frac{1}{3}x^7 + \ldots) < \frac{2}{3}\frac{x^3}{1 - x^2}$; but

$$1 - x^2 = \frac{449^2 - 1}{449^2} = \frac{450\,.\,448}{449^2}.$$

8. $$\log\sec x = \log\frac{1 + \tan^2\frac{x}{2}}{1 - \tan^2\frac{x}{2}} = 2\left\{\tan^2\frac{x}{2} + \tfrac{1}{3}\tan^6\frac{x}{2} + \tfrac{1}{5}\tan^{10}\frac{x}{2} + \ldots\right\}.$$

9. $$\log\frac{1 + x + x^2}{1 - x + x^2} = \log\frac{1 - x^3}{1 - x}\cdot\frac{1 + x}{1 + x^3} = \log\frac{1 + x}{1 - x} - \log\frac{1 + x^3}{1 - x^3}$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right) - 2\left(x^3 + \frac{x^9}{3} + \frac{x^{15}}{5} + \ldots\right).$$

If n is even, there is no term in x^n. If $n=6p+3$, coefficient of x^n is $\dfrac{2}{6p+3}-\dfrac{2}{2p+1}=-\dfrac{4}{6p+3}$. If $n=6p\pm1$, coefficient of x^n is $\dfrac{2}{6p\pm1}$.

10. $\log(1+2x+2x^2+x^3)=\log\dfrac{(1+x)(1-x^3)}{1-x}$

$$=\log\frac{1+x}{1-x}+\log(1-x^3)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right)-\left(x^3+\frac{x^6}{2}+\frac{x^9}{3}+\dots\right)$$

If $n\equiv0 \pmod 6$, coefficient is $-\dfrac{3}{n}$; if $n\equiv\pm1 \pmod 6$, it is $\dfrac{2}{n}$; if $n\equiv3 \pmod 6$, it is $\dfrac{2}{n}-\dfrac{3}{n}$; if $n\equiv\pm2 \pmod 6$, there is no x^n term.

11. Series $=\frac{1}{2}\log\dfrac{1+x}{1-x}$, $x^2<1$, $=\frac{1}{2}\log\dfrac{2ac+1+1}{2ac+1-1}=\frac{1}{2}\log\dfrac{ac+1}{ac}$, but $a=b-1, c=b+1$; $\therefore\ ac=b^2-1$;

$$\therefore\ \text{series}=\tfrac{1}{2}\log\frac{b^2}{ac}=\tfrac{1}{2}\{2\log b-\log a-\log c\}.$$

12. If $x^2<1$, $\log(1+x)-\log(1+x^3)$

$$=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\right)-\left(x^3-\frac{x^6}{2}+\frac{x^9}{3}-\dots\right);$$

$$-\log(1-x+x^2)=\sum_r\frac{1}{r}(x-x^2)^r \text{ if } |x-x^2|<1,$$

e.g. if $-\frac{3}{8}<x<1$; thus $\dfrac{(-1)^{n-1}}{3n}-\dfrac{(-1)^{n-1}}{n}=$ coefficient of of x^{3n} in $\sum_r$, taking terms in order $r=3n$, $3n-1$, $3n-2$, ...,

$$=\frac{1}{3n}-\frac{1}{3n-1}\,\frac{3n-1}{1!}+\frac{1}{3n-2}\,\frac{(3n-2)(3n-3)}{2!}-\dots$$

$$=\frac{1}{3n}-\text{required sum};$$

$$\text{sum}=\frac{1}{3n}+\frac{2(-1)^{n-1}}{3n}.$$

13. $-\log(1-\alpha x)-\log(1-\beta x)=-\log\{1-x(s-px)\}$;

$$\therefore\ \sum\frac{(\alpha x)^n}{n}+\sum\frac{(\beta x)^n}{n}=\sum\frac{x^n(s-px)^n}{n};$$

$\therefore\ \frac{1}{5}(\alpha^5+\beta^5)=$ coefficient x^5 in $\sum\frac{1}{n}x^n(s-px)^n$, and taking $n=5, 4, 3$, this is $\frac{1}{5}s^5-\frac{1}{4}\cdot4s^3p+\frac{1}{3}\cdot3sp^2$.* Similarly

$$\tfrac{1}{13}(\alpha^{13}+\beta^{13})=\tfrac{1}{13}s^{13}-\tfrac{1}{12}\cdot12s^{11}p+\tfrac{1}{11}\cdot\frac{11\cdot10}{2!}s^9p^2$$
$$-\tfrac{1}{10}\,\frac{10\cdot9\cdot8}{3!}s^7p^3+\tfrac{1}{9}\,\frac{9\cdot8\cdot7\cdot6}{4!}s^5p^4-\tfrac{1}{8}\cdot\frac{8\cdot7\cdot6}{3!}s^3p^5+\tfrac{1}{7}\cdot7sp^6.$$

14. $\log(1-\alpha x)+\log(1-\beta x)+\log(1-\gamma x)$
$=\log[(1-\alpha x)(1-\beta x)(1-\gamma x)]$
$=\log(1-sx^2-px^3)=\log[1-x^2(s+px)]$
$=-[x^2(s+px)+\frac{1}{2}x^4(s+px)^2+\frac{1}{3}x^6(s+px)^3+\dots]$.

Equate coefficients of x^3, $-\frac{1}{3}(\alpha^3+\beta^3+\gamma^3)=-p$; equate coefficients of x^5, $-\frac{1}{5}(\alpha^5+\beta^5+\gamma^5)=-(\frac{1}{2}\cdot2sp)$; equate coefficients of x^7, $-\frac{1}{7}(\alpha^7+\beta^7+\gamma^7)=-(\frac{1}{3}\cdot3s^2p)$.

15. Put $\Sigma\alpha=p$, $\Sigma(\alpha\beta)=q$, $\Sigma(\alpha\beta\gamma)=r$, $\alpha\beta\gamma\delta=s$; then
$\Sigma[\log(1-\alpha x)]=\log(1-px+qx^2-rx^3+sx^4)$
$=-[(px-qx^2+rx^3-sx^4)+\frac{1}{2}(px-qx^2\dots)^2+\frac{1}{3}(px\dots)^3+\dots]$;
equate coefficients of x^3, $-\frac{1}{3}\Sigma(\alpha^3)=-[r-pq+\frac{1}{3}p^3]$;
$\therefore\ \Sigma(\alpha^3)-3r=p^3-3pq=p(p^2-3q)$; but $p^2=(\Sigma\alpha)^2$
$=\Sigma(\alpha^2)+2\Sigma(\alpha\beta)$; $\therefore\ p^2-3q=\Sigma(\alpha^2)-\Sigma(\alpha\beta)$.

16. $\alpha+\beta=p$, $\alpha\beta=q$; $(1-\alpha y)(1-\beta y)=1-py+qy^2$;
$\therefore\ \log(1-\alpha y)+\log(1-\beta y)=\log[1-y(p-qy)]=$,
as in No. 12,
$-\sum\frac{1}{r}y^r(p-qy)^r$; equate coefficients of y^n, taking $r=n$, $n-1$, $n-2$, ...;

$$\frac{1}{n}(\alpha^n+\beta^n)=\frac{1}{n}p^n-\frac{1}{n-1}\cdot\frac{n-1}{1!}p^{n-2}q$$
$$+\frac{1}{n-2}\cdot\frac{(n-2)(n-3)}{2!}p^{n-4}q^2-\dots.$$

17. $\log(1+y\sqrt{2}+y^2)=\sum\dfrac{(-1)^{r-1}}{r}y^r(\sqrt{2}+y)^r$; taking $r=2n$, $2n-1$, $2n-2$, ...;

coefficient of y^{2n} is $-\dfrac{1}{2n}(\sqrt{2})^{2n}+\dfrac{1}{2n-1}\,\dfrac{2n-1}{1!}(\sqrt{2})^{2n-2}$
$$-\frac{1}{2n-2}\,\frac{(2n-2)(2n-3)}{2!}(\sqrt{2})^{2n-4}+\dots;$$
coefficient in $\log(1-y\sqrt{2}+y^2)$ is the same, and that in $\log(1+y^4)$ is zero if n is odd.

* Making the same assumption as in No. 12.

18. $\log [(1+x)^{1+x} . (1-x)^{1-x}]$

$$=(1+x)\log(1+x)+(1-x)\log(1-x)$$

$$=\log[(1+x)(1-x)]+x\log\frac{1+x}{1-x}=\log(1-x^2)+x\log\frac{1+x}{1-x}$$

$$=-\sum\frac{x^{2n}}{n}+2x\sum\frac{x^{2n-1}}{2n-1}=\sum\frac{x^{2n}}{n(2n-1)},$$

for $|x|<1$, and this is >0 because each term is >0; but if $\log y>0$, then $y>1$.

Put $x=\dfrac{a-b}{a+b}$, then $|x|<1$ and $x\neq 0$, since a, b are positive and unequal.

Then $1+x=\dfrac{2a}{a+b}$ and $1-x=\dfrac{2b}{a+b}$;

$$\therefore\ \left(\frac{2a}{a+b}\right)^{\frac{2a}{a+b}}\cdot\left(\frac{2b}{a+b}\right)^{\frac{2b}{a+b}}>1;\quad\therefore\ \left(\frac{2a}{a+b}\right)^{a}\cdot\left(\frac{2b}{a+b}\right)^{b}>1.$$

19. $x\log\dfrac{x+z}{x-z}=x\log\dfrac{1+\frac{z}{x}}{1-\frac{1}{x}}=2x\left(\dfrac{z}{x}+\dfrac{z^3}{3x^3}+\ldots\right)$ since $x>z$.

Similarly for $y\log\dfrac{y+z}{y-z}$,

$$x\log\frac{x+z}{x-z}-y\log\frac{y+z}{y-z}=\tfrac{2}{3}z^3\left(\frac{1}{x^2}-\frac{1}{y^2}\right)+\tfrac{2}{5}z^5\left(\frac{1}{x^4}-\frac{1}{y^4}\right)+\ldots<0$$

since each bracket <0; $\therefore\ \log\left(\dfrac{x+z}{x-z}\right)^x<\log\left(\dfrac{y+z}{y-z}\right)^y$.

20. (i) $\sin(\tan^{-1}x)\simeq\sin\left(x-\dfrac{x^3}{3}\right)\simeq x-\dfrac{x^3}{3}-\dfrac{1}{3!}x^3=x-\dfrac{x^3}{2}$;

(ii) $\tan(\sin^{-1}x)$, by V. a, No. 23, $\simeq\tan\left(x+\dfrac{x^3}{6}\right)\simeq$, by p. 82, $x+\dfrac{x^3}{6}+\tfrac{1}{3}x^3=x+\dfrac{x^3}{2}$;

(iii) $\tan^{-1}(\tan^{-1}x)\simeq\tan^{-1}\left(x-\dfrac{x^3}{3}\right)\simeq x-\dfrac{x^3}{3}-\tfrac{1}{3}x^3=x-\dfrac{2x^3}{3}$;

(iv) $\tan(\tan x)\simeq$, by p. 82, $\tan\left(x+\dfrac{x^3}{3}\right)$

$$\simeq x+\frac{x^3}{3}+\tfrac{1}{3}x^3=x+\frac{2x^3}{3};$$

(v) $\sin(\sin x)\simeq\sin\left(x-\dfrac{x^3}{6}\right)\simeq x-\dfrac{x^3}{6}-\dfrac{x^3}{6}=x-\dfrac{x^3}{3}$;

(vi) $\sin^{-1}(\sin^{-1}x)\simeq\sin^{-1}\left(x+\dfrac{x^3}{6}\right)\simeq x+\dfrac{x^3}{6}+\tfrac{1}{6}x^3\simeq x+\dfrac{x^3}{3}$.

21. $u_r=\dfrac{2}{r!}-\dfrac{3}{(r+1)!}+\dfrac{1}{(r+2)!}$; writing out the terms in this form we see that s_n reduces to $\dfrac{2}{1!}-\dfrac{1}{2!}-\dfrac{2}{(n+1)!}+\dfrac{1}{(n+2)!}$;

$\therefore$ when $n\to\infty$, $s_n\to 2-\tfrac{1}{2}$.

22. Since $xe^x>0$, using p. 79, $\displaystyle\int_0^x\left[\int_0^x\left\{\int_0^x(xe^x)\,dx\right\}dx\right]dx>0$; also $\displaystyle\int_0^x xe^x\,dx=xe^x-e^x+1$.

23. $\left(1+\dfrac{1}{x}\right)^x=e^{x\log\left(1+\frac{1}{x}\right)}$; but

$$x\log\left(1+\frac{1}{x}\right)\simeq x\left(\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}-\frac{1}{4x^4}\right)\text{ if } x \text{ is large,}$$

$$=1-\frac{1}{2x}+\frac{1}{3x^2}-\frac{1}{4x^3};$$

$$\therefore\ \left(1+\frac{1}{x}\right)^x\simeq e^{1-\frac{1}{2x}+\frac{1}{3x^2}-\frac{1}{4x^3}}=e\,.\,e^{-\frac{1}{2x}+\frac{1}{3x^2}-\frac{1}{4x^3}}$$

$$\simeq e\left\{1+\left(-\frac{1}{2x}+\frac{1}{3x^2}-\frac{1}{4x^3}\right)+\frac{1}{2!}\cdot\left(-\frac{1}{2x}+\frac{1}{3x^2}\right)^2+\frac{1}{3!}\cdot\left(-\frac{1}{2x}\right)^3\right\}.$$

24. For two approximations see Ex. V. e, No. 41.

3rd approx., $x=ae^{-\frac{p}{2+p}\log a}$

$$\simeq a\left\{1-\frac{p}{2+p}\log a+\frac{1}{2!}\frac{p^2}{(2+p)^2}(\log a)^2\right\}$$

$$\simeq a\left\{1-\frac{p}{2}\left(1+\frac{p}{2}\right)^{-1}\log a+\tfrac{1}{8}p^2(\log a)^2\right\}$$

$$\simeq a\left\{1-\frac{p}{2}\left(1-\frac{p}{2}\right).\log a+\tfrac{1}{8}p^2(\log a)^2\right\}.$$

25. $e^{(e^x)}=1+e^x+\dfrac{1}{2!}e^{2x}+\ldots+\dfrac{1}{r!}e^{rx}+\ldots$; the coefficient of x^n in $\dfrac{1}{r!}e^{rx}$ is $\dfrac{1}{r!}\cdot\dfrac{r^n}{n!}$; $\therefore$ coefficient of x^n in $e^{(e^x)}$ is

$$\frac{1}{1!}\cdot\frac{1^n}{n!}+\frac{1}{2!}\cdot\frac{2^n}{n!}+\frac{1}{3!}\cdot\frac{3^n}{n!}+\ldots=\frac{1}{n!}\left\{\frac{1^n}{1!}+\frac{2^n}{2!}+\frac{3^n}{3!}+\ldots\right\};$$

also $e^{(e^x)} = e^{1+x+\frac{1}{2!}x^2+\dots} = e \cdot e^{x+\frac{1}{2!}x^2+\dots}$

$$= e\left\{1+\left(x+\frac{x^2}{2!}+\dots\right)+\frac{1}{2!}\left(x+\frac{x^2}{2!}+\dots\right)^2 + \frac{1}{3!}\left(x+\frac{x^2}{2!}+\dots\right)^3+\dots\right\};$$

the coefficient of x^4 in this is

$$e\left\{\frac{1}{4!}+\frac{1}{2!}\left[\frac{2}{3!}+\frac{1}{(2!)^2}\right]+\frac{1}{3!}\left(\frac{3}{2!}\right)+\frac{1}{4!}\right\}$$

$$= e\left(\tfrac{1}{24}+\tfrac{7}{24}+\tfrac{3}{12}+\tfrac{1}{24}\right) = \frac{5e}{8};$$

$$\therefore \frac{1}{4!}\left\{\frac{1^4}{1!}+\frac{2^4}{2!}+\frac{3^4}{3!}+\dots\right\} = \frac{5e}{8};$$

$$\therefore \text{ series} = \frac{5e}{8}\times 24 = 15e.$$

26. $\frac{1}{e} = e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}$

$$+\frac{(-1)^{n+1}}{(n+1)!}\left\{1-\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}-\dots\right\};$$

but the terms in the bracket steadily decrease in absolute value and are alternately positive and negative; $\therefore$ the series in the bracket converges to a sum σ, such that

$1 > \sigma > 1 - \frac{1}{n+2}$;

$$\therefore \left|\frac{1}{e}-\left\{\frac{1}{2!}-\frac{1}{3!}-\dots+\frac{(-1)^n}{n!}\right\}\right| < \frac{1}{(n+1)!};$$

to obtain result, multiply each side by $n!$

27. $\log\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\dots\left(1+\frac{n}{n^2}\right)\right]$

$$= \Sigma \log\left(1+\frac{r}{n^2}\right) < \frac{1}{n^2}+\frac{2}{n^2}+\dots+\frac{n}{n^2} = \frac{1+2+\dots+n}{n^2}$$

$$= \frac{n(n+1)}{2n^2} = \tfrac{1}{2}+\frac{1}{2n};$$

$$\therefore \left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\dots\left(1+\frac{n}{n^2}\right) < e^{\frac{1}{2}+\frac{1}{2n}} = \sqrt{e}\cdot\sqrt[2n]{e}.$$

Similarly, $\log\left[\left(1+\frac{1}{n^2}\right)\dots\dots\left(1+\frac{n}{n^2}\right)\right]$

$$> \frac{1}{\frac{n^2}{1}+1}+\frac{1}{\frac{n^2}{2}+1}+\dots+\frac{1}{\frac{n^2}{n}+1}$$

$$= \frac{1}{n^2+1}+\frac{2}{n^2+2}+\dots+\frac{n}{n^2+n} > \frac{1}{n^2+n}+\frac{2}{n^2+n}+\dots+\frac{n}{n^2+n}$$

$$= \frac{n(n+1)}{2(n^2+n)} = \tfrac{1}{2}; \quad \therefore \text{ product} > e^{\frac{1}{2}} = \sqrt{e}.$$

28. $e^{nx} - \frac{n}{1!}e^{(n-1)x} + \frac{n(n-1)}{2!}e^{(n-2)x} - \dots = \left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)^n.$

There is no term in x^{n-1} on the right side; $\therefore$ coefficient of x^{n-1} on left side is zero; this is

$$\frac{1}{(n-1)!}\left\{n^{n-1}-\frac{n}{1!}\cdot(n-1)^{n-1}+\frac{n(n-1)}{2!}(n-2)^{n-1}-\dots\right\}.$$

(i) The right side $= x^n\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\dots\right)^n$

$$= x^n\left[1+n\left(\frac{x}{2}+\frac{x^2}{6}+\dots\right)+\frac{n(n-1)}{2!}\left(\frac{x}{2}+\dots\right)^2+\dots\right]$$

$$= x^n + \frac{n}{2}x^{n+1} + \frac{n(3n+1)}{24}x^{n+2}+\dots$$

equate coefficients of x^n; then

$$\frac{1}{n!}\left[n^n - \frac{n}{1!}(n-1)^n + \frac{n(n-1)}{2!}(n-2)^n - \dots\right] = 1.$$

(ii) Equate coefficients of x^{n+1}

$$\frac{1}{(n+1)!}\left[n^{n+1}-\frac{n}{1!}(n-1)^{n+1}+\frac{n(n-1)}{2!}(n-2)^{n+1}-\dots\right] = \frac{n}{2}.$$

(iii) Equate coefficients of x^{n+2} and proceed as before.

29. $(e^x - e^{-x})^n = e^{nx} - n\cdot e^{(n-2)x} + \frac{n(n-1)}{2!}e^{(n-4)x} - \dots$; but

$$(e^x - e^{-x})^n = \left[2\left(x+\frac{x^3}{3!}+\dots\right)\right]^n = 2^n\cdot x^n\left(1+\frac{x^2}{3!}+\dots\right)^n;$$

equate coefficients of x^n; then

$$\frac{1}{n!}\left[n^n - n.\ (n-2)^n + \frac{n(n-1)}{2!}\cdot(n-4)^n - \dots\right] = 2^n.$$

30. $e^x\cdot(1-e^x)^n = e^x - n\cdot e^{2x} + \frac{n(n-1)}{2!}e^{3x} - \dots$; but

$$e^x(1-e^x)^n = e^x\left(-x-\frac{x^2}{2!}-\dots\right)^n$$

$$= (-1)^n\cdot e^x\cdot x^n\cdot\left(1+\frac{x}{2!}+\dots\right)^n.$$

equate coefficients of x^n; then

$$\frac{1}{n!}\left[1^n - n.\ 2^n + \frac{n(n-1)}{2!}3^n - \dots\right] = (-1)^n.$$

31. $(e^x+1)^n-(e^x-1)^n=2\{c_1 . e^{(n-1)x}+c_3 . e^{(n-3)x}+\ldots\}$; also the expression $=\left(2+x+\frac{x^2}{2!}+\ldots\right)^n-\left(x+\frac{x^2}{2!}+\ldots\right)^n$; equate coefficients of x^3; then

$$\frac{2}{3!}\{c_1(n-1)^3+c_3(n-3)^3+\ldots\}=\text{coefficient of } x^3 \text{ in}$$

$$2^n+n . 2^{n-1}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)$$

$$+\frac{n(n-1)}{2!} . 2^{n-2}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)^2+\ldots, \text{ which is}$$

$$n . 2^{n-1} . \frac{1}{3!}+\frac{n(n-1)}{2!} . 2^{n-2} . \frac{2}{2!}+\frac{n(n-1)(n-2)}{3!} . 2^{n-3} . 1$$

$$=\frac{n . 2^{n-3}}{6}\{4+6(n-1)+(n-1)(n-2)\}=\frac{n . 2^{n-3}}{6} . (n^2+3n).$$

32. Let $y=\frac{(n+1+x)^{n+1}}{(n+x)^n}$;

$$\log y=(n+1)\log(n+1+x)-n\log(n+x);$$

$$\therefore \frac{1}{y}\frac{dy}{dx}=\frac{n+1}{n+1+x}-\frac{n}{n+x}=\frac{x}{(n+1+x)(n+x)}>0,$$

since $x>0$ and $n>0$; but $y>0$; $\therefore \frac{dy}{dx}>0$; $\therefore y$ increases steadily as x increases;

$$\therefore \frac{(n+1+x)^{n+1}}{(n+x)^n}>\frac{(n+1+0)^{n+1}}{(n+0)^n} \text{ for } x>0;$$

$$\therefore \left(\frac{n+x}{n}\right)^n<\left(\frac{n+1+x}{n+1}\right)^{n+1};$$

$$\therefore \left(1+\frac{x}{n}\right)^n<\left(1+\frac{x}{n+1}\right)^{n+1}.$$

33. Let $y=\frac{(n+1-x)^{n+1}}{(n-x)^n}$;

$$\therefore \log y=(n+1)\log(n+1-x)-n\log(n-x);$$

$$\therefore \frac{1}{y}\frac{dy}{dx}=\frac{-(n+1)}{n+1-x}-\frac{-n}{n-x}=\frac{x}{(n+1-x)(n-x)};$$

$$\therefore \frac{dy}{dx}>0 \text{ for } 0<x<n, \text{ since } y>0 \text{ for } 0<x<n;$$

$\therefore y$ increases steadily as x increases;

$$\therefore \frac{(n+1-x)^{n+1}}{(n-x)^n}>\frac{(n+1-0)^{n+1}}{(n-0)^n};$$

$$\therefore \left(\frac{n+1-x}{n+1}\right)^{n+1}>\left(\frac{n-x}{n}\right)^n;$$

$$\therefore \left(1-\frac{x}{n+1}\right)^{n+1}>\left(1-\frac{x}{n}\right)^n.$$

CHAPTER VI

EXERCISE VI. a. (p. 106.)

1-10. All these results are obtained by expressing each function in its exponential form as on p. 105.

11. From eqn. (6), $\frac{\text{ch}^2 x}{\text{sh}^2 x}-1=\frac{1}{\text{sh}^2 x}$.

12. Use $\text{sh } 3x=3 \text{ sh } x+4 \text{ sh}^3 x$. 13. Use eqn. (6).

17. From eqn. (6), $(\text{ch } x+\text{sh } x)(\text{ch } x-\text{sh } x)=1$.

18. $\text{ch } x+\text{sh } x=\frac{1}{2}(e^x+e^{-x})+\frac{1}{2}(e^x-e^{-x})=e^x$;

$$\therefore (\text{ch } x+\text{sh } x)^n=e^{nx}=\text{ch}(nx)+\text{sh}(nx).$$

19. As in No. 18, $(\text{ch } x-\text{sh } x)^n=(e^{-x})^n=e^{-nx}=\text{ch}(nx)-\text{sh}(nx)$.

20. $\text{th}(x+y+z)=\frac{\text{th } x+\text{th}(y+z)}{1+\text{th } x . \text{th}(y+z)}$; put $\text{th}(y+z)=\frac{\text{th } y+\text{th } z}{1+\text{th } y . \text{th } z}$.

21. l.h.s. $=\frac{1}{2}(\text{ch } 2x+\text{ch } 2y)=\frac{1}{2}\{(2 \text{ ch}^2 x-1)+(1+2 \text{ sh}^2 y)\}$.

22. l.h.s. $=\frac{\text{ch } \theta+\text{sh } \theta}{\text{ch } \theta-\text{sh } \theta}=\frac{e^\theta}{e^{-\theta}}$, as in No. 18, $=e^{2\theta}=\text{ch}(2\theta)+\text{sh}(2\theta)$.

23. $\text{ch } \theta$ is essentially +; $\therefore \text{ch } \theta=+\sqrt{(1+\text{sh}^2\theta)}$;

$$\text{th } \theta=\text{sh } \theta \div \text{ch } \theta.$$

24. $\text{ch } \theta=\frac{1}{\text{sech } \theta}=\frac{1}{+\sqrt{(1-\text{th}^2\theta)}}$, see No. 23; $\text{sh } \theta=\text{th } \theta . \text{ch } \theta$.

25. $\text{sh } \theta=\pm\sqrt{(\text{sh}^2\theta)}=\pm\sqrt{\{\frac{1}{2}(\text{ch } 2\theta-1)\}}=\pm\sqrt{\{\frac{1}{2}(k-1)\}}$;

$$\text{ch } \theta=+\sqrt{(\text{ch}^2\theta)}=+\sqrt{\{\tfrac{1}{2}(\text{ch } 2\theta+1)\}}$$

$$=+\sqrt{\{\tfrac{1}{2}(k+1)\}}; \ \text{th } \theta=\text{sh } \theta \div \text{ch } \theta.$$

26. $\text{sh } \theta=\frac{2 \text{ sh } \frac{1}{2}\theta . \text{ch } \frac{1}{2}\theta}{\text{ch}^2\frac{1}{2}\theta-\text{sh}^2\frac{1}{2}\theta}$ and $\text{ch } \theta=\frac{\text{ch}^2\frac{1}{2}\theta+\text{sh}^2\frac{1}{2}\theta}{\text{ch}^2\frac{1}{2}\theta-\text{sh}^2\frac{1}{2}\theta}$; divide numerator and denominator by $\text{ch}^2\frac{1}{2}\theta$.

27. $\left(\frac{x}{\sin u}\right)^2 - \left(\frac{y}{\cos u}\right)^2 = \text{ch}^2 v - \text{sh}^2 v = 1$;

$\left(\frac{x}{\text{ch}\, v}\right)^2 + \left(\frac{y}{\text{sh}\, v}\right)^2 = \sin^2 u + \cos^2 u = 1.$

28. l.h.s. $= \frac{\text{ch}\,\frac12\theta}{\text{sh}\,\frac12\theta} - \frac{\text{ch}\,\theta}{\text{sh}\,\theta} = \frac{2\,\text{ch}^2\frac12\theta}{\text{sh}\,\theta} - \frac{2\,\text{ch}^2\frac12\theta - 1}{\text{sh}\,\theta} = \frac{1}{\text{sh}\,\theta}$.

29. $\tan(\theta+\phi) = \frac{\tan\alpha\,\text{th}\,\beta + \cot\alpha\,\text{th}\,\beta}{1 - \tan\alpha\,\text{th}\,\beta\,.\,\cot\alpha\,\text{th}\,\beta} = \frac{\text{th}\,\beta(\tan\alpha+\cot\alpha)}{1-\text{th}^2\beta}$

$= \frac{\text{th}\,\beta}{\text{sech}^2\beta}\cdot\frac{1}{\sin\alpha\cos\alpha} = \frac{\text{sh}\,\beta\,\text{ch}\,\beta}{\sin\alpha\cos\alpha}$.

30. l.h.s. $=\frac12\{[1+\text{ch}(2\theta+2\phi)] - [1+\text{ch}(2\theta-2\phi)]\}$

$=\frac12[\text{ch}(2\theta+2\phi) - \text{ch}(2\theta-2\phi)]$.

31. l.h.s. $=\frac12\sin^2\theta(1+\text{ch}\,2\phi) + \frac12\cos^2\theta(\text{ch}\,2\phi - 1)$

$=\frac12\,\text{ch}\,2\phi(\sin^2\theta+\cos^2\theta) + \frac12(\sin^2\theta-\cos^2\theta)$

$=\frac12\,\text{ch}\,2\phi + \frac12(-\cos 2\theta)$.

32. Expression $= \frac{2\,\text{ch}^2\frac12\phi + 2\,\text{sh}\,\frac12\phi\,.\,\text{ch}\,\frac12\phi}{-2\,\text{sh}^2\frac12\phi - 2\,\text{sh}\,\frac12\phi\,.\,\text{ch}\,\frac12\phi}$

$= -\frac{\text{ch}\,\frac12\phi(\text{ch}\,\frac12\phi+\text{sh}\,\frac12\phi)}{\text{sh}\,\frac12\phi(\text{sh}\,\frac12\phi+\text{ch}\,\frac12\phi)} = -\frac{\text{ch}\,\frac12\phi}{\text{sh}\,\frac12\phi}$.

33. Expression $=2\,\text{sh}(x+y)\,.\,\text{ch}(x-y) - 2\,\text{ch}(x+y+2z)\,\text{sh}(x+y)$

$=2\,\text{sh}(x+y)\{\text{ch}(x-y) - \text{ch}(x+y+2z)\}$

$=2\,\text{sh}(x+y)\{-2\,\text{sh}(x+z)\,\text{sh}(y+z)\}$.

34. $\sin^2x\,\text{ch}^2y + \cos^2x\,\text{sh}^2y = \cos^2\alpha+\sin^2\alpha = 1$;

∴ $(1-\cos^2x)(1+\text{sh}^2y) + \cos^2x\,\text{sh}^2y = 1$;

∴ $-\cos^2x + \text{sh}^2y = 0$; but $\cos^2x\,\text{sh}^2y = \sin^2\alpha$;

∴ $\cos^2x = \text{sh}^2y = \pm\sin\alpha$.

35. $e^{\log x}=x$; ∴ $\text{sh}(\log x) = \frac12(x-x^{-1})$; $\text{ch}(\log x) = \frac12(x+x^{-1})$.

36. Let sum $=s$, then $2s\,.\,\text{sh}\,\frac12x = \Sigma[\text{sh}\,\frac12(2r+1)x - \text{sh}\,\frac12(2r-1)x]$ for $r=1$ to n, $=\text{sh}\,\frac12(2n+1)x - \text{sh}\,\frac12x$. Cf. p. 127.

EXERCISE VI. b. (p. 108.)

1-9. Use eqns. (11), (12), (13).

10. $\frac{1}{\text{th}\,\frac12x}\cdot\frac12\,\text{sech}^2\frac12x = \frac{1}{2\,\text{sh}\,\frac12x\,\text{ch}\,\frac12x} = \frac{1}{\text{sh}\,x}$.

11. $\frac{1}{1+\coth^2x}\cdot(-\text{cosech}^2x) = -\frac{1}{\text{sh}^2x+\text{ch}^2x} = -\frac{1}{\text{ch}\,2x}$.

12. $\log(\text{sh}\,x+\text{ch}\,x) = \log(e^x) = x$.

13-16. By inspection.

17. Use $\text{sh}^2x = \frac12(\text{ch}\,2x - 1)$.

18. See No. 7.

19. Use $\text{th}^2x = 1-\text{sech}^2x$ and eqn. (13).

20. Use $\coth^2x = 1+\text{cosech}^2x$ and No. 18.

21. Put $\text{th}\,\frac12x = t$, $\frac12\,\text{sech}^2\frac{x}{2}\,.\,dx = dt$; ∴ $dx = \frac{2\,dt}{1-\text{th}^2\frac12x} = \frac{2\,dt}{1-t^2}$;
also from VI. a, No. 26, $\text{ch}\,x = \frac{1+t^2}{1-t^2}$;

∴ integral $=2\int\frac{dt}{1+t^2} = 2\tan^{-1}t$.

22. As in No. 21; $\text{sh}\,x = \frac{2t}{1-t^2}$; integral $=\int\frac{dt}{t} = \log t$.

23. Use $\text{sh}\,x\,.\,\text{sh}\,2x = \frac12(\text{ch}\,3x - \text{ch}\,x)$.

24. Use $\text{ch}^3x = \frac14(\text{ch}\,3x + 3\,\text{ch}\,x)$.

25. $(\cos x\,.\,\text{sh}\,x - \text{ch}\,x\,.\,\sin x) + (\sin x\,.\,\text{ch}\,x + \text{sh}\,x\,.\,\cos x)$.

26. Integrating by parts, $p \equiv \int \text{ch}\,x\sin x\,dx = \sin x\,.\,\text{sh}\,x - q$, where $q = \int\text{sh}\,x\cos x\,dx$, by parts again, $=\cos x\,\text{ch}\,x + p$;

∴ $p = \sin x\,\text{sh}\,x - \cos x\,\text{ch}\,x - p$.

27. $\frac{dy}{dx} = an\,\text{ch}\,(nx) + bn\,\text{sh}\,(nx)$; $\frac{d^2y}{dx^2} = an^2\,\text{sh}\,(nx) + bn^2\,\text{ch}\,(nx) = n^2y$

28. (i) By eqns. (1) and (2);

(ii) $\frac{d}{dx}(\text{sh}\,x) = \text{ch}\,x$ is +; ∴ sh (x) increases with x;

$\frac{d}{dx}(\text{ch}\,x) = \text{sh}\,(x)$

has the same sign as x, so ch (x) decreases with x if x is − and increases with x if x is +;

(iii) $\frac{d}{dx}(\text{ch}\,x) = 0$ if $\text{sh}\,x = 0$, i.e. if $x = 0$;

minimum value $=\text{ch}\,(0) = 1$.

Or $\text{ch}\,x \equiv \frac12(e^{\frac12x} - e^{-\frac12x})^2 + 1$; ∴ $\text{ch}\,x \geqslant 1$;

(iv) $\text{ch}\,x = \frac12(e^x+e^{-x}) \to \infty$ when $|x|\to\infty$;

(v) $\text{sh}\,x = \frac12(e^x - e^{-x})\to\infty$ when $x\to\infty$ and $\text{sh}\,x\to-\infty$ when $x\to-\infty$;

(vi) and (vii) $\frac{\text{ch}\,x}{e^x} = \frac12(1+e^{-2x})\to\frac12$ when $x\to\infty$; similarly $\frac{\text{ch}\,x}{e^{-x}} = \frac12(e^{2x}+1)$.

29. (i) $\text{th}(-x) = -\text{th}(x)$; $\therefore$ $\text{th}(x)$ is an odd function of x; $\coth x = \dfrac{1}{\text{th}\, x}$;

(ii) $\dfrac{d}{dx}(\text{th}\, x) = \text{sech}^2 x$ is +; $\coth x$ steadily decreases as x increases;

(iii) $\text{th}\, x = \dfrac{1-e^{-2x}}{1+e^{-2x}} \to 1$, when $x \to \infty$; $\text{th}\, x = \dfrac{e^{2x}-1}{e^{2x}+1} \to -1$ when $x \to -\infty$; use $\coth x = \dfrac{1}{\text{th}\, x}$;

(iv) $\text{th}\, x = \dfrac{e^{2x}-1}{e^{2x}+1} \to 0$ when $|x| \to 0$;

(v) $\lim\limits_{x\to 0} \dfrac{\text{th}\, x}{x} = 1$; $\therefore$ slope $= 1$;

(vi) $\text{th}^2 x = 1 - \text{sech}^2 x < 1$; $\therefore$ $|\text{th}\, x| < 1$ and $\coth x = \dfrac{1}{\text{th}\, x}$.

30. (i) $0 < \text{sech}\, x \leqslant 1$; also when $|x| \to \infty$, $\text{sech}\, x \to 0$; $\text{sech}\, x$ attains its max. value, 1, at $x = 0$; $\text{sech}(-x) = \text{sech}(x)$.

(ii) $\text{cosech}\, x$ has the same sign as x. When $|x| \to \infty$, $\text{cosech}\, x \to 0$. When $x \to +0$, $\text{cosech}\, x \to +\infty$; $\text{cosech}(-x) = -\text{cosech}(x)$.

EXERCISE VI. c. (p. 112.)

1. If $\text{ch}^{-1} y = x$, $y = \text{ch}\, x = \frac{1}{2}(e^x + e^{-x})$; $\therefore$ $e^{2x} - 2y \,.\, e^x + 1 = 0$;

$\therefore$ $e^x = y \pm \sqrt{(y^2-1)}$, but $y - \sqrt{(y^2-1)} = \dfrac{1}{y + \sqrt{(y^2-1)}}$;

$\therefore$ $x = \log\{y + \sqrt{(y^2-1)}\}$ or $-\log\{y + \sqrt{(y^2-1)}\}$.

2. If $\text{th}^{-1} y = x$, $y = \text{th}\, x = \dfrac{e^{2x}-1}{e^{2x}+1}$;

$$\therefore\ e^{2x} = \frac{1+y}{1-y};\quad \therefore\ 2x = \log\frac{1+y}{1-y}.$$

3. The graph of $y = \text{ch}^{-1} x$ (i.e. $x = \text{ch}\, y$) is the image of $y = \text{ch}\, x$ in the line $y = x$; see VI. b, No. 28; similarly for $y = \text{sh}^{-1} x$.

4. As in No. 3. See VI. b, No. 29.

5. $\text{sech}^{-1} y = \text{ch}^{-1}\left(\dfrac{1}{y}\right)$; use No. 1.

6. By eqn. (14),

$$\text{cosech}^{-1} y = \text{sh}^{-1}\left(\frac{1}{y}\right) = \log\left[\frac{1}{y} + \left|\sqrt{\left(1+\frac{1}{y^2}\right)}\right|\right]$$

$$= \log\left[\frac{1+\sqrt{(y^2+1)}}{y}\right] \text{ if } y > 0$$

and

$$= \log\left[\frac{1-\sqrt{(y^2+1)}}{y}\right] \text{ if } y < 0.$$

7. Either use eqn. (14); **Or** if $\text{sh}^{-1} x = y$, $x = \text{sh}\, y$; $\therefore$ $1 = \text{ch}\, y \,.\, \dfrac{dy}{dx}$; but $\text{ch}\, y$ is +; $\therefore$ $\text{ch}\, y = |\sqrt{(1+\text{sh}^2 y)}| = +\sqrt{(1+x^2)}$.

8. Either use No. 1; **Or** if $\text{ch}^{-1} x = y$, $x = \text{ch}\, y$; $\therefore$ $1 = \text{sh}\, y \,.\, \dfrac{dy}{dx}$; but $\text{sh}\, y = \pm\sqrt{(\text{ch}^2 y - 1)} = \pm\sqrt{(x^2-1)}$; $\text{ch}^{-1} x$ is a two-valued function, see No. 1; the sign of $\dfrac{d}{dx}(\text{ch}^{-1} x)$ is the same as the sign of $\text{ch}^{-1} x$.

9. Either use No. 2; **Or**, if $\text{th}^{-1} x = y$, $x = \text{th}\, y$; $\therefore$ $1 = \text{sech}^2 y \,.\, \dfrac{dy}{dx}$; but $\text{sech}^2 y = 1 - \text{th}^2 y = 1 - x^2$.

10. If $\coth^{-1} x = y$, $x = \coth y$;

$$\therefore\ 1 = -\text{cosech}^2 y \,.\, \frac{dy}{dx} = -(\coth^2 y - 1)\frac{dy}{dx} = -(x^2-1) \,.\, \frac{dy}{dx}.$$

11. (i) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \text{ch}^2 u - \text{sh}^2 u = 1$;

(ii) Put $u + \beta = v$; $\alpha - \beta = \gamma$; then $x = a\,\text{ch}(v+\gamma)$, $y = b\,\text{sh}\, v$;

$\therefore$ $\dfrac{x}{a} = \text{ch}\, v\, \text{ch}\, \gamma + \text{sh}\, v\, \text{sh}\, \gamma$; $\therefore$ $\dfrac{x}{a} - \dfrac{y}{b}\text{sh}\, \gamma = \text{ch}\, v\, \text{ch}\, \gamma$;

but $\dfrac{y}{b}\text{ch}\, \gamma = \text{sh}\, v\, \text{ch}\, \gamma$;

$$\therefore\ \left(\frac{x}{a} - \frac{y}{b}\text{sh}\, \gamma\right)^2 - \left(\frac{y}{b}\text{ch}\, \gamma\right)^2 = \text{ch}^2 \gamma\,(\text{ch}^2 v - \text{sh}^2 v) = \text{ch}^2 \gamma; \text{ expand.}$$

12. $\dfrac{x}{a} + \dfrac{y}{b} = \text{ch}(u+\alpha) + \text{ch}(u+\beta) = 2\,\text{ch}\left(u + \dfrac{\alpha+\beta}{2}\right)\text{ch}\,\dfrac{\alpha-\beta}{2}$, and

$$\frac{x}{a} - \frac{y}{b} = 2\,\text{sh}\left(u + \frac{\alpha+\beta}{2}\right)\text{sh}\,\frac{\alpha-\beta}{2};$$

$$\therefore\ \left(\frac{x}{a} + \frac{y}{b}\right)^2 \text{sech}^2\frac{\alpha-\beta}{2} - \left(\frac{x}{a} - \frac{y}{b}\right)^2 \text{cosech}^2\frac{\alpha-\beta}{2} = 4;$$

a hyperbola because the second degree terms factorize.

13. Chord is

$$\begin{vmatrix} x & y & a \\ \text{ch}\,\theta & \text{sh}\,\theta & 1 \\ \text{ch}\,\phi & \text{sh}\,\phi & 1 \end{vmatrix} = 0$$

or $x(\text{sh}\,\theta - \text{sh}\,\phi) - y(\text{ch}\,\theta - \text{ch}\,\phi)$
$= a\,\text{sh}(\theta - \phi) = 2a\,\text{sh}\,\dfrac{\theta-\phi}{2}\,\text{ch}\,\dfrac{\theta-\phi}{2}$;
now use VI. a, No. 7.

14. $x = a\,\text{ch}\, u$, $y = b\,\text{sh}\, u$;

$$\text{area} = \int y\, dx = \int_0^\theta b\,\text{sh}\, u \,.\, a\,\text{sh}\, u \,.\, du$$

$$= \tfrac{1}{2}ab\int_0^\theta \text{ch}\,(2u - 1)\, du = \tfrac{1}{2}ab\,(\tfrac{1}{2}\text{sh}\, 2\theta - \theta).$$

If O is centre and PN is ordinate,

$$\triangle \text{OPN} = \tfrac{1}{2}ab \text{ ch } \theta \,.\, \text{sh } \theta = \tfrac{1}{4}ab \text{ sh } 2\theta\,;$$

$$\text{and sector} = \triangle\text{OPN} - \tfrac{1}{2}ab(\tfrac{1}{2}\text{ sh } 2\theta - \theta) = \tfrac{1}{2}ab\theta.$$

15. Integral $=\int \frac{a \text{ sech}^2\theta\, d\theta}{a^2 - a^2\text{th}^2\theta} = \frac{1}{a}\int \frac{\text{sech}^2\theta}{\text{sech}^2\theta}d\theta = \frac{1}{a}\int d\theta = \frac{\theta}{a}$;

$$\frac{1}{a^2-x^2} = \frac{1}{2a}\left(\frac{1}{a+x}+\frac{1}{a-x}\right);$$

$$\text{integral} = \frac{1}{2a}\{\log(a+x) - \log(a-x)\}\,; \text{ see No. 2.}$$

16. (i) For $x > 0$, put $x = |a|\,.\,\text{ch }\theta$;

$$\text{integral} = \int \frac{|a|\,.\,\text{sh }\theta\, d\theta}{|a|\,.\sqrt{(\text{ch}^2\theta - 1)}} = \int d\theta = \theta = \text{ch}^{-1}\left(\frac{x}{|a|}\right)$$

$$= \text{ch}^{-1}\left(\left|\frac{x}{a}\right|\right) \text{ since } x > 0\,;$$

(ii) For $x < 0$, put $x = -|a|\,.\,\text{ch }\theta$, then integral reduces to

$$-\int d\theta = -\theta = -\text{ch}^{-1}\left(\frac{-x}{|a|}\right) = -\text{ch}^{-1}\left(\left|\frac{x}{a}\right|\right) \text{ since } x < 0.$$

17. Put $x = |a|\,.\,\text{sh }\theta$. This can be done whether x is + or −.

$$\text{Integral} = \int |a|\,.\,\text{ch }\theta\,.\,|a|\text{ ch }\theta\, d\theta$$

$$= \tfrac{1}{2}a^2\int(1+\text{ch } 2\theta)d\theta = a^2(\tfrac{1}{2}\theta + \tfrac{1}{4}\text{ sh } 2\theta)$$

$$= \tfrac{1}{2}a^2\left[\text{sh}^{-1}\left(\frac{x}{|a|}\right) + \frac{x}{a^2}\cdot\sqrt{(x^2+a^2)}\right].$$

18. As in Example 4, p. 111, if $x = 2\text{ ch }\theta$, integral $= [\text{sh } 2\theta - 2\theta]$

$$= \left[\frac{x}{2}\sqrt{(x^2-4)} - 2\text{ ch}^{-1}(\tfrac{1}{2}x)\right]_2^{2\frac{1}{2}}$$

$$= \tfrac{5}{4}\,.\,\tfrac{3}{2} - 2\text{ ch}^{-1}(\tfrac{5}{4}) = \tfrac{15}{8} - 2\log(\tfrac{5}{4}+\tfrac{3}{4}).$$

19. Put $x = 2\text{ ch }\theta$; integral $= \int \frac{2\text{ sh }\theta\, d\theta}{2\text{ sh }\theta} = \int d\theta = \theta$

$$= \left[\text{ch}^{-1}(\tfrac{1}{2}x)\right]_2^{2\frac{1}{2}} = \text{ch}^{-1}(\tfrac{5}{4}) = \log(\tfrac{5}{4}+\tfrac{3}{4}).$$

20. Put $x = -y$; integral $= \int_4^5 \frac{1}{\sqrt{(y^2+9)}}dy$; put $y = 3\text{ sh }\theta$; then as in Example 3, p. 111,

$$\text{integral} = \left[\text{sh}^{-1}\frac{y}{3}\right]_4^5 = \text{sh}^{-1}(\tfrac{5}{3}) - \text{sh}^{-1}(\tfrac{4}{3})$$

$$= \log\left(\tfrac{5}{3} + \frac{\sqrt{34}}{3}\right) - \log(\tfrac{4}{3}+\tfrac{5}{3}) \text{ by eqn. (14),}$$

$$= \log(5+\sqrt{34}) - \log 9.$$

21. Put $x = 3\text{ sh }\theta$;

$$\text{integral} = \int 3\text{ ch }\theta\,.\,3\text{ ch }\theta\, d\theta = \tfrac{9}{2}\int(1+\text{ch } 2\theta)\,d\theta$$

$$= \tfrac{9}{2}(\theta + \tfrac{1}{2}\text{ sh } 2\theta) = \tfrac{9}{2}\left\{\text{sh}^{-1}\left(\frac{x}{3}\right) + \frac{x}{3}\cdot\sqrt{\left(1+\frac{x^2}{9}\right)}\right\}$$

$$= \tfrac{9}{2}\left\{\text{sh}^{-1}\left(\frac{x}{3}\right) + \frac{x}{9}\cdot\sqrt{(9+x^2)}\right\}.$$

22. Put $x = |a|\,.\,\text{ch }\theta$, since x is +;

$$\text{integral} = a^2\int \text{sh}^2\theta\, d\theta = \frac{a^2}{2}\int(\text{ch } 2\theta - 1)\,d\theta$$

$$= \frac{a^2}{2}\{\tfrac{1}{2}\text{ sh } 2\theta - \theta\}$$

$$= \frac{a^2}{2}\left\{\frac{x}{|a|}\cdot\sqrt{\left(\frac{x^2}{a^2}-1\right)} - \text{ch}^{-1}\left(\frac{x}{|a|}\right)\right\}.$$

23. Put $x = \text{sh }\theta$;

$$\text{integral} = \int \frac{\text{sh}^2\theta\,.\,\text{ch }\theta\, d\theta}{\text{ch }\theta} = \tfrac{1}{2}\int(\text{ch } 2\theta - 1)d\theta$$

$$= \tfrac{1}{2}(\tfrac{1}{2}\text{ sh } 2\theta - \theta) = \tfrac{1}{2}\{x\sqrt{(1+x^2)} - \text{sh}^{-1}x\}.$$

24. Put $\tan\frac{x}{2} = t$; $\sec x = \frac{1+t^2}{1-t^2}$ and $\sec^2\frac{x}{2}\,.\,\tfrac{1}{2}dx = dt$;

$$\therefore\ \tfrac{1}{2}(1+t^2)\,.\,dx = dt\,;\ \therefore\ \text{integral} = \int\frac{2dt}{1-t^2} = 2\text{ th}^{-1}(t),$$

by No. 9, if $|t| < 1$; and $= 2\text{ coth}^{-1}(t)$, by No. 10, if $|t| > 1$.

EXERCISE VI. d. (p. 113.)

1. $\text{ch } x + \text{sh } x = e^x$; etc.; l.h.s. $= e^x\,.\,e^y$; r.h.s. $= e^{x+y}$.
2. l.h.s. $= (\text{ch}^2x - \text{sh}^2x)[(\text{ch}^2x - \text{sh}^2x)^2 + 3\text{ ch}^2x\,.\,\text{sh}^2x]$; use eqns. (6), (7); $\text{ch}(4x) = 1 + 2\text{ sh}^2 2x$;

$$\therefore\ \text{expression} = 1 + \tfrac{3}{8}[\text{ch}(4x) - 1].$$

3. 1st bracket $= 2\text{ sh } x\text{ ch } y + \text{sh } x = \text{sh } x(2\text{ ch } y + 1)$;
   2nd bracket $= 2\text{ ch } x\text{ ch } y + \text{ch } x = \text{ch } x(2\text{ ch } y + 1)$.
4. l.h.s. $= \frac{(\text{ch }\theta + \text{sh }\theta)^3}{(\text{ch }\theta - \text{sh }\theta)^3} = \frac{(e^\theta)^3}{(e^{-\theta})^3} = e^{6\theta} = $ r.h.s.
5. By VI. c, No. 7,

$$\text{diff. coeff.} = \sqrt{(1+x^2)} + \frac{x^2}{\sqrt{(1+x^2)}} + \frac{1}{\sqrt{(1+x^2)}}$$

$$= \sqrt{(1+x^2)} + \frac{1+x^2}{\sqrt{(1+x^2)}}.$$

6. Diff. coeff. $=\sqrt{(x^2-a^2)}+\dfrac{x^2}{\sqrt{(x^2-a^2)}}\mp a^2\cdot\dfrac{1}{\sqrt{\left(\dfrac{x^2}{a^2}-1\right)}}\cdot\dfrac{1}{a}$, see

VI. c, No. 8, $=\dfrac{1}{\sqrt{(x^2-a^2)}}\{x^2-a^2+x^2\mp a^2\}$; $-$ or $+$

according as $\text{ch}^{-1}\left(\dfrac{x}{a}\right)$ is $+$ or $-$.

7. If $y=\text{sech}^{-1}x$, $\text{sech}\, y=x$; $\therefore\ -\dfrac{\text{sh}\, y}{\text{ch}^2 y}\cdot\dfrac{dy}{dx}=1$;

but $\text{ch}\, y=\dfrac{1}{x}$ and $\text{sh}\, y=\pm\sqrt{(\text{ch}^2 y-1)}=\pm\sqrt{\left(\dfrac{1}{x^2}-1\right)}$

$=\pm\dfrac{1}{x}\sqrt{(1-x^2)}$; $\therefore\ \dfrac{dy}{dx}=\mp\dfrac{1}{x^2}\cdot\dfrac{x}{\sqrt{(1-x^2)}}$.

8. If $y=\text{cosech}^{-1}\dfrac{x}{a}$, $a\,\text{cosech}\, y=x$; $\therefore\ -a\cdot\dfrac{\text{ch}\, y}{\text{sh}^2 y}\cdot\dfrac{dy}{dx}=1$;

but $\text{sh}\, y=\dfrac{a}{x}$ and $\text{ch}\, y=|\sqrt{(1+\text{sh}^2 y)}|$

$=\dfrac{1}{|x|}\cdot\sqrt{(x^2+a^2)}$ since $\text{ch}\, y$ is $+$.

9. If $y=x^{\text{sh}\, x}$, $\log y=\text{sh}\, x\,.\log x$;

$\therefore\ \dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{1}{x}\,.\,\text{sh}\, x+\text{ch}\, x\,.\log x.$

10. Diff. coeff. $=\text{sh}(bx)\,.\,ae^{ax}+e^{ax}.\,b\,\text{ch}(bx)$.

11. By inspection, using the formula for differentiating a product; **Or** by parts,

$$\int e^x\text{sech}^2 x\,dx=e^x\,\text{th}\, x-\int e^x\,\text{th}\, x\,dx.$$

12. $\text{sh}\, x\,.\,\text{sh}\, 2x\,.\,\text{sh}\, 3x=\frac{1}{2}\text{sh}\, 2x\,.\,(\text{ch}\, 4x-\text{ch}\, 2x)$

$=\frac{1}{4}[\text{sh}\, 6x-\text{sh}\, 2x-\text{sh}\, 4x]$.

13. $p\equiv\int e^{ax}\text{sh}\, bx\,dx=\text{sh}\, bx\cdot\dfrac{e^{ax}}{a}-\int\dfrac{e^{ax}}{a}\cdot b\,\text{ch}\, bx\,dx$, and

$q\equiv\int e^{ax}\text{ch}\, bx\,dx=\text{ch}\, bx\cdot\dfrac{e^{ax}}{a}-\int\dfrac{e^{ax}}{a}\cdot b\,\text{sh}\, bx\,dx$;

$\therefore\ p=\dfrac{e^{ax}}{a}\,.\,\text{sh}\, bx-\dfrac{b}{a}\left\{\dfrac{e^{ax}}{a}\,.\,\text{ch}\, bx-\dfrac{b}{a}\,.\,p\right\}$; solve for p.

14. Put $\text{ch}\, x=u$;

$$\text{integral}=\int\frac{du}{(1+u)(2+u)}=\int\left(\frac{1}{1+u}-\frac{1}{2+u}\right)du$$
$$=\log(1+u)-\log(2+u).$$

15. From eqns. (1), (3), $y=1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\ldots\simeq 1+\frac{1}{2}x^2$ when x is small;

radius of curvature $=\lim\limits_{x\to 0}\dfrac{x^2}{2(y-1)}=1$ (Newton's formula).

16. Curves cut where $1+\text{ch}\, x=e^x=\text{ch}\, x+\text{sh}\, x$;

$\therefore\ \text{sh}\, x=1$; $\therefore\ \text{ch}^2 x=1+\text{sh}^2 x=2$; $\therefore\ \text{ch}\, x=+\sqrt{2}$;

slope of $y=1+\text{ch}\, x$ at this point is $\dfrac{\pi}{4}$; slope of $y=e^x$ is

$\tan^{-1}(e^x)=\tan^{-1}(\text{ch}\, x+\text{sh}\, x)=\tan^{-1}(\sqrt{2}+1)=\dfrac{3\pi}{8}$;

difference of slopes $=\dfrac{3\pi}{8}-\dfrac{\pi}{4}$.

17. $\text{ch}\,\theta=\frac{1}{2}(e^\theta+e^{-\theta})>\frac{1}{2}(e^\theta-e^{-\theta})$

$=\text{sh}\,\theta=\theta+\dfrac{\theta^3}{3!}+\ldots=\theta\left(1+\dfrac{\theta^2}{3!}+\ldots\right)>\theta$;

$\dfrac{d}{d\theta}(\theta-\text{th}\,\theta)=1-\text{sech}^2\theta=\text{th}^2\theta>0$;

$\therefore\ \theta-\text{th}\,\theta$ increases as θ increases, but $\theta-\text{th}\,\theta=0$ if $\theta=0$;

$\therefore\ \theta-\text{th}\,\theta>0$ if $\theta>0$.

18. By No. 17,

$$1>\frac{x}{\text{sh}\, x}>\frac{\text{th}\, x}{\text{sh}\, x}=\frac{1}{\text{ch}\, x},$$

for $x>0$; but $\text{ch}\, x\to 1$ when $x\to 0$; $\therefore\ \dfrac{x}{\text{sh}\, x}\to 1$ when $x\to 0$

for $x>0$; but $\dfrac{x}{\text{sh}\, x}=\dfrac{(-x)}{\text{sh}(-x)}$; $\therefore\ \dfrac{x}{\text{sh}\, x}\to 1$ when $x\to 0$ in any manner; also $\dfrac{x}{\text{th}\, x}=\dfrac{x}{\text{sh}\, x}\,.\,\text{ch}\, x\to 1$ when $x\to 0$, since $\dfrac{x}{\text{sh}\, x}$ and $\text{ch}\, x$ each $\to 1$.

19. Proceed as in V. a, No. 16. Here, from eqn. (5),

$\dfrac{1}{x^2}\left(\dfrac{\text{sh}\, x}{x}-1\right)=\dfrac{1}{3!}+\dfrac{x^2}{5!}+\dfrac{x^4}{7!}+\ldots$, which lies between $\dfrac{1}{3!}$

and $\dfrac{1}{3!}+\dfrac{x^2}{5!}\left(1-\dfrac{x^2}{6^2}\right)^{-1}$ if $|x|<6$, and hence $\to\dfrac{1}{3!}$.

20. For x small, $\text{sh}\, x-\sin x\simeq\left(x+\dfrac{x^3}{6}\right)-\left(x-\dfrac{x^3}{6}\right)=\dfrac{x^3}{3}$ with errors $<|Ax^5|$ as in No. 19 and by p. 80.

21. By eqns. (1) and (2), for x small,

$$\text{th}\, x \simeq \left(x+\frac{x^3}{6}\right)\cdot\left(1+\frac{x^2}{2}\right)^{-1}$$
$$\simeq\left(x+\frac{x^3}{6}\right)\left(1-\tfrac{1}{2}x^2\right)\simeq x-\frac{x^3}{3};$$
$$\therefore\ \text{sh}\, x-\text{th}\, x\simeq\left(x+\frac{x^3}{6}\right)-\left(x-\frac{x^3}{3}\right)=\tfrac{1}{2}x^3.$$

22. By eqn. (2), for x small,

$$x\,\text{cosech}\, x\simeq\left(1+\frac{x^2}{6}+\frac{x^4}{120}\right)^{-1}\simeq 1-\left(\frac{x^2}{6}+\frac{x^4}{120}\right)+\left(\frac{x^2}{6}\right)^2.$$

23. By eqn. (1), for $|x|<1$,

$$\text{ch}\, x<1+\frac{1}{2\,!}+\frac{1}{4\,!}+\frac{1}{6\,!}+\ldots<1+(\tfrac{1}{2})+(\tfrac{1}{2})^3+(\tfrac{1}{2})^5+\ldots$$
$$<1+\frac{\frac{1}{2}}{1-\frac{1}{4}}=1\tfrac{2}{3}.$$

24. By eqn. (2) for $0<x<1$,

$$\text{sh}\, x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots<x+\frac{x}{3\,!}+\frac{x}{5\,!}+\ldots$$
$$<x\left(1+\tfrac{1}{6}+\frac{1}{6^2}+\frac{1}{6^3}+\ldots\right)=\frac{6x}{5}.$$

25. By No. 17, for $x>0$, $\text{th}\,\frac{x}{2}<\frac{x}{2}$;

$$\therefore\ 2(\text{ch}\, x-1)\equiv 4\,\text{sh}^2\frac{x}{2}$$
$$\equiv 4\,\text{th}\,\frac{x}{2}\cdot\text{sh}\,\frac{x}{2}\cdot\text{ch}\,\frac{x}{2}<4\cdot\frac{x}{2}\cdot\tfrac{1}{2}\,\text{sh}\, x=x\,\text{sh}\, x.$$

For $x<0$, put $x=-y$, then $2(\text{ch}\, y-1)<y\,\text{sh}\, y$;

$$\therefore\ 2[\text{ch}\,(-x)-1]<(-x)\,\text{sh}\,(-x)=x\,\text{sh}\, x;$$

but $\text{ch}\,(-x)=\text{ch}\, x$.

26. $x=2\,\text{th}\, x=2\cdot\dfrac{e^{2x}-1}{e^{2x}+1}$;

$$\therefore\ e^{2x}(x-2)+x+2=0;\quad \therefore\ x=2-(x+2)e^{-2x};$$

1st approx., $x=2$;

2nd approx., $x=2-(2+2)e^{-4}=2-4e^{-4}$
$=2-4\times\cdot02=1\cdot92\simeq1\cdot9$;

3rd approx., $x=2-(1\cdot92+2)e^{-2\times1\cdot92}$
$=2-3\cdot92e^{-3\cdot84}=2-0\cdot084.$

27. $\sin 2x=\dfrac{2\tan x}{1+\tan^2 x}=\dfrac{2\,\text{th}\, y}{1+\text{th}^2 y}=\text{th}\, 2y$;

$$\therefore\ 2\tan^{-1}(\sin 2x)=\tan^{-1}\frac{2\,\text{th}\,2y}{1-\text{th}^2 2y}$$
$$=\tan^{-1}(2\,\text{sh}\,2y\,\text{ch}\,2y)=\tan^{-1}(\text{sh}\,4y).$$

28. By eqn. (14), $\text{sh}^{-1}(\cot\theta)=\log\{\cot\theta+\sqrt{(1+\cot^2\theta)}\}$
$=\log\{\cot\theta+|\text{cosec}\,\theta|\}$,

since the positive value of $\sqrt{(1+\cot^2\theta)}$ must be taken.

29. (i) $\text{th}\,(u+v)=\dfrac{\text{th}\, u+\text{th}\, v}{1+\text{th}\, u\,.\,\text{th}\, v}$; put $\text{th}\, u=x$, $\text{th}\, v=y$; then

$$\text{th}^{-1}x+\text{th}^{-1}y=u+v=\text{th}^{-1}\left(\frac{x+y}{1+xy}\right);$$

(ii) From (i), $\dfrac{x+y}{1+xy}=\text{th}\, c=\dfrac{1}{k}$, say;

$$\therefore\ xy-k(x+y)+1=0\ \text{or}\ (x-k)(y-k)=k^2-1,$$

a hyperbola with asymptotes parallel to Ox, Oy.

30. (i) Ordinates PN, QM; area of trapezium QMNP is

$\tfrac{1}{2}ab(\text{sh}\,\theta+\text{sh}\,\phi)(\text{ch}\,\theta-\text{ch}\,\phi)$
$=\tfrac{1}{2}ab\cdot 2\,\text{sh}\,\dfrac{\theta+\phi}{2}\,\text{ch}\,\dfrac{\theta-\phi}{2}\cdot 2\,\text{sh}\,\dfrac{\theta+\phi}{2}\,\text{sh}\,\dfrac{\theta-\phi}{2}$
$=\tfrac{1}{2}ab\cdot\text{sh}\,(\theta-\phi)\cdot\{\text{ch}(\theta+\phi)-1\}$;

area under curve

$=\displaystyle\int y\,dx=\int_\phi^\theta b\,\text{sh}\,u\cdot a\,\text{sh}\,u\,du$
$=\tfrac{1}{2}ab\displaystyle\int_\phi^\theta(\text{ch}\,2u-1)\,du=\tfrac{1}{2}ab\left[\tfrac{1}{2}\,\text{sh}\,2u-u\right]_\phi^\theta$
$=\tfrac{1}{4}\,ab(\text{sh}\,2\theta-\text{sh}\,2\phi)-\tfrac{1}{2}ab(\theta-\phi)$
$=\tfrac{1}{2}ab\{\text{ch}(\theta+\phi)\,\text{sh}\,(\theta-\phi)-(\theta-\phi)\}$;

segment = difference of these two areas;

(ii) As in VI. c, No. 13, chord is

$$\frac{x}{a}\,\text{ch}\,\tfrac{1}{2}(\theta+\phi)-\frac{y}{b}\,\text{sh}\,\tfrac{1}{2}(\theta+\phi)=\text{ch}\,\tfrac{1}{2}(\theta-\phi)$$

and tangent is $\dfrac{x}{a}\,\text{ch}\,\tfrac{1}{2}(\theta+\phi)-\dfrac{y}{b}\,\text{sh}\,\tfrac{1}{2}(\theta+\phi)=1$; these are parallel;

(iii) Polar of (ξ,η) is $\dfrac{x\xi}{a^2}-\dfrac{y\eta}{b^2}=1$; this is the same line as the chord in (ii) if $\xi=a\,\text{ch}\,\tfrac{1}{2}(\theta+\phi)\,\text{sech}\,\tfrac{1}{2}(\theta-\phi)$ and $\eta=b\,\text{sh}\,\tfrac{1}{2}(\theta+\phi)\,\text{sech}\,\tfrac{1}{2}(\theta-\phi)$.

31. $\operatorname{ch}\theta = \dfrac{\operatorname{ch}^2\frac{\theta}{2} + \operatorname{sh}^2\frac{\theta}{2}}{\operatorname{ch}^2\frac{\theta}{2} - \operatorname{sh}^2\frac{\theta}{2}} = \dfrac{1+t^2}{1-t^2}$; $\operatorname{sh}\theta = \dfrac{2\operatorname{sh}\frac{\theta}{2}\operatorname{ch}\frac{\theta}{2}}{\operatorname{ch}^2\frac{\theta}{2} - \operatorname{sh}^2\frac{\theta}{2}} = \dfrac{2t}{1-t^2}$.

One branch of hyperbola is given by $x = a\operatorname{ch}\theta$, $y = b\operatorname{sh}\theta$, the other branch by $x = -a\operatorname{ch}\theta$, $y = b\operatorname{sh}\theta$, since $\operatorname{ch}\theta$ must be positive; $\therefore$ one set of equations in θ cannot give the whole curve. But if $t = \lambda$ is a point P on one branch, $t = -\dfrac{1}{\lambda}$ is the image P′ of P in the y-axis; $\therefore$ the equations in t give the whole curve.

32. $\left(\dfrac{ds}{dx}\right)^2 = 1 + \operatorname{sh}^2 x = \operatorname{ch}^2 x$; $\therefore \dfrac{ds}{dx} = \pm\operatorname{ch}x$; $\therefore s = \pm\operatorname{sh}x$ if $s = 0$ when $x = 0$; $\therefore s^2 = \operatorname{sh}^2 x = \operatorname{ch}^2 x - 1 = y^2 - 1$.

33. $y^2 = 4a^2\operatorname{sh}^2\theta = 4ax$;

$$\left(\frac{ds}{d\theta}\right)^2 = (2a\operatorname{sh}\theta\operatorname{ch}\theta)^2 + (2a\operatorname{ch}\theta)^2$$
$$= (2a\operatorname{ch}\theta)^2 \cdot \{\operatorname{sh}^2\theta + 1\} = 4a^2\operatorname{ch}^4\theta;$$

$\therefore \dfrac{ds}{d\theta} = \pm 2a\operatorname{ch}^2\theta = \pm a(1 + \operatorname{ch}2\theta)$; $\therefore s = \pm a(\theta + \frac{1}{2}\operatorname{sh}2\theta)$ if $s = 0$ when $\theta = 0$.

EXERCISE VI. e. (p. 115.)

1. $\dfrac{\operatorname{sh}2x}{a} = \dfrac{\sin 2y}{b} = \dfrac{\operatorname{ch}2x + \cos 2y}{1} = k$;

$$\therefore k^2 = \frac{\operatorname{sh}^2 2x + \sin^2 2y + (\operatorname{ch}2x + \cos 2y)^2}{a^2 + b^2 + 1}$$
$$= \frac{2\operatorname{ch}^2 2x + 2\operatorname{ch}2x\cos 2y}{a^2+b^2+1} = \frac{2\operatorname{ch}2x \cdot k}{a^2+b^2+1};$$
$$\therefore \frac{2\operatorname{ch}2x}{a^2+b^2+1} = k = \frac{\operatorname{sh}2x}{a}.$$

Also $$k^2 = \frac{(\operatorname{ch}2x + \cos 2y)^2 - \operatorname{sh}^2 2x - \sin^2 2y}{1 - a^2 - b^2}$$
$$= \frac{2\cos^2 2y + 2\operatorname{ch}2x\cos 2y}{1-a^2-b^2} = \frac{2\cos 2y \cdot k}{1-a^2-b^2};$$
$$\therefore \frac{2\cos 2y}{1-a^2-b^2} = k = \frac{\sin 2y}{b}.$$

2. $$\tan(x+y) = \frac{\tan\lambda \cdot \operatorname{th}\mu + \cot\lambda\operatorname{th}\mu}{1 - \operatorname{th}^2\mu}$$
$$= \frac{\operatorname{th}\mu \cdot (\tan\lambda + \cot\lambda) \cdot \operatorname{ch}^2\mu}{\operatorname{ch}^2\mu - \operatorname{sh}^2\mu}$$
$$= \operatorname{sh}\mu \cdot \operatorname{ch}\mu \cdot \frac{\sin^2\lambda + \cos^2\lambda}{\sin\lambda\cos\lambda} = \frac{\operatorname{sh}2\mu}{\sin 2\lambda}.$$

3. Left side $$= \tan\left\{\tan^{-1}\frac{2\tan\alpha\operatorname{th}\beta}{1 - \tan^2\alpha\operatorname{th}^2\beta}\right\}$$
$$= \frac{2\sin\alpha\operatorname{sh}\beta\cos\alpha\operatorname{ch}\beta}{\cos^2\alpha\operatorname{ch}^2\beta - \sin^2\alpha\operatorname{sh}^2\beta}$$
$$= \frac{\sin 2\alpha\operatorname{sh}2\beta}{(1+\cos 2\alpha)\cdot\operatorname{ch}^2\beta - (1-\cos 2\alpha)\operatorname{sh}^2\beta} = \text{right side.}$$

4. $\operatorname{sh}^2 u = \operatorname{ch}^2 u - 1 = \sec^2\theta - 1 = \tan^2\theta$; $\therefore \operatorname{sh}u = \pm\tan\theta$; also $\sec\theta = \operatorname{ch}u = +$; $\therefore -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$; also u and θ have the same sign; $\therefore$ if $u < 0$, $-\dfrac{\pi}{2} < \theta < 0$, and if $u > 0$, $0 < \theta < \dfrac{\pi}{2}$; $\therefore \operatorname{sh}u$ and $\tan\theta$ have the same sign; $\therefore \operatorname{sh}u = \tan\theta$; $\therefore e^u = \operatorname{ch}u + \operatorname{sh}u = \sec\theta + \tan\theta$; $\therefore u = \log(\sec\theta + \tan\theta)$; also $\operatorname{th}\dfrac{u}{2} = \dfrac{\operatorname{ch}u - 1}{\operatorname{sh}u} = \dfrac{\sec\theta - 1}{\tan\theta} = \dfrac{1 - \cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2}$. Similarly, if $u\theta$ is negative, when $u < 0$, $0 < \theta < \dfrac{\pi}{2}$ and when $u > 0$, $-\dfrac{\pi}{2} < \theta < 0$; $\therefore \operatorname{sh}u = -\tan\theta$, etc.

5. Put $ax + b = v$, $ax_1 + b = v_1$, $ac - b^2 = k$; then $ay^2 = v^2 + k$, $ay_1{}^2 = v_1{}^2 + k$; also $a \cdot dx = dv$, $ay \cdot dy = v\,dv$; put $\dfrac{axx_1 + b(x + x_1) + c}{yy_1} = \operatorname{th}z$, then $ayy_1\operatorname{th}z = vv_1 + k$;

$\therefore ayy_1\operatorname{sech}^2 z \cdot dz + ay_1\operatorname{th}z \cdot dy = v_1 dv$;

but $$(ayy_1\operatorname{sech}z)^2 = a^2y^2y_1{}^2 - (ayy_1\operatorname{th}z)^2$$
$$= (v^2+k)(v_1{}^2+k) - (vv_1+k)^2 = k(v - v_1)^2;$$

$$\therefore ky(v-v_1)^2 dz = y \cdot (ayy_1\operatorname{sech}z)^2 dz$$
$$= ay^2y_1(v_1\,dv - ay_1\operatorname{th}z \cdot dy)$$
$$= (v^2+k) \cdot y_1v_1 dv - y_1 \cdot ayy_1\operatorname{th}z \cdot ay\,dy$$
$$= (v^2+k) \cdot y_1v_1 \cdot a\,dx - y_1(vv_1+k) \cdot v\,dv$$
$$= (v^2+k) \cdot ay_1v_1 dx - (vv_1+k) \cdot vy_1 \cdot a\,dx$$
$$= ay_1 dx\{v_1(v^2+k) - v(vv_1+k)\}$$
$$= ay_1 dx \cdot k(v_1 - v);$$

$\therefore y(v_1 - v)dz = ay_1 dx$; but $v_1 - v = a(x_1 - x)$;

$\therefore y(x_1 - x)dz = y_1 dx$; expression $= \dfrac{dz}{dx}$.

6. From No. 5, $\dfrac{1}{(x-x_1)\sqrt{(ax^2+2bx+c)}}$

$$=-\frac{1}{y_1}\cdot\frac{d}{dx}\,\text{th}^{-1}\frac{axx_1+b(x+x_1)+c}{yy_1};$$

hence result.

7. Put $x=\text{sh}\,\theta$;

$$\text{integral}=\int_0^\infty(\text{ch}\,\theta-\text{sh}\,\theta)^n\,.\,\text{ch}\,\theta\,d\theta=\int_0^\infty e^{-n\theta}\,.\,\tfrac{1}{2}(e^\theta+e^{-\theta})d\theta$$

$$=\tfrac{1}{2}\int_0^\infty\{e^{-(n-1)\theta}+e^{-(n+1)\theta}\}\,d\theta$$

$$=\tfrac{1}{2}\left[-\frac{e^{-(n-1)\theta}}{n-1}-\frac{e^{-(n+1)\theta}}{n+1}\right]_0^\infty;$$

but $n>1$; $\therefore$ $e^{-(n-1)\theta}$ and $e^{-(n+1)\theta}$ tend to 0 when $\theta\to\infty$; $\therefore$ $\text{integral}=\tfrac{1}{2}\left\{0-\left(-\dfrac{1}{n-1}-\dfrac{1}{n+1}\right)\right\}=\dfrac{n}{n^2-1}$.

8. By VI. d, No. 17, $x>\text{th}\,x$, i.e. $x\,\text{ch}\,x-\text{sh}\,x>0$, for $x>0$;

$$\therefore\ \int_0^x(u\,\text{ch}\,u-\text{sh}\,u)\,du>0;$$

but $\int u\,\text{ch}\,u\,du=u\,\text{sh}\,u-\int\text{sh}\,u\,du$;

$$\therefore\ \Big[u\,\text{sh}\,u-\text{ch}\,u-\text{ch}\,u\Big]_0^x>0,\text{ for }x>0;$$

$$\therefore\ x\,\text{sh}\,x-2\,\text{ch}\,x+2>0;$$

$$\therefore\ \int_0^x(u\,\text{sh}\,u-2\,\text{ch}\,u+2)\,du>0,\ x>0;$$

$$\therefore\ \Big[(u\,\text{ch}\,u-\text{sh}\,u)-2\,\text{sh}\,u+2u\Big]_0^x>0,\ x>0;$$

$$\therefore\ x\,\text{ch}\,x-3\,\text{sh}\,x+2x>0.$$

9. $\dfrac{x}{\text{sh}\,x}-\dfrac{\text{th}\,x}{x}=\dfrac{x^2\,\text{ch}\,x-\text{sh}^2x}{\frac{1}{2}x\,\text{sh}\,2x}=\dfrac{2x^2\,\text{ch}\,x-(\text{ch}\,2x-1)}{x\,\text{sh}\,2x}\simeq$, if x is small,

$$\frac{2x^2(1+\frac{1}{2}x^2)-\left(\dfrac{4x^2}{2}+\dfrac{16x^4}{4!}\right)}{2x^2}\simeq\frac{x^2}{6}>0.$$

10. For θ small, by V. a, No. 11, $\text{cosec}\,\theta\simeq\dfrac{1}{\theta}\left(1+\dfrac{\theta^2}{6}+\dfrac{7\theta^4}{360}\right)$; by VI. d, No. 22, $\text{cosech}\,\theta\simeq\dfrac{1}{\theta}\left(1-\dfrac{\theta^2}{6}+\dfrac{7\theta^4}{360}\right)$;

$$\therefore\ \frac{1}{\theta^3}(\text{cosec}\,\theta+\text{cosech}\,\theta)\simeq\frac{1}{\theta^4}\left(2+\frac{7\theta^4}{180}\right).$$

11. From VI. a, No. 35, equation becomes

$$\tfrac{1}{2}\left(x+\frac{1}{x}\right)=\tfrac{1}{2}\left(\frac{x}{2}-\frac{2}{x}\right)+\tfrac{7}{4};\quad\therefore\ x^2-7x+6=0.$$

12. $2e^x\,.\,\tfrac{1}{2}(e^x-e^{-x})=3+a$;

$$\therefore\ e^{2x}=4+a;$$

$$\therefore\ 2x=\log(4+a)=\log 4+\log\left(1+\frac{a}{4}\right)$$

$$=2\log 2+\left(\frac{a}{4}-\tfrac{1}{2}\cdot\frac{a^2}{16}+\tfrac{1}{3}\cdot\frac{a^3}{64}-\dots\right)$$

since a is small.

13. As x increases from 0 to ∞, $\text{th}\,mx$ increases steadily from 0 to 1, but $\sin x$ oscillates between $+1$ and -1; $\therefore$ there are 2 roots of $\sin x=\text{th}\,mx$ in each interval,

$$2n\pi<x<(2n+1)\pi.$$

If mx is large,

$$\text{th}\,mx=\frac{1-e^{-2mx}}{1+e^{-2mx}}\simeq1-2e^{-2mx};$$

$\therefore$ for a 1st approximation, $\sin x=1$; $\therefore$ $x=(2n+\tfrac{1}{2})\pi$. Put $x=y+(2n+\tfrac{1}{2})\pi$; equation becomes $\cos y=\text{th}\,(my+u)$ where $u=(2n+\tfrac{1}{2})m\pi$. Then

$$\sin y=\sqrt{(1-\cos^2y)}=\pm\,\text{sech}\,(my+u);$$

$\therefore$ 2nd approximation, since u is large compared with y, is $y=\pm\,\text{sech}\,u=\pm\,a$. For 3rd approximation,

$$\sin y=\pm\frac{1}{\text{ch}\,my\,\text{ch}\,u+\text{sh}\,my\,\text{sh}\,u}\simeq\pm\frac{1}{\text{ch}\,u}\cdot\frac{1}{1+my\,\text{th}\,u}$$

$$\simeq\pm\,\text{sech}\,u\cdot\frac{1}{1+my},$$

since $\text{th}\,u\simeq1-2e^{-2mu}$;

$$\therefore\ y\simeq\pm\,\text{sech}\,u\,(1-my)\simeq\pm\,\text{sech}\,u\,(1\mp m\,\text{sech}\,u)$$

$$=\pm\,a(1\mp ma)=\pm\,a-ma^2.$$

14. l.h.s. $=(e^\theta)^n=e^{n\theta}=\text{ch}\,n\theta+\text{sh}\,n\theta$.

15. $(\text{ch}\,a-\text{sh}\,a)^n=(e^{-a})^n=e^{-na}=\text{ch}\,na-\text{sh}\,na$; now use No. 14.

16. From No. 15, $2\,\text{ch}\,5x=(\text{ch}\,x+\text{sh}\,x)^5+(\text{ch}\,x-\text{sh}\,x)^5$; expand and put $\text{sh}^2x=\text{ch}^2x-1$, $\text{sh}^4x=(\text{ch}^2x-1)^2$.

17. As in No. 15, $2\,\text{sh}\,5x=(\text{ch}\,x+\text{sh}\,x)^5-(\text{ch}\,x-\text{sh}\,x)^5$; expand and put $\text{ch}^2x=1+\text{sh}^2x$, $\text{ch}^4x=(1+\text{sh}^2x)^2$.

18. As in No. 17, $2\,\text{sh}\,6x=(\text{ch}\,x+\text{sh}\,x)^6-(\text{ch}\,x-\text{sh}\,x)^6$; expand and divide each side by $\text{ch}\,x$; then put $\text{ch}^2x=1+\text{sh}^2x$, etc.

19. $64\,\text{ch}^7 x = \frac{1}{2}(2\,\text{ch}\,x)^7 = \frac{1}{2}(e^x + e^{-x})^7$
$= \frac{1}{2}\{e^{7x} + 7e^{5x} + 21e^{3x} + 35e^x + 35e^{-x} + 21e^{-3x} + 7e^{-5x} + e^{-7x}\}$
$= \frac{1}{2}\{(e^{7x} + e^{-7x}) + 7(e^{5x} + e^{-5x}) + \ldots\}.$
Similarly $64\,\text{sh}^7 x = \frac{1}{2}(2\,\text{sh}\,x)^7 = \frac{1}{2}(e^x - e^{-x})^7$; expand and proceed as before.

20. $32\,\text{sh}^6 x = \frac{1}{2}(2\,\text{sh}\,x)^6 = \frac{1}{2}(e^x - e^{-x})^6$; expand and proceed as in No. 19.

21. $\text{l.h.s.} = \left(2\,\text{ch}^2\frac{\theta}{2} + 2\,\text{sh}\frac{\theta}{2}\,\text{ch}\frac{\theta}{2}\right)^n$
$= \left(2\,\text{ch}\frac{\theta}{2}\right)^n . \left(\text{ch}\frac{\theta}{2} + \text{sh}\frac{\theta}{2}\right)^n = \text{r.h.s., by No. 14.}$

22. $\text{l.h.s.} = \left(2\,\text{sh}^2\frac{\theta}{2} + 2\,\text{sh}\frac{\theta}{2}\,\text{ch}\frac{\theta}{2}\right)^n = \left(2\,\text{sh}\frac{\theta}{2}\right)^n . \left(\text{sh}\frac{\theta}{2} + \text{ch}\frac{\theta}{2}\right)^n$; use No. 14.

23. Apply the formula, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$;
$$\text{l.h.s.} = \left\{\tan^{-1}\left(\frac{\text{th}\,2\phi}{\tan 2\theta}\right) + \tan^{-1}1\right\} + \left\{\tan^{-1}\left(\frac{\tan\theta}{\text{th}\,\phi}\right) - \tan^{-1}1\right\}$$
$$= \tan^{-1}\left(\frac{\text{th}\,2\phi}{\tan 2\theta}\right) + \tan^{-1}\left(\frac{\tan\theta}{\text{th}\,\phi}\right)$$
$$= \tan^{-1}\frac{\text{th}\,2\phi\,\text{th}\,\phi + \tan 2\theta\tan\theta}{\tan 2\theta\,\text{th}\,\phi - \tan\theta\,\text{th}\,2\phi};$$
but $\text{th}\,2\phi = \frac{2\,\text{th}\,\phi}{1+\text{th}^2\phi}$, $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$;
$$\therefore \text{expression} = \tan^{-1}\frac{2\,\text{th}^2\phi(1-\tan^2\theta) + 2\tan^2\theta(1+\text{th}^2\phi)}{2\tan\theta\,\text{th}\,\phi\{(1+\text{th}^2\phi) - (1-\tan^2\theta)\}}$$
$$= \tan^{-1}\frac{\text{th}^2\phi + \tan^2\theta}{\tan\theta\,\text{th}\,\phi\,(\text{th}^2\phi + \tan^2\theta)}$$
$$= \tan^{-1}\left(\frac{1}{\tan\theta\,\text{th}\,\phi}\right).$$

24. $S = \Sigma\,\text{sh}\,r\alpha$, for $r = 1$ to n; then
$$2S . \text{sh}\frac{\alpha}{2} = 2\Sigma\,\text{sh}\,r\alpha . \text{sh}\frac{\alpha}{2} = \Sigma[\text{ch}\,(r+\tfrac{1}{2})\alpha - \text{ch}\,(r-\tfrac{1}{2})\alpha]$$
$=$, as on p. 127, $\text{ch}\,(n+\frac{1}{2})\alpha - \text{ch}\,\frac{1}{2}\alpha = 2\,\text{sh}\,(n+1)\frac{\alpha}{2}\,\text{sh}\,\frac{n\alpha}{2}$.

25. By method of p. 127, if $S = \Sigma\,\text{ch}\,[\alpha + (r-1)\beta]$ for $r = 1$ to n, then
$$2S . \text{sh}\frac{\beta}{2} = \Sigma\left\{2\,\text{ch}\,[\alpha + (r-1)\beta] . \text{sh}\frac{\beta}{2}\right\}$$
$$= \Sigma\{\text{sh}\,[\alpha + (r-\tfrac{1}{2})\beta] - \text{sh}\,[\alpha + (r-\tfrac{3}{2})\beta]\}$$

$$= \text{sh}\,[\alpha + (n-\tfrac{1}{2})\beta] - \text{sh}\,[\alpha - \tfrac{1}{2}\beta]$$
$$= 2\,\text{ch}\left[\alpha + (n-1)\frac{\beta}{2}\right] . \text{sh}\frac{n\beta}{2}.$$

26. By No. 24, $\text{sh}\,\theta + \text{sh}\,2\theta + \ldots + \text{sh}\,n\theta = \frac{\text{ch}\,(n+\frac{1}{2})\theta}{2\,\text{sh}\,\frac{1}{2}\theta} - \frac{1}{2}\coth\frac{1}{2}\theta$
$$= \tfrac{1}{2}\,\text{ch}\,n\theta\coth\tfrac{1}{2}\theta + \tfrac{1}{2}\,\text{sh}\,n\theta - \tfrac{1}{2}\coth\tfrac{1}{2}\theta.$$
Differentiate w.r.t. θ; then given series
$$= \frac{n}{2}\,\text{sh}\,n\theta\coth\tfrac{1}{2}\theta - \tfrac{1}{4}\,\text{ch}\,n\theta\,\text{cosech}^2\tfrac{1}{2}\theta$$
$$+ \frac{n}{2}\,\text{ch}\,n\theta + \tfrac{1}{4}\,\text{cosech}^2\tfrac{1}{2}\theta$$
$$= \tfrac{1}{4}\,\text{cosech}^2\tfrac{1}{2}\theta\,\{2n\,\text{sh}\,n\theta\,\text{ch}\frac{\theta}{2}\,\text{sh}\frac{\theta}{2} - \text{ch}\,n\theta$$
$$+ 2n\,\text{ch}\,n\theta\,\text{sh}^2\tfrac{1}{2}\theta + 1\}$$
$$= \tfrac{1}{4}\,\text{cosech}^2\tfrac{1}{2}\theta\,\{2n\,\text{sh}\frac{\theta}{2}\left(\text{sh}\,n\theta\,\text{ch}\frac{\theta}{2} + \text{ch}\,n\theta\,\text{sh}\frac{\theta}{2}\right)$$
$$- (\text{ch}\,n\theta - 1)\}.$$
Or reduce given series to form of Nos. 24, 25 by multiplying it by $\text{ch}\,\theta - 1$.

27. $e^x + n . e^{2x} + \frac{n(n-1)}{1 . 2}e^{3x} + \ldots = e^x(1+e^x)^n$
$$= e^x . e^{\frac{1}{2}nx}(e^{-\frac{1}{2}x} + e^{\frac{1}{2}x})^n = e^{(\frac{1}{2}n+1)x} . \left(2\,\text{ch}\frac{x}{2}\right)^n.$$
Change x to $-x$, and add; $\text{ch}\left(-\frac{x}{2}\right) = \text{ch}\left(\frac{x}{2}\right)$;
$$\therefore \text{twice given series} = \left(2\,\text{ch}\frac{x}{2}\right)^n . \{e^{(\frac{1}{2}n+1)x} + e^{-(\frac{1}{2}n+1)x}\}$$
$$= \left(2\,\text{ch}\frac{x}{2}\right)^n . \{2\,\text{ch}\,(\tfrac{1}{2}n+1)x\}.$$

28. $1 + e^\theta + \frac{e^{2\theta}}{2!} + \ldots = e^{e^\theta}$ and $1 + e^{-\theta} + \frac{e^{-2\theta}}{2!} + \ldots = e^{e^{-\theta}}$; adding,
$$\text{twice given series} = e^{e^\theta} + e^{e^{-\theta}} = e^{\text{ch}\,\theta + \text{sh}\,\theta} + e^{\text{ch}\,\theta - \text{sh}\,\theta}$$
$$= e^{\text{ch}\,\theta} . \{e^{\text{sh}\,\theta} + e^{-\text{sh}\,\theta}\} = e^{\text{ch}\,\theta} . \{2\,\text{ch}\,(\text{sh}\,\theta)\}$$

29. As in No. 28; subtracting, twice given series
$$= e^{\text{ch}\,\theta}\{e^{\text{sh}\,\theta} - e^{-\text{sh}\,\theta}\} = e^{\text{ch}\,\theta} . \{2\,\text{sh}\,(\text{sh}\,\theta)\}.$$

30. Put $e^\alpha = x$, $e^\beta = y$, then $1<x<y$; $\therefore\ 0<\frac{1}{xy}<\frac{x}{y}<1$.

$$\text{Series} = \tfrac12\left\{\frac{x-\frac1x}{y} - \tfrac12\cdot\frac{x^2-\frac{1}{x^2}}{y^2} + \tfrac13\cdot\frac{x^3-\frac{1}{x^3}}{y^3} - \dots\right\}$$
$$=\tfrac12\left\{\left(\frac{x}{y}-\tfrac12\cdot\frac{x^2}{y^2}+\dots\right)-\left(\frac{1}{xy}-\tfrac12\cdot\frac{1}{x^2y^2}+\dots\right)\right\}$$
$$=\tfrac12\left\{\log\left(1+\frac xy\right)-\log\left(1+\frac1{xy}\right)\right\},$$

since $0<\frac1{xy}<\frac xy<1$,

$$=\tfrac12\log\left[\left(1+\frac xy\right)\cdot\left(\frac{xy}{xy+1}\right)\right]=\tfrac12\log\frac{x+y}{\frac1x+y}.$$

31. For $|x|<1$, $\tan^{-1}x = x-\frac13x^3+\frac15x^5-\dots$. Hence, as in No. 30,

$$\text{series} =\tfrac12\left\{\tan^{-1}\left(\frac xy\right)-\tan^{-1}\left(\frac1{xy}\right)\right\}$$
$$=\tfrac12\tan^{-1}\frac{\frac xy-\frac1{xy}}{1+\frac xy\cdot\frac1{xy}}=\tfrac12\tan^{-1}\frac{x-\frac1x}{y+\frac1y}=\tfrac12\tan^{-1}\frac{e^\alpha-e^{-\alpha}}{e^\beta+e^{-\beta}}.$$

32. $x+\frac{\sin\theta}{1!}x^2+\frac{\sin^2\theta}{2!}x^3+\dots = x\cdot e^{x\sin\theta}$; first put $x=e^\theta$, then put $x=e^{-\theta}$, and add; $\therefore$ twice given series

$$=e^\theta\cdot e^{e^\theta\sin\theta}+e^{-\theta}\cdot e^{e^{-\theta}\sin\theta}$$
$$=e^\theta\cdot e^{\sin\theta(\operatorname{ch}\theta+\operatorname{sh}\theta)}+e^{-\theta}\cdot e^{\sin\theta(\operatorname{ch}\theta-\operatorname{sh}\theta)}$$
$$=e^{\sin\theta\cdot\operatorname{ch}\theta}\{e^{\theta+\sin\theta\operatorname{sh}\theta}+e^{-\theta-\sin\theta\operatorname{sh}\theta}\}$$
$$=e^{\sin\theta\cdot\operatorname{ch}\theta}\{2\operatorname{ch}(\theta+\sin\theta\operatorname{sh}\theta)\}.$$

33. Put $e^\alpha=a$, then, for $\alpha>0$, $a>1$ and $0<|x|<\frac1a<1$. [If $a=1$, we have $\alpha=0$ and $\operatorname{sh}\alpha=0$, so that each side is zero.]

$$\text{Series} =\tfrac12\left\{\left(a-\frac1a\right)+x\left(a^2-\frac1{a^2}\right)+x^2\left(a^3-\frac1{a^3}\right)+\dots\right\}$$
$$=\tfrac12\left\{a(1+xa+x^2a^2+\dots)-\frac1a\left(1+\frac xa+\frac{x^2}{a^2}+\dots\right)\right\}$$
$$=\tfrac12\left\{a\cdot\frac1{1-xa}-\frac1a\cdot\frac1{1-\frac xa}\right\}$$

since $|xa|<1$ and $\left|\frac xa\right|<|x|<1$.

$$\therefore\ \text{series} =\tfrac12\left\{\frac{a}{1-xa}-\frac1{a-x}\right\}$$
$$=\tfrac12\cdot\frac{a-\frac1a}{1-x\left(a+\frac1a\right)+x^2}=\tfrac12\cdot\frac{2\operatorname{sh}\alpha}{1-x(2\operatorname{ch}\alpha)+x^2}.$$

If $-\alpha$ is written for α, the sign of each term of the series is changed, and so is the sign of the right side, but the necessary condition becomes $|x|<e^\alpha$.

CHAPTER VII

EXERCISE VII. a. (p. 121.)

5. (i) $4\cos(\phi+\pi)$, $5\cos(\pi-\theta)$, $3\cos\frac{\pi}{2}$; (ii) for cos read sin.

6. (i) $c\cos\left(\phi+\frac{\pi}{2}\right)$, $c\cos(\phi-\pi)$, $c\cos\left(\phi-\frac{3\pi}{4}\right)$;
(ii) for cos read sin.

7. AD $=2a$, $\angle$(AD, AK) $=\alpha+\frac{\pi}{3}$; CF $=2a$, $\angle$(CF, AK) $=\alpha+\pi$.

8. (i) $3\cos\alpha+2\cos\left(\alpha+\frac{\pi}{2}\right)$; (ii) for cos read sin.

9. $4\cos 0+2\cos(\pi-\alpha)+3\cos\left(\frac{\pi}{2}-\alpha\right)+2\cos\left(\frac{3\pi}{2}-\alpha-\beta\right)$ the x-coordinate of D. For C, omit the last term. For the y-coordinates change cos into sin.

13. Use cosine formula for $\triangle$ABC; $\angle$ABC $=\pm\pi\pm(\theta_1-\theta_2)$.

14. $0+(-p)+h\cos\alpha+k\cos\left(\frac{\pi}{2}-\alpha\right)$.

EXERCISE VII. b. (p. 126.)

1. The same proof holds for all figures.

2. Draw perp. PH from P to Ox. ξ coordinate of P = projection of OP on Oξ = sum of projections of OH and HP on

$$O\xi = x\cos(2\pi - A) + y\cos\left(\frac{\pi}{2} - A\right); \ \eta \text{ coordinate of P}$$
$$= x\sin(2\pi - A) + y\sin\left(\frac{\pi}{2} - A\right).$$

3. The same expressions as in No. 2.

4. In the proof on pp. 123, 124, write $-B$ for B everywhere;
$$\cos(A - B) = \cos A\cos(-B) - \sin A\sin(-B)$$
$$= \cos A\cos B - \sin A \cdot (-\sin B).$$

5. $OA = OB = l$, $\angle AOM = \frac{1}{2}\angle AOB = \frac{1}{2}(\theta - \phi)$; $\therefore$ $OM = l\cos\frac{\theta - \phi}{2}$; also $\angle xOM = \frac{1}{2}(\theta + \phi)$. Projection of OA + projection of OB = projection of OM + projection of MA + projection of OM + projection of MB = 2 . projection of OM, since projection of MA + projection of MB = 0; for projections on Ox,
$$l\cos\theta + l\cos\phi = 2l\cos\frac{\theta - \phi}{2} \cdot \cos\frac{\theta + \phi}{2}.$$
For (ii), take projections on Oy.

6. From No. 5, projection of OA − projection of OB = projection of MA − projection of MB = 2 . projection of MA; for projections on Ox,
$$l\cos\theta - l\cos\phi = 2l\sin\frac{\theta - \phi}{2} \cdot \cos\left(\frac{\theta + \phi}{2} + \frac{\pi}{2}\right),$$
for projections on Oy,
$$l\sin\theta - l\sin\phi = 2l\sin\frac{\theta - \phi}{2} \cdot \sin\left(\frac{\theta + \phi}{2} + \frac{\pi}{2}\right).$$

7. Pentagon ABCDE; let AB make 5° with x axis, then BC, CD, DE, EA makes angles 77°, 149°, 221°, 293° with x axis; sum of projections on x axis of AB, BC, CD, DE, EA is zero. For (ii) project on y axis.

8. (i) $\dfrac{\cos(5° + 144°) \cdot \sin 180°}{\sin 36°} = 0$; (ii) $\dfrac{\sin(5° + 144°) \cdot \sin 180°}{\sin 36°}$.

9. $A_1A_2A_3 \ldots A_n$ is a regular n sided polygon; let A_1A_2 make angle θ with x axis; then A_2A_3 makes $\theta + \frac{2\pi}{n}$ with x axis, etc.; project on x axis. For sines, project on y axis.

10. (i) $\dfrac{\cos\left(\theta + \frac{n-1}{2} \cdot \frac{2\pi}{n}\right) \cdot \sin\left(\frac{n}{2} \cdot \frac{2\pi}{n}\right)}{\sin\frac{\pi}{n}} = 0$ since $\sin\pi = 0$;

(ii) as in (i).

11. (i) l.h.s. $= \Sigma\{2\sin\theta \cdot \cos(2r-1)\theta\}$ for $r = 1$ to n,
$$= \Sigma\{\sin 2r\theta - \sin(2r-2)\theta\}$$
$$= (\sin 2\theta - 0) + (\sin 4\theta - \sin 2\theta) + (\sin 6\theta - \sin 4\theta) + \ldots + [\sin 2n\theta - \sin(2n-2)\theta] = \sin 2n\theta;$$

(ii) $\cos\theta + \cos 3\theta + \ldots + \cos(2n-1)\theta$
$$= \frac{\cos n\theta \cdot \sin n\theta}{\sin\theta} = \frac{\sin 2n\theta}{2\sin\theta}.$$

12. (i) As in No. 11, $\Sigma[2\sin\theta \cdot \sin(2r-1)\theta]$
$$= \Sigma\{\cos(2r-2)\theta - \cos 2r\theta\}$$
$$= (\cos 0 - \cos 2\theta) + (\cos 2\theta - \cos 4\theta) + \ldots + [\cos(2n-2)\theta - \cos 2n\theta] = 1 - \cos 2n\theta;$$
$$\therefore \text{ series} = \frac{1 - \cos 2n\theta}{2\sin\theta}; \quad \text{(ii) } \frac{\sin n\theta \cdot \sin n\theta}{\sin\theta} = \frac{\sin^2 n\theta}{\sin\theta}.$$

EXERCISE VII. c. (p. 129.)

1. By (13), $\dfrac{\cos(n+1)\frac{\theta}{4} \cdot \sin\frac{n\theta}{4}}{\sin\frac{\theta}{4}} = \dfrac{\frac{1}{2}\left[\sin(2n+1)\frac{\theta}{4} - \sin\frac{\theta}{4}\right]}{\sin\frac{\theta}{4}}$.

2. By (13), sum $= \dfrac{\cos\frac{4\pi}{7} \cdot \sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}} = \dfrac{\frac{1}{2}\left(\sin\pi - \sin\frac{\pi}{7}\right)}{\sin\frac{\pi}{7}}$.

3. By (13), sum $= \dfrac{\cos\frac{5\pi}{11} \cdot \sin\frac{5\pi}{11}}{\sin\frac{\pi}{11}}$;
$$\text{numerator} = \tfrac{1}{2}\sin\frac{10\pi}{11} = \tfrac{1}{2}\sin\frac{\pi}{11}.$$

4. As in VII. b, No. 9, with $\theta = \frac{2\pi}{n}$.

5. By (13) $= \dfrac{\cos\frac{(n+1)\pi}{2n+1} \cdot \sin\frac{n\pi}{2n+1}}{\sin\frac{\pi}{2n+1}}$;
$$\text{numerator} = \tfrac{1}{2}\left(\sin\pi - \sin\frac{\pi}{2n+1}\right).$$

6. By (14), numerator $=\dfrac{\sin\frac{n\theta}{2}\cdot\sin(n-1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}$;

by (13), denominator $=\dfrac{\cos\frac{n\theta}{2}\cdot\sin(n-1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}$.

7. Series $=\cos\alpha+\cos[\alpha+(\beta+\pi)]+\cos[\alpha+2(\beta+\pi)]+\ldots$;

(i) sum $=\cos[\alpha+\frac{1}{2}(2n-1)(\beta+\pi)]$

$$.\ \sin 2n\left(\frac{\beta+\pi}{2}\right).\ \mathrm{cosec}\left(\frac{\beta+\pi}{2}\right)$$

$$=(-1)^n.\sin\left[\alpha+(2n-1)\frac{\beta}{2}\right].(-1)^n\sin n\beta.\sec\frac{\beta}{2};$$

(ii) sum $=\cos[\alpha+\frac{1}{2}(2n)(\beta+\pi)]$

$$.\ \sin(2n+1)\left(\frac{\beta+\pi}{2}\right).\ \mathrm{cosec}\left(\frac{\beta+\pi}{2}\right)$$

$$=(-1)^n.\cos(\alpha+n\beta).(-1)^n.\cos(n+\tfrac{1}{2})\beta.\sec\frac{\beta}{2};$$

(iii) see answer.

8. Series $=\sin\alpha+\sin[\alpha+(\beta+\pi)]+\sin[\alpha+2(\beta+\pi)]+\ldots$; use (14).

9. $2\cos\frac{\theta}{2}(\cos\theta-\cos2\theta+\cos3\theta-\ldots)$

$$=\left(\cos\frac{\theta}{2}+\cos\frac{3\theta}{2}\right)-\left(\cos\frac{3\theta}{2}+\cos\frac{5\theta}{2}\right)+\left(\cos\frac{5\theta}{2}+\cos\frac{7\theta}{2}\right)$$

$$-\ldots+(-1)^{n-1}\{\cos(n-\tfrac{1}{2})\theta+\cos(n+\tfrac{1}{2})\theta\}$$

$$=\cos\frac{\theta}{2}+(-1)^{n-1}\cos(n+\tfrac{1}{2})\theta.$$

10. $\frac{1}{2}\Sigma(1-\cos2r\theta)=\frac{n}{2}-\frac{1}{2}(\cos2\theta+\cos4\theta+\ldots+\cos2n\theta)=$, as in Ex. 2, p. 129, $\frac{n}{2}-\frac{1}{4\sin\theta}[\sin(2n+1)\theta-\sin\theta]$.

11. Last term is $\cos\theta$; sum $=\dfrac{\cos n\theta.\sin n\theta}{\sin\theta}$.

12. $\Sigma[\cos r\theta.\sin(r+1)\theta]=\frac{1}{2}\Sigma[\sin(2r+1)\theta+\sin\theta]$;

sum $=\frac{n}{2}\sin\theta+\frac{1}{2}[\sin3\theta+\sin5\theta+\ldots+\sin(2n+1)\theta]$;

use (14).

13. Series $=\cos\theta+\cos\left(2\theta+\frac{\pi}{2}\right)+\cos(3\theta+\pi)+\cos\left(4\theta+\frac{3\pi}{2}\right)+\ldots$; use (13).

14. Series $=\Sigma\cos^2[\theta+(r-1)\phi]=\frac{1}{2}\Sigma\{1+\cos[2\theta+2(r-1)\phi]\}$

$$=\frac{n}{2}+\tfrac{1}{2}\{\cos2\theta+\cos(2\theta+2\phi)+\cos(2\theta+4\phi)+\ldots\};$$

use (13).

15. Series $=\Sigma\sin^2r\theta.\sin(r+1)\theta=\frac{1}{2}\Sigma(1-\cos2r\theta).\sin(r+1)\theta$

$$=\tfrac{1}{4}\Sigma\{2\sin(r+1)\theta-\sin(3r+1)\theta+\sin(r-1)\theta\};$$

use (14) for each group.

16. Series $=\Sigma\cos^3r\theta=\frac{1}{4}\Sigma(\cos3r\theta+3\cos r\theta)$

$$=\tfrac{1}{4}\left\{\cos3(n+1)\frac{\theta}{2}.\sin\frac{3n\theta}{2}\ \mathrm{cosec}\ \frac{3\theta}{2}\right.$$

$$\left.+3\cos(n+1)\frac{\theta}{2}.\sin\frac{n\theta}{2}.\ \mathrm{cosec}\ \frac{\theta}{2}\right\}$$

$$=\tfrac{1}{8}\left\{\left[\sin(3n+\tfrac{3}{2})\theta-\sin\frac{3\theta}{2}\right]\mathrm{cosec}\ \frac{3\theta}{2}\right.$$

$$\left.+3\left[\sin(n+\tfrac{1}{2})\theta-\sin\frac{\theta}{2}\right]\mathrm{cosec}\ \frac{\theta}{2}\right\}.$$

17. Series $=\Sigma\cos^4r\theta=\Sigma[\frac{1}{2}(1+\cos2r\theta)]^2$

$$=\Sigma\tfrac{1}{4}(1+2\cos2r\theta+\cos^22r\theta)$$

$$=\Sigma\tfrac{1}{8}(2+4\cos2r\theta+1+\cos4r\theta)$$

$$=\Sigma\tfrac{1}{8}(3+4\cos2r\theta+\cos4r\theta)$$

$$=\frac{3n}{8}+\tfrac{1}{2}.\frac{\cos(n+1)\theta\sin n\theta}{\sin\theta}+\tfrac{1}{8}\frac{\cos(2n+2)\theta.\sin2n\theta}{\sin2\theta}$$

$$=\frac{3n}{8}+\tfrac{1}{4}\ \mathrm{cosec}\ \theta\ [\sin(2n+1)\theta-\sin\theta]$$

$$+\tfrac{1}{16}\ \mathrm{cosec}\ 2\theta\ [\sin(4n+2)\theta-\sin2\theta].$$

18. By (14), sum $=\sin(n+1)\frac{\theta}{2}\sin\frac{n\theta}{2}\ \mathrm{cosec}\ \frac{\theta}{2}$

$$=\tfrac{1}{2}\left[\cos\frac{\theta}{2}-\cos(2n+1)\frac{\theta}{2}\right].\ \mathrm{cosec}\ \frac{\theta}{2}$$

$$=\tfrac{1}{2}\left[\cot\frac{\theta}{2}-\frac{\cos(2n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right];$$

$$\therefore \text{ second series} = \frac{d}{d\theta}\,\tfrac{1}{2}\left[\cot\frac{\theta}{2} - \frac{\cos(2n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right]$$

$$= -\tfrac{1}{4}\operatorname{cosec}^2\frac{\theta}{2} - \tfrac{1}{2}\cdot\frac{-\frac{1}{2}(2n+1)\sin\frac{\theta}{2}\cdot\sin(2n+1)\frac{\theta}{2} - \frac{1}{2}\cos(2n+1)\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}$$

$$= \tfrac{1}{8}\operatorname{cosec}^2\frac{\theta}{2}\{-2 + (2n+1)[\cos n\theta - \cos(n+1)\theta] + [\cos(n+1)\theta + \cos n\theta]\}.$$

19. $\cos\theta + \cos 3\theta + \cos 5\theta + \dots$ to n terms $= \dfrac{\cos n\theta\,.\,\sin n\theta}{\sin\theta}$

$$= \tfrac{1}{2}\,\frac{\sin 2n\theta}{\sin\theta};\quad \therefore \text{ series} = -\frac{d}{d\theta}\left(\frac{\sin 2n\theta}{2\sin\theta}\right).$$

20. $2(1-\cos\beta)\,.\,C = \Sigma 2r\,\{\cos[\alpha+(r-1)\beta] - \cos\beta\,.\,\cos[\alpha+(r-1)\beta]\}$

$$= \Sigma\{2r\,.\,u_{r-1} - ru_r - ru_{r-2}\},$$

where $u_r \equiv \cos(\alpha + r\beta)$,

$$= \Sigma\{[(r+1)u_{r-1} - ru_r] - [ru_{r-2} - (r-1)u_{r-1}]\}$$

$$= [(n+1)u_{n-1} - nu_n] - u_{-1};\ \text{hence result.}$$

Or $\Sigma\sin(\gamma + r\beta)$, for $r = 1$ to n, by (14),

$$= \sin\left(\gamma + \frac{n+1}{2}\beta\right)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2}.$$

Diffg. w.r.t. β, $\Sigma r\cos(\gamma + r\beta)$

$$= \tfrac{1}{2}\,\frac{d}{d\beta}\left(\frac{\cos(\gamma+\frac{1}{2}\beta) - \cos(\gamma+\frac{1}{2}\beta+n\beta)}{\sin\frac{1}{2}\beta}\right)$$

$$= \frac{1}{2\sin^2\frac{1}{2}\beta}\{-\tfrac{1}{2}\sin(\gamma+\tfrac{1}{2}\beta)\sin\tfrac{1}{2}\beta + (n+\tfrac{1}{2})\sin(\gamma+\tfrac{1}{2}\beta+n\beta)\sin\tfrac{1}{2}\beta - \tfrac{1}{2}\cos(\gamma+\tfrac{1}{2}\beta)\cos\tfrac{1}{2}\beta + \tfrac{1}{2}\cos(\gamma+\tfrac{1}{2}\beta+n\beta)\cos\tfrac{1}{2}\beta\}$$

$$= \frac{1}{2(1-\cos\beta)}\{-\cos\gamma + n(\cos\overline{\gamma+n\beta} - \cos\overline{\gamma+n\beta+\beta}) + \cos(\gamma+n\beta)\}$$

and put $\gamma = \alpha - \beta$.

21. Sum to n terms $= C_n = \cos\theta + \cos(3\theta+\pi) + \cos(5\theta+2\pi) + \dots$

$$= \cos\left[n\theta + (n-1)\frac{\pi}{2}\right].\sin\left(n\theta + \frac{n\pi}{2}\right)\operatorname{cosec}\left(\theta+\frac{\pi}{2}\right)$$

$$= \tfrac{1}{2}\left\{\sin\left(2n\theta + n\pi - \frac{\pi}{2}\right) + \sin\frac{\pi}{2}\right\}.\sec\theta$$

$$= \tfrac{1}{2}\sec\theta\,\{(-1)^{n+1}\cos 2n\theta + 1\};$$

also for $-\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4}$, $1 \leqslant \sec\theta < \sqrt{2}$, and

$$0 \leqslant (-1)^{n+1}\cos 2n\theta + 1 \leqslant 2;$$

$\therefore 0 \leqslant C_n < \frac{1}{2}\sqrt{2}\,.\,2 = \sqrt{2}$; (ii) follows by integrating w.r.t. θ.

EXERCISE VII. d. (p. 132.)

1. $\tan 2\theta - \tan\theta = \dfrac{\sin 2\theta\cos\theta - \sin\theta\cos 2\theta}{\cos 2\theta\cos\theta}$

$$= \frac{\sin(2\theta-\theta)}{\cos 2\theta\cos\theta} = \frac{\sin\theta}{\cos\theta}.\sec 2\theta;$$

$$\text{series} = \Sigma[\tan(2^{r-1}\theta)\,.\,\sec(2^r\theta)]$$

$$= \Sigma[\tan(2^r\theta) - \tan(2^{r-1}\theta)]$$

=, as in Example 5, $\tan(2^n\theta) - \tan\theta$.

2. Identity is proved in Example 4;

$$\text{series} = (\cot\theta - 2\cot 2\theta) + \tfrac{1}{2}\left(\cot\frac{\theta}{2} - 2\cot\theta\right) + \dots + \frac{1}{2^{n-1}}\left(\cot\frac{\theta}{2^{n-1}} - 2\cot\frac{\theta}{2^{n-2}}\right)$$

$$= -2\cot 2\theta + \frac{1}{2^{n-1}}\cot\left(\frac{\theta}{2^{n-1}}\right).$$

3. $\cot\beta = \dfrac{1}{\tan\{[\alpha+r\beta] - [\alpha+(r-1)\beta]\}}$

$$= \frac{1 + \tan[\alpha+r\beta]\,.\,\tan[\alpha+(r-1)\beta]}{\tan[\alpha+r\beta] - \tan[\alpha+(r-1)\beta]};$$

hence $\Sigma\{\tan[\alpha+r\beta]\,.\,\tan[\alpha+(r-1)\beta]\}$,

for $r = 1$ to n, = result in Answers.

4. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$; $\therefore \tan 2\theta - \tan^2\theta\,.\,\tan 2\theta = 2\tan\theta$; hence

$$\tan^2\theta\tan 2\theta + \tfrac{1}{2}\tan^2 2\theta\tan 4\theta + \tfrac{1}{4}\tan^2 4\theta\tan 8\theta + \dots$$

$$= (\tan 2\theta - 2\tan\theta) + \tfrac{1}{2}(\tan 4\theta - 2\tan 2\theta) + \dots$$

$$+ \frac{1}{2^{n-1}}[\tan(2^n\theta) - 2\tan(2^{n-1}\theta)] = \text{result in Answers.}$$

5. $\sin^2\theta - 2\sin^2\frac{\theta}{2} = 4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} - 2\sin^2\frac{\theta}{2}$

$$= 2\sin^2\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2} - 1\right) = 2\sin^2\frac{\theta}{2} \cdot \cos\theta;$$

hence $\cos\theta \cdot \sin^2\frac{\theta}{2} + 2\cos\frac{\theta}{2}\sin^2\frac{\theta}{4} + 4\cos\frac{\theta}{4} \cdot \sin^2\frac{\theta}{8} + \ldots$

$$= \tfrac{1}{2}\left(\sin^2\theta - 2\sin^2\frac{\theta}{2}\right) + \left(\sin^2\frac{\theta}{2} - 2\sin^2\frac{\theta}{4}\right) + \ldots$$

$$+ 2^{n-2}\left(\sin^2\frac{\theta}{2^{n-1}} - 2\sin^2\frac{\theta}{2^n}\right) = \text{result in Answers.}$$

6. (i) l.h.s. $= \dfrac{\cos r\theta\sin(r+1)\theta - \cos(r+1)\theta\sin r\theta}{\sin r\theta \cdot \sin(r+1)\theta}$

$$= \frac{\sin\{(r+1)\theta - r\theta\}}{\sin r\theta \cdot \sin(r+1)\theta};$$

(ii) hence, series $= \operatorname{cosec}\theta\,\{(\cot\theta - \cot 2\theta)$

$$+ (\cot 2\theta - \cot 3\theta) + \ldots + [\cot n\theta - \cot(n+1)\theta]$$

$$= \operatorname{cosec}\theta\,\{\cot\theta - \cot(n+1)\theta\}$$

$$= \operatorname{cosec}\theta \cdot \frac{\sin\{(n+1)\theta - \theta\}}{\sin\theta \cdot \sin(n+1)\theta}.$$

7. Use the identity of No. 1 with $\frac{\theta}{2}$ for θ, then

$$\tan\frac{\theta}{2}\sec\theta = \tan\theta - \tan\frac{\theta}{2};$$

hence, series $= \left(\tan\theta - \tan\frac{\theta}{2}\right) + \left(\tan\frac{\theta}{2} - \tan\frac{\theta}{4}\right) + \ldots$

$$+ \left(\tan\frac{\theta}{2^{n-1}} - \tan\frac{\theta}{2^n}\right).$$

8. $\frac{3}{8}(\frac{1}{3}\tan 3\theta - \tan\theta) = \frac{1}{8}\left(\dfrac{\sin 3\theta}{\cos 3\theta} - \dfrac{3\sin\theta}{\cos\theta}\right)$

$$= \tfrac{1}{8}\left\{\frac{\sin 3\theta - 3\sin\theta\,(4\cos^2\theta - 3)}{\cos 3\theta}\right\}$$

$$= \tfrac{1}{8}\left\{\frac{(3\sin\theta - 4\sin^3\theta) - 3\sin\theta\,(1 - 4\sin^2\theta)}{\cos 3\theta}\right\} = \frac{\sin^3\theta}{\cos 3\theta};$$

hence, series $= \frac{3}{8}\{(\frac{1}{3}\tan 3\theta - \tan\theta) + \frac{1}{3}(\frac{1}{3}\tan 9\theta - \tan 3\theta)$

$$+ \tfrac{1}{9}(\tfrac{1}{3}\tan 27\theta - \tan 9\theta) + \ldots\}.$$

9. As in No. 6 (i), $\tan(r+1)\theta - \tan r\theta = \dfrac{\sin\theta}{\cos(r+1)\theta\cos r\theta}$;

hence series $= \dfrac{1}{\sin\theta}\{(\tan 2\theta - \tan\theta) + (\tan 3\theta - \tan 2\theta) + \ldots$

$$+ [\tan(n+1)\theta - \tan n\theta]\}$$

$$= \frac{1}{\sin\theta}\{\tan(n+1)\theta - \tan\theta\}$$

$$= \frac{1}{\sin\theta} \cdot \frac{\sin\{(n+1)\theta - \theta\}}{\cos(n+1)\theta\cos\theta}.$$

10. In No. 3, put $\alpha = \beta = \theta$,

$$\tan r\theta \cdot \tan(r+1)\theta = \cot\theta\,\{\tan(r+1)\theta - \tan r\theta\} - 1;$$

$$\therefore\ \text{series} = \cot\theta\,\{\tan(n+1)\theta - \tan\theta\} - n.$$

11. $\cot\{(r+1)\theta - r\theta\} = \dfrac{\cot(r+1)\theta \cdot \cot r\theta + 1}{\cot r\theta - \cot(r+1)\theta}$;

$$\therefore\ \cot r\theta \cdot \cot(r+1)\theta = \cot\theta\{\cot r\theta - \cot(r+1)\theta\} - 1;$$

$$\therefore\ \text{series} = \cot\theta\,\{\cot\theta - \cot(n+1)\theta\} - n.$$

12. $\dfrac{1}{\cos\theta + \cos(2r+1)\theta} = \dfrac{1}{2\cos(r+1)\theta \cdot \cos r\theta}$;

$\therefore$ series $= \frac{1}{2}$ series in No. 9.

13. $\tan 3\theta - \tan\theta = \dfrac{\sin(3\theta - \theta)}{\cos 3\theta\cos\theta} = \dfrac{\sin 2\theta}{\cos 3\theta\cos\theta} = \dfrac{2\sin\theta}{\cos 3\theta}$;

$$\therefore\ \text{series} = \tfrac{1}{2}\{(\tan 3\theta - \tan\theta) + (\tan 9\theta - \tan 3\theta) + \ldots\}.$$

14. By No. 2, $\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{4}\tan\frac{\theta}{4} + \ldots$, n terms,

$$= -2\cot 2\theta + 2^{1-n}\cot(2^{1-n}\theta);$$

differentiate w.r.t. θ, given series

$$= \frac{d}{d\theta}\{-2\cot 2\theta + 2^{1-n}\cot(2^{1-n}\theta)\};$$

Or use identity $4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta = \sec^2\theta$.

15. Series $= (\sec^2\theta - 1) + \dfrac{1}{2^2}\left(\sec^2\dfrac{\theta}{2} - 1\right) + \ldots =$, from No. 14,

$$4\operatorname{cosec}^2 2\theta - 4^{1-n}\operatorname{cosec}^2(2^{1-n}\theta) - \left\{1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots + \frac{1}{2^{2n-2}}\right\};$$

$$\text{G.P.} = \{1 - (\tfrac{1}{4})^n\} \div \{1 - \tfrac{1}{4}\}.$$

16. l.h.s. $= \tan^{-1}\dfrac{(n+1) - n}{1 + n(n+1)} = \tan^{-1}\dfrac{1}{1 + n + n^2}$; hence,

$$\text{series} = (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + \ldots$$

$$= \tan^{-1}(n+1) - \tan^{-1}1 = \tan^{-1}\frac{(n+1) - 1}{1 + (n+1)}.$$

17. $\tan^{-1}\left\{\dfrac{2}{1 + (2r-1)(2r+1)}\right\} = \tan^{-1}\left\{\dfrac{(2r+1) - (2r-1)}{1 + (2r-1)(2r+1)}\right\}$

$$= \tan^{-1}(2r+1) - \tan^{-1}(2r-1);$$

hence,

$$\text{series} = (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}5 - \tan^{-1}3) + \ldots$$
$$= \tan^{-1}(2n+1) - \tan^{-1}1 = \tan^{-1}\frac{(2n+1)-1}{1+(2n+1)}.$$

18. $\tan^{-1}(n+1) - \tan^{-1}(n-1)$
$$= \tan^{-1}\frac{(n+1)-(n-1)}{1+(n+1)(n-1)} = \tan^{-1}\left(\frac{2}{n^2}\right);$$

hence,

$$\text{series} = (\tan^{-1}2 - 0) + (\tan^{-1}3 - \tan^{-1}1)$$
$$+ (\tan^{-1}4 - \tan^{-1}2) + \ldots, \ n \text{ terms},$$
$$= \{\tan^{-1}2 + \tan^{-1}3 + \ldots + \tan^{-1}(n+1)\}$$
$$- \{0 + \tan^{-1}1 + \tan^{-1}2 + \ldots + \tan^{-1}(n-1)\}$$
$$= \tan^{-1}n + \tan^{-1}(n+1) - \tan^{-1}1.$$

19. $\tan^{-1}\frac{n+2}{2} - \tan^{-1}\frac{n-2}{2} = \tan^{-1}\frac{\frac{1}{2}(n+2)-\frac{1}{2}(n-2)}{1+\frac{1}{4}(n^2-4)}$
$$= \tan^{-1}\frac{8}{n^2} = \cot^{-1}\left(\frac{n^2}{8}\right)$$

hence,

$$\text{series} = \{\tan^{-1}\tfrac{3}{2} - \tan^{-1}(-\tfrac{1}{2})\} + \{\tan^{-1}\tfrac{4}{2} - \tan^{-1}0\}$$
$$+ \{\tan^{-1}\tfrac{5}{2} - \tan^{-1}\tfrac{1}{2}\} + \ldots + \left\{\tan^{-1}\frac{n+2}{2} - \tan^{-1}\frac{n-2}{2}\right\}$$
$$= \left\{\tan^{-1}\tfrac{3}{2} + \tan^{-1}\tfrac{4}{2} + \ldots + \tan^{-1}\frac{n+2}{2}\right\}$$
$$- \left\{\tan^{-1}(-\tfrac{1}{2}) + \tan^{-1}0 + \tan^{-1}\tfrac{1}{2} + \ldots + \tan^{-1}\frac{n-2}{2}\right\}$$
$$= \tan^{-1}\frac{n-1}{2} + \tan^{-1}\frac{n}{2} + \tan^{-1}\frac{n+1}{2} + \tan^{-1}\frac{n+2}{2}$$
$$- \tan^{-1}(-\tfrac{1}{2}) - \tan^{-1}0 - \tan^{-1}\tfrac{1}{2} - \tan^{-1}1.$$

20. $\tan^{-1}\frac{n}{n+1} - \tan^{-1}\frac{n-1}{n} = \tan^{-1}\left\{\frac{\frac{n}{n+1} - \frac{n-1}{n}}{1+\frac{n}{n+1}\cdot\frac{n-1}{n}}\right\}$
$$= \tan^{-1}\frac{n^2-(n^2-1)}{n(n+1)+n(n-1)} = \tan^{-1}\left(\frac{1}{2n^2}\right) = \cot^{-1}(2n^2);$$

hence, series $= (\tan^{-1}\frac{1}{2} - \tan^{-1}0)$
$$+ (\tan^{-1}\tfrac{2}{3} - \tan^{-1}\tfrac{1}{2}) + \ldots + \left(\tan^{-1}\frac{n}{n+1} - \tan^{-1}\frac{n-1}{n}\right).$$

EXERCISE VII. e. (p. 133.)

1. Projections of AB, BC on Ax are $\cos\alpha$, $\cos\left(\alpha+\frac{\pi}{2}\right)$; AC $=\sqrt{2}$, projection of AC on Ax is $\sqrt{2}\cos\left(\alpha+\frac{\pi}{4}\right)$ = sum of projections of AB, BC on Ax. Maximum value is $\sqrt{2}$ when $\alpha+\frac{\pi}{4}=0$.

2. In Fig. 63, let AB = 24, BC = 7; then AC $=\sqrt{(7^2+24^2)}=25$; $24\sin\alpha+7\cos\alpha$ = projection of AC on Ay (perp. to Ax); $\therefore$ max. value = 25.

3. As in VII. b, No. 5, projection of ZC $=\frac{1}{2}$ sum of projections of AC, BC; but projection of AC on AD = AD = $b\sin$ C and projection of BC on AD = 0.

4. Put $a=r\sin\alpha$, $b=r\cos\alpha$, then $r=+\sqrt{(a^2+b^2)}$ and
$$\sin\alpha : \cos\alpha : 1 = a : b : +\sqrt{(a^2+b^2)};$$
$$a\cos\theta + b\sin\theta = r(\sin\alpha\cos\theta + \cos\alpha\sin\theta);$$
r, α are the polar coordinates of point (b, a); max. value is r, when $\theta+\alpha=\frac{\pi}{2}$; min. value is $-r$ when $\theta+\alpha=-\frac{\pi}{2}$.

5. Sum $=\frac{1}{2}\Sigma\sin\frac{2r\pi}{n}$ =, by (14), $\frac{1}{2}\cdot\sin\frac{(n+1)\pi}{n}\cdot\sin\pi\cdot\operatorname{cosec}\frac{\pi}{n}=0$.

6. Sum $=\frac{1}{2}\Sigma\left\{1+\cos\left(2\theta+\frac{4r\pi}{n}\right)\right\}$ =, by (13),
$$\frac{n}{2}+\tfrac{1}{2}\cdot\cos\left[2\theta+\frac{2(n+1)\pi}{n}\right]\cdot\sin 2\pi\cdot\operatorname{cosec}\frac{2\pi}{n}=\frac{n}{2}+0.$$

7. Series $=\frac{1}{2}\Sigma\{\cos(2r+1)\theta+\cos\theta\}$
$$=\tfrac{1}{2}\cdot\cos(n+2)\theta\cdot\sin n\theta\operatorname{cosec}\theta+\tfrac{1}{2}n\cos\theta$$
$$=\tfrac{1}{4}\{\sin(2n+2)\theta-\sin 2\theta\}\operatorname{cosec}\theta+\tfrac{1}{2}n\cos\theta.$$

8. Series $=\frac{1}{2}\Sigma\{\cos\beta-\cos[2\alpha+(2r+1)\beta]\}$; use (13).

9. $\Sigma\sin(r+1)\theta\cdot\sin^2\frac{1}{2}(r+1)\theta=\frac{1}{2}\Sigma\sin(r+1)\theta\cdot[1-\cos(r+1)\theta]$
$$=\tfrac{1}{2}\Sigma\sin(r+1)\theta-\tfrac{1}{4}\Sigma\sin(2r+2)\theta;$$
use (14).

10. $2\sin\frac{\theta}{2}\{\frac{1}{2}+\Sigma(\cos r\theta)\}$
$$=\sin\frac{\theta}{2}+\Sigma\{\sin(r+\tfrac{1}{2})\theta-\sin(r-\tfrac{1}{2})\theta\}=\sin(n+\tfrac{1}{2})\theta;$$
$\therefore$ series $=1+\Sigma\{\cos r(\alpha+\beta)+\cos r(\alpha-\beta)\}$
$$=\{\tfrac{1}{2}+\Sigma\cos r(\alpha+\beta)\}+\{\tfrac{1}{2}+\Sigma\cos r(\alpha-\beta)\}$$

$$=\frac{\sin(n+\frac12)(\alpha+\beta)}{2\sin\frac{\alpha+\beta}{2}}+\frac{\sin(n+\frac12)(\alpha-\beta)}{2\sin\frac{\alpha-\beta}{2}}$$

$$=\frac{1}{2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}\times$$

$$\left\{\sin\frac{\alpha-\beta}{2}\sin(n+\tfrac12)(\alpha+\beta)+\sin\frac{\alpha+\beta}{2}\sin(n+\tfrac12)(\alpha-\beta)\right\}$$

$$=\frac{1}{2(\cos\beta-\cos\alpha)}\{\cos(n\alpha+\overline{n+1}\beta)-\cos(\overline{n+1}\alpha+n\beta)$$
$$+\cos(n\alpha-\overline{n+1}\beta)-\cos(\overline{n+1}\alpha-n\beta)\}$$

$$=\frac{1}{\cos\beta-\cos\alpha}\{\cos n\alpha\cos\overline{n+1}\beta-\cos\overline{n+1}\alpha\cos n\beta\}.$$

11. $2(1-\cos\theta)\,r\sin r\theta=2r\sin r\theta-r\{\sin(r+1)\theta+\sin(r-1)\theta\}$
$=\{(r+1)\sin r\theta-r\sin(r+1)\theta\}-\{r\sin(r-1)\theta-(r-1)\sin r\theta\}$.
Put $r=1, 2, \dots n$ and add;
∴ $2(1-\cos\theta)\Sigma(r\sin r\theta)=(n+1)\sin n\theta-n\sin(n+1)\theta$.
Or find $\Sigma(\cos r\theta)$ by (13) and differentiate w.r.t. θ.

12. $2(1-\cos\theta)r^2\cos r\theta=2r^2\cos r\theta-r^2\{\cos(r+1)\theta+\cos(r-1)\theta\}$
$=\{(r+1)^2\cos r\theta-r^2\cos(r+1)\theta\}$
$-\{r^2\cos(r-1)\theta-(r-1)^2\cos r\theta\}-2\cos r\theta$.
Put $r=1, 2, \dots n$ and add;
∴ $2(1-\cos\theta)\Sigma(r^2\cos r\theta)$
$=(n+1)^2\cos n\theta-n^2\cos(n+1)\theta-1-2\Sigma(\cos r\theta)$.
But $\sin\frac{\theta}{2}(1+2\Sigma\cos r\theta)$

$$=\sin\frac{\theta}{2}+\Sigma\{\sin(r+\tfrac12)\theta-\sin(r-\tfrac12)\theta\}=\sin(n+\tfrac12)\theta;$$

∴ $2(1-\cos\theta)\Sigma(r^2\cos r\theta)$
$=(n+1)^2\cos n\theta-n^2\cos(n+1)\theta-\sin(n+\frac12)\theta\,.\,\operatorname{cosec}\frac12\theta$.
Or differentiate w.r.t. θ the value of $\Sigma(r\sin r\theta)$ in No. 11.

13. $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2^2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\cos\frac{\theta}{2}$

$$=2^3\sin\frac{\theta}{8}\cos\frac{\theta}{8}\cos\frac{\theta}{4}\cos\frac{\theta}{2}=\text{etc.}$$

Take logs of each side;

∴ $\log\sin\theta-\log\sin\left(\frac{\theta}{2^n}\right)=n\log2+\Sigma\log\cos\left(\frac{\theta}{2^r}\right)$;

∴ $\Sigma\log\cos\left(\frac{\theta}{2^r}\right)=\log\sin\theta-\log\sin\left(\frac{\theta}{2^n}\right)-n\log2$;

differentiate w.r.t. θ;

∴ $\Sigma\frac{1}{2^r}(-1)\,.\tan\left(\frac{\theta}{2^r}\right)=\cot\theta-\frac{1}{2^n}\cot\left(\frac{\theta}{2^n}\right)$.

Note the alternative method in VII. d, No. 2.

14. Series $=\log(\cos\theta\,.\cos2\theta\,.\cos4\theta\dots n$ factors)

$$=\log\left\{\frac{\sin2\theta}{2\sin\theta}\cdot\frac{\sin4\theta}{2\sin2\theta}\cdots\frac{\sin(2^n\theta)}{2\sin(2^{n-1}\theta)}\right\}=\log\left\{\frac{\sin(2^n\theta)}{2^n\sin\theta}\right\}.$$

15. $s_1=s_3=s_5=\dots=1$; $s_2=s_4=s_6=\dots=0$;
∴ $s_1+s_2+\dots+s_{2p-1}=s_1+s_2+\dots+s_{2p}$
$=s_1+s_3+\dots+s_{2p-1}=p$;
∴ $\frac{s_1+s_2+\dots+s_n}{n}=\frac12$ if n is even, and
$=\frac{n+1}{2n}$ if n is odd; ∴ it $\to\frac12$.

16. $s_n=\sin\frac12(n+1)\theta\,.\sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2}$, by eqn. (14),

$=\frac12\operatorname{cosec}\frac{\theta}{2}\cdot\left\{\cos\frac{\theta}{2}-\cos(n+\frac12)\theta\right\}$;

∴ $s_1+s_2+\dots+s_n=\frac12\operatorname{cosec}\frac{\theta}{2}\left\{n\cos\frac{\theta}{2}-\Sigma\cos(r+\frac12)\theta\right\}$;

∴ $\frac{s_1+s_2+\dots+s_n}{n}$

$=\frac12\cot\frac{\theta}{2}-\frac{1}{2n}\operatorname{cosec}\frac{\theta}{2}\cdot\cos(\frac12n+1)\theta\,.\sin\frac12n\theta\,.\operatorname{cosec}\frac{\theta}{2}$;

but modulus of second term $<\left|\frac{1}{2n}\operatorname{cosec}^2\frac{\theta}{2}\right|$, and this $\to0$ when $n\to\infty$, for $\theta\neq2r\pi$.

(ii) By eqn. 13, unless $\theta=k\pi$,
$s_n=\cos n\theta\sin n\theta\operatorname{cosec}\theta=\frac12\operatorname{cosec}\theta\sin2n\theta$;
∴ $S\equiv s_1+s_2+\dots+s_n$
$=\frac12\operatorname{cosec}\theta(\sin(n+1)\theta\sin n\theta\operatorname{cosec}\theta)$,
but $|\sin(n+1)\theta|\leqslant1$, $|\sin n\theta|\leqslant1$; ∴ $\lim(S/n)=0$.

17. $\cot\theta-2\cot2\theta=\frac{1}{\tan\theta}-\frac{2(1-\tan^2\theta)}{2\tan\theta}=\tan\theta$;

∴ series $=(\cot\theta-2\cot2\theta)+2(\cot2\theta-2\cot4\theta)+\dots$
$+2^{n-1}[\cot(2^{n-1}\theta)-2\cot(2^n\theta)]$.

18. From No. 17, by differentiating w.r.t. θ,

$$\sec^2\theta + 4\sec^2 2\theta + 4^2\sec^2 4\theta + 4^3\sec^2 8\theta + \ldots + 4^{n-1}\sec^2(2^{n-1}\theta)$$
$$= -\operatorname{cosec}^2\theta + 4^n \operatorname{cosec}^2(2^n\theta);$$

but $\tan^2\theta = \sec^2\theta - 1$;

$\therefore$ given series $= -\operatorname{cosec}^2\theta + 4^n\operatorname{cosec}^2(2^n\theta)$
$-\{1 + 4 + 4^2 + \ldots + 4^{n-1}\}$.

19. $\sin\theta = 3\sin\frac{\theta}{3} - 4\sin^3\frac{\theta}{3}$;

$$\therefore\ 4\left[3^{r-1}\,.\,\sin^3\left(\frac{\theta}{3^r}\right)\right] = 3^r\,.\,\sin\left(\frac{\theta}{3^r}\right) - 3^{r-1}\sin\left(\frac{\theta}{3^{r-1}}\right);$$

$$\therefore\ \text{series} = \left(3\sin\frac{\theta}{3} - \sin\theta\right) + \left(3^2\sin\frac{\theta}{3^2} - 3\sin\frac{\theta}{3}\right) + \ldots.$$

Differentiate w.r.t. θ, then

$$4\,.\,\Sigma\left\{3^{r-1}\,.\,\frac{1}{3^r}\,.\,3\sin^2\frac{\theta}{3^r}\,.\,\cos\frac{\theta}{3^r}\right\} \equiv 4\,.\,\Sigma\left(\sin^2\frac{\theta}{3^r}\,.\,\cos\frac{\theta}{3^r}\right)$$
$$= 3^n\,.\,\frac{1}{3^n}\cos\frac{\theta}{3^n} - \cos\theta = \cos\frac{\theta}{3^n} - \cos\theta.$$

20. Take A_1x along A_1A_2 and A_1y perp. to A_1A_2; each exterior angle of polygon is $\frac{2\pi}{n} - a \equiv \theta$. Projection of A_1A_{n+1} on Ax = sum of projections of

A_1A_2, A_2A_3, ... on $A_1x = c[1 + \cos\theta + \cos 2\theta + \ldots + \cos(n-1)\theta]$
$= c\cos\frac{1}{2}(n-1)\theta\sin\frac{1}{2}n\theta\operatorname{cosec}\frac{1}{2}\theta \equiv k$, say.

Projection of A_1A_{n+1} on Ay = sum of projections of

A_1A_2, A_2A_3, ... on $A_1y = c\{\sin\theta + \sin 2\theta + \ldots + \sin(n-1)\theta\}$
$= c\sin\frac{1}{2}n\theta\,.\,\sin\frac{1}{2}(n-1)\theta\,.\,\operatorname{cosec}\frac{1}{2}\theta \equiv l$, say.

Then $A_1A_{n+1}{}^2 = k^2 + l^2 = c^2\sin^2\frac{1}{2}n\theta\,.\,\operatorname{cosec}^2\frac{1}{2}\theta$;

$\therefore\ A_1A_{n+1} = c\sin\frac{1}{2}n\theta\operatorname{cosec}\frac{1}{2}\theta$, where $\theta = \frac{2\pi}{n} - a$.

21. If OA_1 makes angle θ with the line and if circumradius = R, sum of projections

$$= R\left[\cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \cos\left(\theta + \frac{4\pi}{n}\right) + \ldots\ n\text{ terms}\right] = 0,$$

as in VII. b, No. 9. **Or** $\overline{OA_1} + \overline{OA_2} + \ldots + \overline{OA_n} = n\,.\,\overline{OO} = 0$ by *M.G.* p. 52.

22. If a_r is projection of A_r on OP, $PA_r{}^2 = PO^2 + OA_r{}^2 - 2PO\,.\,Oa_r$; but from No. 21, $\Sigma Oa_r = 0$; $\therefore\ \Sigma PA_r{}^2 = n\,.\,PO^2 + n\,.\,R^2$. **Or** by *M.G.* p. 62.

23. If O is circumcentre, R circumradius, and $\angle POA_1 = 2\theta$, then

$\angle POA_{r+1} = 2\theta + 2ra$ where $a = \frac{\pi}{2n+1}$; $PA_{r+1} = 2R\sin(\theta + ra)$;

$\therefore$ expression

$= 2R\{\sin\theta - \sin(\theta + a) + \sin(\theta + 2a) - \ldots + \sin(\theta + 2na)\}$

$$= \frac{2R}{2\cos\frac{1}{2}a}\{[\sin(\theta + \tfrac{1}{2}a) + \sin(\theta - \tfrac{1}{2}a)]$$
$$-\left[\sin\left(\theta + \frac{3a}{2}\right) + \sin(\theta + \tfrac{1}{2}a)\right] + \ldots$$
$$+ [\sin(\theta + \tfrac{1}{2}(4n+1)a) + \sin(\theta + \tfrac{1}{2}(4n-1)a)]\}$$

$= R\sec\frac{1}{2}a\{\sin(\theta - \frac{1}{2}a) + \sin(\theta + \frac{1}{2}(4n+1)a)\} = 0$, since
$\theta + \frac{1}{2}(4n+1)a = \theta - \frac{1}{2}a + \pi$.

24. PM_1, PM_2, ... and ON_1, ON_2, ... are the perps. from P, O to A_1A_2, A_2A_3, ... If $\angle PON_1 = \theta$, $\angle PON_r = \theta + \frac{2(r-1)\pi}{n}$;

$$PM_1{}^2 = (a - OP\cos\theta)^2 = a^2 - 2a\,.\,OP\cos\theta + OP^2\cos^2\theta$$
$$= a^2 - 2a\,.\,OP\cos\theta + \tfrac{1}{2}OP^2(1 + \cos 2\theta)$$
$$= a^2 + \tfrac{1}{2}OP^2 - 2a\,.\,OP\cos\theta + \tfrac{1}{2}OP^2\cos 2\theta;$$

$$\therefore\ \Sigma PM_r{}^2 = n(a^2 + \tfrac{1}{2}OP^2) - 2a\,.\,OP\,.\,\Sigma\cos\left[\theta + \frac{2(r-1)\pi}{n}\right]$$
$$+ \tfrac{1}{2}OP^2\,.\,\Sigma\cos\left[2\theta + \frac{4(r-1)\pi}{n}\right]$$

for $r = 1$ to n; but as in VII. b, No. 9, the sum of each series is zero.

EXERCISE VII. f. (p. 135.)

1. $q\,.\,BK = p\,.\,KC$; $\therefore\ q\,.\,\overline{KB} + p\,.\,\overline{KC} = 0$; but $\overline{AB} = \overline{AK} + \overline{KB}$, $\overline{AC} = \overline{AK} + \overline{KC}$; $\therefore\ q\,.\,\overline{AB} + p\,.\,\overline{AC}$
$= q\,.\,\overline{AK} + p\,.\,\overline{AK} + q\,.\,\overline{KB} + p\,.\,\overline{KC} = (q+p)\,.\,\overline{AK}$;
$\therefore$ projecting on BC, $(q+p)\,.\,DK = q\,.\,DB + p\,.\,DC$, taking account of sense of lines;
$\therefore\ (p+q)\,.\,AD\cot AKC = q\,.\,AD\cot B - p\,.\,AD\cot C$.

2. (i) Let BOAE be base of box; proj. of OD on OP = sum of proj. of OA, AE, ED on OP = sum of proj. of OA, OB, OC on OP = OA . $\cos a$ + OB . $\cos\beta$ + OC . $\cos\gamma$.

(ii) If OA′, OB′, OC′ are edges of box for which OP is diagonal, A′, B′, C′, being on OA, OB, OC.

OA′ = OP $\cos a$, OB′ = OP $\cos\beta$, etc.;

then from (i),

OP . $\cos 0$ = OP $\cos a\,.\,\cos a$
+ OP $\cos\beta\,.\,\cos\beta$ + OP $\cos\gamma\,.\,\cos\gamma$.

3. Proj. of AD on BC $=b-a\cos B-c\cos C$; proj. of AD on line perp. to BC $=a\sin B-c\sin C$;

$$\therefore\ d^2=AD^2=(b-a\cos B-c\cos C)^2+(a\sin B-c\sin C)^2$$
$$=b^2+a^2(\cos^2 B+\sin^2 B)+c^2(\cos^2 C+\sin^2 C)$$
$$-2ab\cos B-2bc\cos C+2ac(\cos B\cos C-\sin B\sin C);$$

but $\cos B\cos C-\sin B\sin C=\cos(B+C)$
$$=\cos(2\pi-A-D)=\cos(A+D).$$

Or, use scalar products of vectors; thus $d^2=\bar{d}^2=(\bar{a}+\bar{b}+\bar{c})^2$ $=a^2+b^2+c^2+2ab\cos(\pi-B)+2bc\cos(\pi-C)+2ac\cos\theta$, where $\theta=$ the angle between AB, CD $=(\pi-B)+(\pi-C)=A+D$ [cf. Lamb, *Statics*, Ch. VII].

4. From No. 3,
$$AC^2=e^2+d^2+c^2-2ed\cos E-2cd\cos D+2ec\cos(D+E);$$
but $AC^2=a^2+b^2-2ab\cos B$. **Or,** as in No. 3,
$$(\bar{a}+\bar{b})^2=AC^2=(\bar{c}+\bar{d}+\bar{e})^2, \text{ etc.}$$

5. rth term $=\frac{1}{2}\sin rx\{\cos r(y-z)-\cos r(y+z)\}$
$$=\tfrac{1}{4}\{\sin r(x+y-z)+\sin r(x-y+z)$$
$$-\sin r(x+y+z)+\sin r(y+z-x)\}.$$

But $\Sigma\sin r\alpha=\sin(n+1)\frac{\alpha}{2}\sin\frac{n\alpha}{2}\cdot\operatorname{cosec}\frac{\alpha}{2}$; hence result in Answers.

6. $\cos^2 r\theta\sin^3 r\theta=\frac{1}{4}\sin^2 2r\theta\sin r\theta=\frac{1}{8}(1-\cos 4r\theta).\sin r\theta$
$$=\tfrac{1}{16}(2\sin r\theta-\sin 5r\theta+\sin 3r\theta);$$
then as in No. 5.

7. $\cos^4 r\theta=\frac{1}{4}(1+\cos 2r\theta)^2=\frac{1}{8}(2+4\cos 2r\theta+1+\cos 4r\theta)$; hence $\cos^5 r\theta=\frac{1}{8}(3\cos r\theta+4\cos 2r\theta\cos r\theta+\cos 4r\theta\cos r\theta)$
$$=\tfrac{1}{16}(10\cos r\theta+5\cos 3r\theta+\cos 5r\theta).$$

But $\Sigma\cos r\theta=\cos\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta\operatorname{cosec}\frac{\theta}{2}$
$$=\tfrac{1}{2}\{\sin(n+\tfrac{1}{2})\theta-\sin\tfrac{1}{2}\theta\}\operatorname{cosec}\frac{\theta}{2}$$
$$=\tfrac{1}{2}\sin(n+\tfrac{1}{2})\theta\,.\operatorname{cosec}\frac{\theta}{2}-\tfrac{1}{2};$$
hence result in Answers.

8. $S\equiv n+(n-1)\cos\theta+\ldots+\cos(n-1)\theta$
$$=\Sigma(n-r)\cos r\theta\ \text{ for } r=0 \text{ to } n-1;$$
$\therefore\ 2S(1-\cos\theta)$
$$=\Sigma(n-r)\{2\cos r\theta-\cos(r+1)\theta-\cos(r-1)\theta\};$$
in general, coefficient of $\cos r\theta$ is
$$2(n-r)-(n-r+1)-(n-r-1)=0,$$

except at beginning and end;
$$2S(1-\cos\theta)=2n(1-\cos\theta)+(n-1)(2\cos\theta-\cos 2\theta-1)$$
$$+(n-2)(2\cos 2\theta-\cos 3\theta-\cos\theta)+\ldots$$
$$+2\{2\cos(n-2)\theta-\cos(n-1)\theta-\cos(n-3)\theta\}$$
$$+\{2\cos(n-1)\theta-\cos n\theta-\cos(n-2)\theta\}=2n-(n-1)$$
$$-2n\cos\theta+2(n-1)\cos\theta-(n-2)\cos\theta-\cos n\theta$$
$$=n+1-n\cos\theta-\cos n\theta=n(1-\cos\theta)+(1-\cos n\theta).$$

Or, use result in Ex. VII. e, No. 18, and subtract it from $n\{\cos n\theta+\cos(n-1)\theta+\cos(n-2)\theta+\ldots+1\}$
$$=n\,.\cos\frac{n\theta}{2}\,.\sin\tfrac{1}{2}(n+1)\theta\operatorname{cosec}\frac{\theta}{2}.$$

9. Series $=\cos(n-1)\theta+2\cos(n-2)\theta+\ldots$
$$+(n-1)\cos\theta+n=\text{same as in No. 8.}$$

10. Series $=n+(n-1)\cos^2\theta+(n-2)\cos^2 2\theta+\ldots$
$$=\tfrac{1}{2}\{2n+(n-1)(1+\cos 2\theta)+(n-2)(1+\cos 4\theta)+\ldots\}$$
$$=\tfrac{1}{2}\{[n+(n-1)+(n-2)+\ldots+1]+[n+(n-1)\cos 2\theta$$
$$+(n-2)\cos 4\theta+\ldots+\cos(2n-2)\theta]\}$$
$$=\tfrac{1}{2}\left\{\frac{n(n+1)}{2}+\tfrac{1}{2}\left[n+\frac{1-\cos 2n\theta}{1-\cos 2\theta}\right]\right\},$$
putting 2θ for θ in No. 8, $=\frac{1}{4}\left\{n^2+n+n+\dfrac{\sin^2 n\theta}{\sin^2\theta}\right\}$.

11. Double series in No. 8 and subtract n.

12. Use method of No. 8. **Or,** from Ex. VII, e, No. 11,
$$\sin\theta+2\sin 2\theta+\ldots+n\sin n\theta=\frac{(n+1)\sin n\theta-n\sin(n+1)\theta}{2(1-\cos\theta)};$$
$$\text{also } n(\sin\theta+\sin 2\theta+\ldots+\sin n\theta)=\frac{n\,.\sin\frac{1}{2}(n+1)\theta\,.\sin\frac{1}{2}n\theta}{\sin\dfrac{\theta}{2}}$$
$$=\frac{n}{2\sin^2\dfrac{\theta}{2}}\cdot\{\cos\tfrac{1}{2}\theta-\cos(n+\tfrac{1}{2})\theta\}\,.\sin\tfrac{1}{2}\theta$$
$$=\frac{n}{2(1-\cos\theta)}\cdot\{\sin\theta-\sin(n+1)\theta+\sin n\theta\};$$
subtract.

13. In general, coefficient of x^r is
$$\cos r\theta-2\cos\theta\,.\cos(r-1)\theta+\cos(r-2)\theta=0,$$
except at beginning and end;
product $=1+x(\cos\theta-2\cos\theta)$
$$+x^{n+1}[\cos(n-1)\theta-2\cos\theta\cos n\theta]+x^{n+2}\cos n\theta.$$

14. In general, coefficient of x^r is

$$\sin r\theta - 2\cos\theta \,.\, \sin(r-1)\theta + \sin(r-2)\theta = 0,$$

except at beginning and end;

$$\text{product} = x\sin\theta + x^2(\sin 2\theta - 2\sin\theta\cos\theta)$$
$$+ x^{n+1}[\sin(n-1)\theta - 2\cos\theta \,.\, \sin n\theta] + x^{n+2}\sin n\theta.$$

15. In result of No. 13, put $x = \cos\theta$;

$$\therefore\ (1 - 2\cos^2\theta + \cos^2\theta)\,.\,(1 + \text{given series})$$
$$= 1 - \cos^2\theta - \cos^{n+1}\theta\cos(n+1)\theta + \cos^{n+2}\theta\cos n\theta$$
$$= \sin^2\theta + \cos^{n+1}\theta[\cos\theta\cos n\theta - \cos(n+1)\theta]$$
$$= \sin^2\theta + \cos^{n+1}\theta \,.\, \sin n\theta\sin\theta;$$

$$\therefore\ \text{given series} = -1 + \frac{1}{\sin^2\theta}[\sin^2\theta + \cos^{n+1}\theta\sin n\theta\sin\theta].$$

16. In result of No. 13, put $x = \sec\phi$, $\theta = \phi$;

$$\therefore\ (1 - 2\sec\phi\cos\phi + \sec^2\phi)\,.\,(\text{given series})$$
$$= 1 - \sec\phi\cos\phi - \sec^{n+1}\phi\cos(n+1)\phi + \sec^{n+2}\phi\cos n\phi$$
$$= \sec^{n+2}\phi[\cos n\phi - \cos\phi\cos(n+1)\phi];\ \text{ but}$$
$$\cos n\phi = \cos[(n+1)\phi - \phi]$$
$$= \cos(n+1)\phi\cos\phi + \sin(n+1)\phi\sin\phi;$$

$$\therefore\ \text{series} = \frac{1}{\sec^2\phi - 1}\,.\,\sec^{n+2}\phi\,.\,\sin(n+1)\phi\sin\phi.$$

17. $\sin^3\alpha\cos\alpha = \frac{1}{2}\sin^2\alpha\sin 2\alpha$

$$= \tfrac{1}{4}(1 - \cos 2\alpha)\sin 2\alpha = \tfrac{1}{4}\sin 2\alpha - \tfrac{1}{8}\sin 4\alpha;$$

$$\therefore\ \text{series} = (\tfrac{1}{4}\sin 2\theta - \tfrac{1}{8}\sin 4\theta) + \tfrac{1}{2}(\tfrac{1}{4}\sin 4\theta - \tfrac{1}{8}\sin 8\theta) + \dots$$
$$+ \frac{1}{2^{n-1}}[\tfrac{1}{4}\sin(2^n\theta) - \tfrac{1}{8}\sin(2^{n+1}\theta)].$$

18. $\sec(2r+2)\theta - \sec 2r\theta$

$$= \frac{\cos 2r\theta - \cos(2r+2)\theta}{\cos 2r\theta \,.\, \cos(2r+2)\theta} = \frac{2\sin(2r+1)\theta \,.\, \sin\theta}{\cos 2r\theta \,.\, \cos(2r+2)\theta};$$

$$\therefore\ \sin(2r+1)\theta \,.\, \sec 2r\theta \,.\, \sec(2r+2)\theta$$
$$= \tfrac{1}{2}\operatorname{cosec}\theta\{\sec(2r+2)\theta - \sec 2r\theta\};$$

$$\therefore\ \text{series} = \tfrac{1}{2}\operatorname{cosec}\theta\{(\sec 4\theta - \sec 2\theta)$$
$$+ (\sec 6\theta - \sec 4\theta) + \dots + [\sec(2n+2)\theta - \sec 2n\theta]\}.$$

19. $\dfrac{\sin 4r\theta}{\sin r\theta} = \dfrac{2\sin 2r\theta\cos 2r\theta}{\sin r\theta} = 4\cos r\theta\cos 2r\theta = 2(\cos 3r\theta + \cos r\theta)$;

hence as in No. 7.

20. $\cot\alpha - \cot 3\alpha = \dfrac{\sin 3\alpha\cos\alpha - \cos 3\alpha\sin\alpha}{\sin\alpha\sin 3\alpha}$

$$= \frac{\sin(3\alpha - \alpha)}{\sin\alpha\sin 3\alpha} = \frac{2\cos\alpha}{\sin 3\alpha};$$

$$\therefore\ \cos\alpha\operatorname{cosec} 3\alpha = \tfrac{1}{2}(\cot\alpha - \cot 3\alpha);$$

$$\therefore\ \text{series} = \tfrac{1}{2}\{(\cot\theta - \cot 3\theta) + \cot 3\theta - \cot 9\theta) + \dots$$
$$+ [\cot(3^{n-1}\theta) - \cot(3^n\theta)]\}.$$

21. $\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$;

$$\therefore\ \operatorname{cosech}^2 x - \operatorname{sech}^2 x = \operatorname{cosech}^2 x\operatorname{sech}^2 x = 4\operatorname{cosech}^2 2x;$$
$$\therefore\ \operatorname{th}^2 x \equiv 1 - \operatorname{sech}^2 x = 1 + 4\operatorname{cosech}^2 2x - \operatorname{cosech}^2 x;$$
$$\therefore\ \text{sum to } n \text{ terms of series} = \tfrac{1}{4} + \frac{1}{4^2} + \dots + \frac{1}{4^n}$$
$$+ \tfrac{1}{4}\left(4\operatorname{cosech}^2 x - \operatorname{cosech}^2\frac{x}{2}\right)$$
$$+ \frac{1}{4^2}\left(4\operatorname{cosech}^2\frac{x}{2} - \operatorname{cosech}^2\frac{x}{2^2}\right) + \dots$$
$$+ \frac{1}{4^n}\left(4\operatorname{cosech}^2\frac{x}{2^{n-1}} - \operatorname{cosech}^2\frac{x}{2^n}\right)$$
$$= \tfrac{1}{3}\left(1 - \frac{1}{4^n}\right) + \operatorname{cosech}^2 x - \frac{1}{4^n}\operatorname{cosech}^2\frac{x}{2^n};$$

but when $n \to \infty$, $\dfrac{1}{4^n} \to 0$ and $\dfrac{1}{4^n}\operatorname{cosech}^2\dfrac{x}{2^n} \to \dfrac{1}{x^2}$;

$$\therefore\ \text{sum to } n \text{ terms} \to \tfrac{1}{3} + \operatorname{cosech}^2 x - \frac{1}{x^2}.$$

22. R = circumradius; $A_1A_2 = 2R\sin\dfrac{\pi}{n}$, $A_1A_3 = 2R\sin\dfrac{2\pi}{n}$, etc.;

$$\therefore\ A_1A_2^{\,2} + A_1A_3^{\,2} + \dots + A_1A^2_{n-1} + A_1A_n^{\,2} = 4R^2 \,.\, \Sigma\sin^2\frac{r\pi}{n}$$

for $r = 1$ to $n-1$,

$$= 2R^2 \cdot \Sigma\left(1 - \cos\frac{2r\pi}{n}\right)$$
$$= 2R^2 \cdot \left\{n - 1 - \cos\pi \cdot \frac{\sin(n-1)\dfrac{\pi}{n}}{\sin\dfrac{\pi}{n}}\right\} = 2R^2\{n - 1 + 1\};$$

since $\sin(n-1)\dfrac{\pi}{n} = \sin\left(\pi - \dfrac{\pi}{n}\right) = \sin\dfrac{\pi}{n}$;

$$\therefore\ \text{given expression} = 2R^2 \,.\, n - 2A_1A_2^{\,2},$$

and $R = \tfrac{1}{2}A_1A_2\operatorname{cosec}\dfrac{\pi}{n}$.

23. Let $OP = \epsilon$, $\angle POA_1 = \theta$, then $\angle POA_r = \theta + \dfrac{2(r-1)\pi}{n}$;

$$PA_1^2 = R^2 - 2R\epsilon\cos\theta + \epsilon^2 = R^2\left\{1 - \epsilon\left(\frac{2\cos\theta}{R} - \frac{\epsilon}{R^2}\right)\right\};$$

$$\therefore\ PA_1 \simeq R\left\{1 - \tfrac{1}{2}\epsilon\left(\frac{2\cos\theta}{R} - \frac{\epsilon}{R^2}\right) - \tfrac{1}{8}\epsilon^2\left(\frac{2\cos\theta}{R}\right)^2\right\}$$

$$= R - \epsilon\cos\theta + \frac{\epsilon^2}{2R}\sin^2\theta = R - \epsilon\cos\theta + \frac{\epsilon^2}{4R}(1 - \cos 2\theta);$$

$$\therefore\ \Sigma PA_r \simeq n\left(R + \frac{\epsilon^2}{4R}\right) - \epsilon\Sigma\cos\left[\theta + \frac{2(r-1)\pi}{n}\right]$$

$$- \frac{\epsilon^2}{4R}\Sigma\cos\left[2\theta + \frac{4(r-1)\pi}{n}\right] = n\left(R + \frac{\epsilon^2}{4R}\right),$$

since the sum of each series is zero, as in VII. b, Nos. 9, 10.

24. PM_1, PM_2, ... are perps. to A_1A_2, A_2A_3, ... ; PA_2 is diameter of circumcircle of quad. $PM_1A_2M_2$, i.e. of circumcircle of $\triangle PM_1M_2$;

$$\therefore\ M_1M_2 = PA_2 \,.\, \sin M_1PM_2 = PA_2 \,.\, \sin\frac{2\pi}{n};$$

$$\therefore\ \text{required sum} = \sin^2\frac{2\pi}{n} \,.\, \Sigma PA_r^2 = \sin^2\frac{2\pi}{n} \,.\, n(R^2 + c^2),$$

from VII. e, No. 22.

25. R = radius of circle; $\angle AOQ_0 = 2\theta$,

$$\angle AOQ_1 = \theta,\ \angle AOQ_r = \frac{\theta}{2^{r-1}};\quad \therefore\ \angle ABQ_r = \frac{\theta}{2^r};$$

$$\therefore\ BQ_1 \,.\, BQ_2 \ldots BQ_n = \left(2R\cos\frac{\theta}{2}\right)\cdot\left(2R\cos\frac{\theta}{2^2}\right)\ldots\left(2R\cos\frac{\theta}{2^n}\right)$$

$$= (2R)^n \,.\, \cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\ldots\cos\frac{\theta}{2^n} = R^n \,.\, \frac{\sin\theta}{\sin\left(\frac{\theta}{2^n}\right)},$$

by VII. e, No. 13; but $AQ_0 = 2R\sin\theta$ and $AQ_n = 2R\sin\dfrac{\theta}{2^n}$; $\therefore\ \dfrac{AQ_0}{AQ_n} = \dfrac{\sin\theta}{\sin\left(\frac{\theta}{2^n}\right)}$.

CHAPTER VIII

EXERCISE VIII. a. (p. 139.)

1. Commutative Law of addition,

$$[a, b] + [c, d] = [a + c, b + d] = [c + a, d + b]$$
$$= [c, d] + [a, b];\ [1 + 2, 5 + 3].$$

2. Commutative Law of multiplication,

$$[a, b] \times [c, d] = [ac - bd, ad + bc] = [ca - db, cb + da]$$
$$= [c, d] \times [a, b];\ [6 - 20, 8 + 15].$$

3. l.h.s. $= [e(a + c) - f(b + d),\ f(a + c) + e(b + d)]$
$= [ae - bf,\ af + be] + [ce - df,\ cf + de]$
$= [a, b] \times [e, f] + [c, d] \times [e, f]$.

4. (i) $[5, 0]$; (ii) $[a + c, 0]$; (iii) $[6, 0]$; (iv) $[ac, 0]$; (v) $[5a, 5b]$ $[5a, 5b]$.

5. $[3 + a, 5 + b] = [7, 8]$; $\therefore\ 3 + a = 7$ and $5 + b = 8$.

6 and 7 and 8. Use eqns. (1) to (4).

9. By eqn. (3), $[\cos\theta\cos\phi - \sin\theta\sin\phi,\ \cos\theta\sin\phi + \sin\theta\cos\phi]$.

10. By eqn. (3), $[ac - bd,\ ad + bc] = [0, 0]$; $\therefore\ ac - bd = 0$. $ad + bc = 0$; $\therefore\ (a^2 + b^2)(c^2 + d^2) \equiv (ac - bd)^2 + (ad + bc)^2 = 0$; $\therefore$ either $a^2 + b^2 = 0$ or $c^2 + d^2 = 0$; but, for all values of k, $k^2 \geqslant 0$; $\therefore\ a^2 + b^2 = 0$ requires that $a^2 = 0$ and $b^2 = 0$; similarly $c^2 + d^2 = 0$ requires that $c^2 = 0$ and $d^2 = 0$.

EXERCISE VIII. b. (p. 144.)

1 and 2. Use $[a, b] = a + bi$.

3 to 12. Use eqns. (1) to (4) or ordinary algebraic operations.

13 to 18. See pp. 150, 151.

19. $(x - \cos\theta)^2 - (i\sin\theta)^2 = x^2 - 2x\cos\theta + \cos^2\theta + \sin^2\theta$.

20. $(x + \sin\phi)^2 - (i\cos\phi)^2 = x^2 + 2x\sin\phi + \sin^2\phi + \cos^2\phi$.

21. $1 + 2i + i^2 = 1 + 2i - 1$.

22. $(1 - i)^2 = 1 - 2i + i^2 = 1 - 2i - 1 = -2i$;

$$\therefore\ (1 - i)^3 = -2i(1 - i) = -2i + 2i^2.$$

23 to 25. Use the method of Example 2, p. 142.

26. By No. 21, $\dfrac{(1 + i)^2}{1 - i} = \dfrac{2i}{1 - i}$; use Example 2, p. 142.

27. As in No. 9. 28 and 29. Use Example 2, p. 142.

30. $$\frac{1 + x + iy}{1 - x - iy} = \frac{\{(1 + x) + iy\}\{(1 - x) + iy\}}{\{(1 - x) - iy\}\{(1 - x) + iy\}}$$
$$= \frac{1 - x^2 - y^2 + i(y + xy + y - xy)}{(1 - x)^2 + y^2}.$$

31. (i) By No. 21, $(1 + i)^2 = 2i$; $\therefore\ (1 + i)^{-2} = \dfrac{1}{2i} = \dfrac{i}{2i^2} = -\tfrac{1}{2}i$; similarly $(1 - i)^{-2} = +\tfrac{1}{2}i$;

(ii) $(1 + i)^{-4} = (-\tfrac{1}{2}i)^2 = -\tfrac{1}{4}$ and $(1 - i)^{-4} = (\tfrac{1}{2}i)^2 = -\tfrac{1}{4}$.

32. By p. 143,

$$a^2+b^2=(a+bi)(a-bi)=(2+3i)(3-4i)(2-3i)(3+4i)$$
$$=(2+3i)(2-3i)\,.\,(3+4i)(3-4i)$$
$$=(4+9)\,.\,(9+16)=13\times 25.$$

33. Squares are $\frac{1}{4}(-2\mp 2i\sqrt{3})=\frac{1}{2}(-1\mp i\sqrt{3})$; cubes are

$$\tfrac{1}{2}(-1\mp i\sqrt{3})\,.\,\tfrac{1}{2}(-1\pm i\sqrt{3})=\tfrac{1}{4}(1+3).$$

34. (ii) Write $x-1$ for x and $y-2$ for y in (i).

35. By p. 143, $a-bi=(x-yi)^n$;

$$\therefore\ a^2+b^2=(a+bi)(a-bi)=(x+yi)^n(x-yi)^n$$
$$=\{(x+yi)(x-yi)\}^n.$$

36. If $n=2p$, term $=\{1+1\}\{1+(-1)^p\}=2\{1+(-1)^p\}=4$ if p is even and $=0$ if p is odd; if $n=2p+1$, $i^{2n}=i^{4p+2}=i^2=-1$; $\therefore$ term $=(1-1)(1+i^n)=0$.

37. $2x+y+2+i(3y-x)=0$; $\therefore\ 2x+y+2=0$ and $3y-x=0$; solve.

38. $z-3=[3,\ 4]-[3,\ 0]=[0,\ 4]$;

$$\therefore\ (z-3)^2=[0,\ 4]^2=[-16,\ 0]=-16;$$

similarly $z=[3,\ -4]$ satisfies $(z-3)^2=-16$.

39. $x+yi=(A+Bi)^2=A^2-B^2+2ABi$; $\therefore\ x=A^2-B^2$ and $y=2AB$; Now solve $5=A^2-B^2$, $12=2AB$; these give $A^4-5A^2-36=0$, i.e. $(A^2-9)(A^2+4)=0$; $\therefore$ the only (positive) value of A is $A=3$; so $B=2$.

If $\sqrt{i}=A+Bi$, as before $A^2-B^2=0$, $2AB=1$;

$$\therefore\ A=B=\frac{1}{\sqrt{2}}.$$

40. $k^2+2(a+ib)k+c+id=0$, i.e. $k^2+2ak+c+i(2bk+d)=0$, requires that $k^2+2ak+c=0$ and $2bk+d=0$ for the same k. If $b\neq 0$, $k=-\dfrac{d}{2b}$; $\therefore\ \dfrac{d^2}{4b^2}-\dfrac{2ad}{2b}+c=0$. If $b=0$, then $d=0$ and $(k+a)^2=a^2-c$, this requires $a^2\geqslant c$.

EXERCISE VIII. c. (p. 147.)

1 to 24. Use the method on p. 146 or write down the answer by inspection.

25. (i) For $-\dfrac{\pi}{2}<\alpha<0$, $\cos\alpha>0$;

expression $=\cos\alpha\,(\cos\alpha+i\sin\alpha)$;

$\therefore$ the modulus is $\cos\alpha$;

(ii) For $\dfrac{\pi}{2}<\alpha<\pi$, $\cos\alpha<0$; expression must be written

$$(-\cos\alpha)\{-\cos\alpha-i\sin\alpha\}$$
$$=(-\cos\alpha)\{\cos(\alpha\pm\pi)+i\sin(\alpha\pm\pi)\}$$

since the modulus is essentially positive; also here $|\alpha-\pi|<|\alpha+\pi|$; $\therefore$ the principal value of the amplitude is $\alpha-\pi$;

(iii) For $-\pi<\alpha<-\dfrac{\pi}{2}$, as in (ii), $\cos\alpha<0$; $\therefore$ modulus is $(-\cos\alpha)$; but $|\alpha-\pi|>|\alpha+\pi|$; $\therefore$ the principal value of the amplitude is $\alpha+\pi$.

26. $1+i\tan\alpha=\sec\alpha(\cos\alpha+i\sin\alpha)$

$$=(-\sec\alpha)\{\cos(\alpha+\pi)+i\sin(\alpha+\pi)\},$$

according as $\sec\alpha$ is positive or negative,

i.e. as $2n\pi-\dfrac{\pi}{2}<\alpha<2n\pi+\dfrac{\pi}{2}$ or $2n\pi+\dfrac{\pi}{2}<\alpha<2n\pi+\dfrac{3\pi}{2}$.

27. $1+i\cot\alpha=\operatorname{cosec}\alpha\,(\sin\alpha+i\cos\alpha)$

$$=\operatorname{cosec}\alpha\left\{\cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)\right\}$$
$$=(-\operatorname{cosec}\alpha)\left\{\cos\left(\frac{3\pi}{2}-\alpha\right)+i\sin\left(\frac{3\pi}{2}-\alpha\right)\right\}$$

according as, (see No. 26),

$$2n\pi<\alpha<(2n+1)\pi \text{ or } (2n+1)\pi<\alpha<(2n+2)\pi.$$

28. $\tan\beta-i=\sec\beta(\sin\beta-i\cos\beta)$

$$=\sec\beta\left\{\cos\left(\beta-\frac{\pi}{2}\right)+i\sin\left(\beta-\frac{\pi}{2}\right)\right\}$$
$$=(-\sec\beta)\left\{\cos\left(\beta+\frac{\pi}{2}\right)+i\sin\left(\beta+\frac{\pi}{2}\right)\right\}$$

according as etc., see No. 26.

30. Expression $=2\cos^2\dfrac{\theta}{2}+2i\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

$$=2\cos\frac{\theta}{2}\cdot\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$$
$$=\left(-2\cos\frac{\theta}{2}\right)\cdot\left\{\cos\left(\frac{\theta}{2}+\pi\right)+i\sin\left(\frac{\theta}{2}+\pi\right)\right\}$$

according as, (see No. 26),

$$2n\pi-\frac{\pi}{2}<\frac{\theta}{2}<2n\pi+\frac{\pi}{2},\ \text{i.e. } (4n-1)\pi<\theta<(4n+1)\pi \text{ or etc.}$$

31. Expression, as in No. 30, $=2\cos\frac{\theta}{2}\cdot\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)$

$$=2\cos\frac{\theta}{2}\left\{\cos\left(-\frac{\theta}{2}\right)+i\sin\left(-\frac{\theta}{2}\right)\right\}$$

$$=\left(-2\cos\frac{\theta}{2}\right)\cdot\left\{\cos\left(\pi-\frac{\theta}{2}\right)+i\sin\left(\pi-\frac{\theta}{2}\right)\right\}$$

according as etc., see No. 30.

32. Write $\frac{\pi}{2}-\theta$ for θ in No. 30.

33. $(\cos\alpha+\cos\beta)+i(\sin\alpha+\sin\beta)$

$$=2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)+2i\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)$$

$$=2\cos\tfrac{1}{2}(\alpha-\beta)\,.\,\{\cos\tfrac{1}{2}(\alpha+\beta)+i\sin\tfrac{1}{2}(\alpha+\beta)\}$$

$$=\{-2\cos\tfrac{1}{2}(\alpha-\beta)\}\cdot\{\cos\tfrac{1}{2}(\alpha+\beta+2\pi)+i\sin\tfrac{1}{2}(\alpha+\beta+2\pi)\},$$

according as etc., see No. 30.

34. $(\cos\alpha-\cos\beta)+i(\sin\alpha-\sin\beta)$. Write $\beta+\pi$ for β in No. 33.

35. If $1+r\cos\phi+ir\sin\phi=s(\cos\psi+i\sin\psi)$, by p. 145,

$$s=+\sqrt{\{(1+r\cos\phi)^2+r^2\sin^2\phi\}}$$

$$=+\sqrt{\{1+2r\cos\phi+r^2(\cos^2\phi+\sin^2\phi)\}}$$

and $\cos\psi=\frac{1}{s}(1+r\cos\phi)$, $\sin\psi=\frac{1}{s}\cdot r\sin\phi$.

36. Origin O; $\overline{\text{OA}}=1$,

$$\overline{\text{OB}}=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right),\ \overline{\text{OC}}=\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right);$$

then ABC is an equilateral triangle, circumcentre O; and $\overline{\text{OA}}+\overline{\text{OB}}+\overline{\text{OC}}=0$; i.e. O is centroid of A, B, C. This holds for any equilateral triangle in the circle;

$$\therefore\ \{\cos\theta+i\sin\theta\}+\left\{\cos\left(\theta+\frac{2\pi}{3}\right)+i\sin\left(\theta+\frac{2\pi}{3}\right)\right\}$$

$$+\left\{\cos\left(\theta+\frac{4\pi}{3}\right)+i\sin\left(\theta+\frac{4\pi}{3}\right)\right\}=0.$$

It also holds for any regular polygon in the circle;

$$\therefore\ \Sigma\left\{\cos\left(\theta+\frac{2r\pi}{n}\right)+i\sin\left(\theta+\frac{2r\pi}{n}\right)\right\},$$

for $r=0$ to $n-1$, $=0$.

37. Origin O; $\overline{\text{OA}}=1$, $\overline{\text{OP}}=\cos\theta+i\sin\theta$; AP is a chord of the unit circle, centre O; the mid-point M of AP is given by $\overline{\text{OM}}=\cos\frac{\theta}{2}\cdot\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$; relation is $\overline{\text{OA}}+\overline{\text{OP}}=2\overline{\text{OM}}$.

38. Origin O; $\overline{\text{OP}_1}=r_1(\cos\theta_1+i\sin\theta_1)$; $\overline{\text{OP}_2}=r_2(\cos\theta_2+i\sin\theta_2)$; complete parallelogram $\text{P}_2\text{OP}_1\text{N}$; then

$$\overline{\text{ON}}=\overline{\text{OP}_1}+\overline{\text{OP}_2}=\sin(\theta_2-\theta_1)\cdot\left\{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right\};$$

$\therefore$ ON is of length $\sin(\theta_2-\theta_1)$ and ON is perp. to Ox;

$$\therefore\ \angle\text{NOP}_1=\frac{\pi}{2}-\theta_1,\ \angle\text{P}_1\text{NO}=\angle\text{NOP}_2=\theta_2-\frac{\pi}{2},$$

$$\angle\text{OP}_1\text{N}=\pi-(\theta_2-\theta_1);$$

$\therefore$ from $\triangle\text{OP}_1\text{N}$,

$$\frac{r_1}{\sin\left(\theta_2-\frac{\pi}{2}\right)}=\frac{\sin(\theta_2-\theta_1)}{\sin(\theta_2-\theta_1)}=1=\frac{r_2}{\sin\left(\frac{\pi}{2}-\theta_1\right)}.$$

EXERCISE VIII. d. (p. 149.)

1. (i) Draw AP equal and parallel to BO;

$$\overline{\text{OP}}=\overline{\text{OA}}+\overline{\text{AP}}=\overline{\text{OA}}-\overline{\text{OB}}=\alpha-\beta;$$

(ii) Bisect AB at Q; $\overline{\text{OQ}}=\frac{1}{2}(\alpha+\beta)$;

(iii) Draw AR parallel to OB and $=2$OB; $\overline{\text{OR}}=\overline{\text{OA}}+\overline{\text{AR}}=\alpha+2\beta$;

(iv) Draw AS parallel to BO and $=3$BO; $\overline{\text{OS}}=\overline{\text{OA}}+\overline{\text{AS}}=\alpha-3\beta$.

2. As in 1 (ii), $\overline{\text{OC}}+\overline{\text{OA}}=2\overline{\text{OB}}$; $\therefore$ $\overline{\text{OC}}=2\overline{\text{OB}}-\overline{\text{OA}}$.

3. $\overline{\text{AB}}=\overline{\text{AO}}+\overline{\text{OB}}=\overline{\text{OB}}-\overline{\text{OA}}=\beta-\alpha$; $\overline{\text{BC}}=\overline{\text{OC}}-\overline{\text{OB}}=\gamma-\beta$; but $\overline{\text{AB}}=2\overline{\text{BC}}$; $\therefore$ $\beta-\alpha=2(\gamma-\beta)$.

4. $\overline{\text{OP}}=z$; to represent $az+b$, produce OP to Q so that OQ $=a\,.$ OP and draw QR parallel to Ox and $=b$, then $\overline{\text{OR}}=\overline{\text{OQ}}+\overline{\text{QR}}=a\,.\,z+b$. Each part of No. 4 is a special case of this construction.

5. In Fig. 69,

(i) $|z_1+z_2|+|z_2|=\text{OR}+\text{OQ}>\text{QR}=\text{OP}=|z_1|$;

(ii) $\text{SP}=\text{PR}=\text{OQ}=|z_2|$; $\therefore$ $|z_1-z_2|+|z_2|=\text{OS}+\text{SP}>\text{OP}=|z_1|$.

6. (i) In Fig. 69, if R lies on OP between O and P, i.e. if Q lies on PO produced and if OQ $\leqslant$ PO, then $|z_1+z_2|=|z_1|-|z_2|$; this is equivalent to $\text{am}(z_1)=\pm\pi+\text{am}(z_2)$ and $|z_2|\leqslant|z_1|$;

(ii) In Fig. 69, if S lies on OP between O and P, i.e. if Q lies on OP between O and P, then $|z_1-z_2|=|z_1|-|z_2|$; this is equivalent to $\text{am}(z_1)=\text{am}(z_2)$ and $|z_2|\leqslant|z_1|$.

7. As in No. 4, if $|z|=1$, az is represented by a point on a circle centre O, rad. a; $az+b$ is represented by a point on the circle obtained by moving the former circle a distance b parallel to Ox, i.e. a circle, rad. a, centre $(b, 0)$.

8. If $|az - b - ci| = d$, then $\left|z - \left(\frac{b}{a} + \frac{c}{a}i\right)\right| = \frac{d}{|a|}$; this means that the distance of P from the point which represents $\frac{b}{a} + \frac{c}{a}i$, i.e. coordinates $\left(\frac{b}{a}, \frac{c}{a}\right)$ equals $\frac{d}{|a|}$; $\therefore$ P lies on a circle, centre $\left(\frac{b}{a}, \frac{c}{a}\right)$, rad. $\frac{d}{|a|}$. No. 8 (i)-(v) are special cases of this fact. (vi) am $(z) = 0$ means that OP makes a zero angle with Ox; $\therefore$ P lies on the positive half of the x-axis; the negative half is given by am $(z) = \pi$.

9. Use the general statement in the solution of No. 8.

10 to 14. If $|z - a - ib| < c$, the point P representing z is inside a circle, centre (a, b), rad. c. Since $|z - d|$ is the distance of P from $(d, 0)$, the greatest and least values of $|z - d|$ are the distances from P to the circle, centre (a, b), rad. c, measured along the diameter through P; Nos. 10-14 are special cases of this statement; e.g. in No. 14, P lies inside or on the circle, centre $(0, -3)$, rad. 1; the distance of the centre of this circle from $(4, 0)$ is $\sqrt{(4^2 + 3^2)} = 5$; $\therefore$ the greatest and least values of $|z - 4|$ are $5 + 1$ and $5 - 1$.

15. $|z + 2 - 3i| = |z - (-2 + 3i)| =$ distance of P from $(-2, 3)$.

16. As in No. 4, the point representing $1 + z$ is at distance 1 unit, measured parallel to Ox, from the point representing z, which by the data lies inside the circle, centre 0, rad. 1.

17. By No. 16, the point representing $1 + z$ lies inside the circle, centre $(1, 0)$, rad. 1; $\therefore$ the angle OP makes with Ox lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

18. As in No. 16, since $|z| = \frac{1}{2}$, the point representing $1 + z$ lies on the circle, centre $(1, 0)$, rad. $\frac{1}{2}$; the tangents OP, OQ to this circle make angles $+\frac{\pi}{6}$, $-\frac{\pi}{6}$ with Ox, these are the extreme limits for am $(1 + z)$.

19. The point is $\left(\frac{m_1x_1 + m_2x_2}{m_1 + m_2}, \frac{m_1y_1 + m_2y_2}{m_1 + m_2}\right)$; it divides the line joining (x_1, y_1) to (x_2, y_2) in the ratio $m_2 : m_1$ and is the centre of mass of m_1 at P_1 and m_2 at P_2.

20. The point $\left(\frac{\Sigma mx}{\Sigma m}, \frac{\Sigma my}{\Sigma m}\right)$, i.e. the point which represents $\frac{\Sigma mz}{\Sigma m}$, is the centre of mass of m_1 at (x_1, y_1), m_2 at (x_2, y_2), etc., or m_1 at z_1, m_2 at z_2, etc.

EXERCISE VIII. e. (p. 153.)

Numbers 1-14 are direct applications of eqns. (12) *to* (17). *As explained on p. 152, cis θ is a convenient abbreviation for* $\cos\theta + i\sin\theta$.

1. $\frac{\text{cis } 2\alpha}{\text{cis } \alpha} = \text{cis } (2\alpha - \alpha)$.

2. $\frac{\text{cis } \beta}{\text{cis } (-\beta)} = \text{cis } \{\beta - (-\beta)\}$.

3. Eqn. (17).

4. $\frac{\text{cis } (-\phi)}{\text{cis } 2\phi} = \text{cis } (-\phi - 2\phi)$.

5. $\frac{\text{cis } 3\alpha}{\text{cis } (-\alpha)} = \text{cis } (3\alpha + \alpha)$.

6. $\frac{\text{cis } (-4\theta)}{\text{cis } (-2\theta)} = \text{cis } (-4\theta + 2\theta)$.

7. $\frac{\text{cis } (\alpha + \beta)}{\text{cis } \gamma}$.

8. $\frac{\text{cis } 2\theta}{\text{cis } (-\phi)} = \text{cis } \{2\theta - (-\phi)\}$.

9. $\{\text{cis } (-\theta)\}^3$.

10. Eqn. (14).

11. $\frac{\{\text{cis } (-\theta)\}^2}{\text{cis } 3\theta} = \text{cis } \{-2\theta - 3\theta\}$.

12. $\frac{\{\text{cis } (-2\theta)\}^3}{\text{cis } 12\theta} = \text{cis } (-6\theta - 12\theta)$.

13. Eqn. (14), $\cos\pi + i\sin\pi$.

14. Eqn. (14), $\cos 3\pi + i\sin 3\pi$.

15. $\left\{\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right\}^3 = \text{cis}\left(\frac{3\pi}{2} - 3\theta\right)$.

16. $\left\{\cos\left(\theta - \frac{\pi}{2}\right) + i\sin\left(\theta - \frac{\pi}{2}\right)\right\}^5 = \text{cis}\left(5\theta - \frac{5\pi}{2}\right) = \text{cis}\left(5\theta - \frac{\pi}{2}\right)$.

17. By VIII. c, No. 30,

$$(1 + \cos\theta + i\sin\theta)^3 = \left\{2\cos\frac{\theta}{2} \,.\, \text{cis}\frac{\theta}{2}\right\}^3 = \left(2\cos\frac{\theta}{2}\right)^3 . \text{ cis}\frac{3\theta}{2}.$$

18. $1 + i\sin\theta - \cos\theta = 2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$= 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}\left(\cos\frac{\pi - \theta}{2} + i\sin\frac{\pi - \theta}{2}\right)$$

$$= 2\sin\frac{\theta}{2} \,.\, \text{cis}\frac{\pi - \theta}{2};$$ then as in No. 17.

19. As in No. 17, numerator $= 2\cos\theta \,.\, \text{cis } \theta$;

$$\text{fraction} = \frac{2\cos\theta \,.\, \text{cis } \theta}{\text{cis } 2\theta} = 2\cos\theta \,.\, \text{cis } (\theta - 2\theta).$$

20. $$\frac{2\sin^2\theta + 2i\sin\theta\cos\theta}{2\cos^2\theta - 2i\sin\theta\cos\theta} = \frac{2\sin\theta(\sin\theta + i\cos\theta)}{2\cos\theta(\cos\theta - i\sin\theta)} = \frac{\sin\theta \,.\, i(\cos\theta - i\sin\theta)}{\cos\theta(\cos\theta - i\sin\theta)}.$$

21. By VIII. c, No. 32,

$$\text{numerator} = \left\{2\cos\left(\frac{\pi}{4}-\theta\right).\operatorname{cis}\left(\frac{\pi}{4}-\theta\right)\right\}^4;$$

$$\therefore\ \text{denominator} = \left\{2\cos\left(\frac{\pi}{4}-\theta\right).\operatorname{cis}\left(-\frac{\pi}{4}+\theta\right)\right\}^4,$$

writing $-i$ for i;

$$\therefore\ \text{fraction} = \operatorname{cis}\left\{4\left(\frac{\pi}{4}-\theta\right)-4\left(-\frac{\pi}{4}+\theta\right)\right\} = \operatorname{cis}(2\pi-8\theta).$$

22. $1-\sin\theta-i\cos\theta = 1-\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)$

$$= 2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-2i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= 2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left\{\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-i\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

$$= 2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left\{\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\};$$

$$\therefore\ \text{expression} = \frac{1}{\left\{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^3.\left\{\operatorname{cis}\left(-\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^3}$$

$$= \frac{\operatorname{cis}\left(\frac{3\pi}{4}+\frac{3\theta}{2}\right)}{8\sin^3\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}.$$

23. $\dfrac{\operatorname{cis}15\theta\cdot\operatorname{cis}(-3\theta)}{\operatorname{cis}10\theta\cdot\operatorname{cis}6\theta} = \operatorname{cis}(15\theta-3\theta-10\theta-6\theta).$

24. By Example 5, p. 146,

$$1-\operatorname{cis}\theta = 1-\cos\theta-i\sin\theta = 2\sin\frac{\theta}{2}\cdot\operatorname{cis}\frac{\theta-\pi}{2};$$

by VIII. c, No. 31,

$$1+\operatorname{cis}(-\theta) = 1+\cos\theta-i\sin\theta = 2\cos\frac{\theta}{2}\cdot\operatorname{cis}\left(-\frac{\theta}{2}\right);$$

$$\therefore\ \text{fraction} = \left\{\frac{2\sin\frac{\theta}{2}\cdot\operatorname{cis}\frac{\theta-\pi}{2}}{2\cos\frac{\theta}{2}\cdot\operatorname{cis}\left(-\frac{\theta}{2}\right)}\right\}^3$$

$$= \left\{\tan\frac{\theta}{2}\cdot\operatorname{cis}\left(\frac{\theta-\pi}{2}+\frac{\theta}{2}\right)\right\}^3 = \tan^3\frac{\theta}{2}\cdot\operatorname{cis}\left(3\theta-\frac{3\pi}{2}\right).$$

25. By eqn. (14), $\operatorname{cis}\theta$, $\operatorname{cis}(\theta+2\pi)$, $\operatorname{cis}(\theta+4\pi)$; each $=\operatorname{cis}\theta$;
$\therefore\ \operatorname{cis}\frac{\theta}{3}, \operatorname{cis}\frac{\theta+2\pi}{3}, \operatorname{cis}\frac{\theta+4\pi}{3}$ are the 3 cube roots of $\operatorname{cis}\theta$.

26. By eqn. (14), $\operatorname{cis}4\alpha$, $\operatorname{cis}(4\alpha+2\pi)$, $\operatorname{cis}(4\alpha+4\pi)$, $\operatorname{cis}(4\alpha+6\pi)$; each $=\operatorname{cis}4\alpha$; $\operatorname{cis}\alpha$, $\operatorname{cis}\left(\alpha+\frac{\pi}{2}\right)$, $\operatorname{cis}(\alpha+\pi)$, $\operatorname{cis}\left(\alpha+\frac{3\pi}{2}\right)$ are the 4 fourth roots of $\operatorname{cis}4\alpha$.

27. $\operatorname{cis}(A+B+C) = \operatorname{cis}\pi = \cos\pi + i\sin\pi$.

28. By eqn. (14) and p. 152, $(\cos n\theta+i\sin n\theta)+(\cos n\theta-i\sin n\theta)$.

29. By eqn. (17), $\frac{1}{z} = \cos\theta-i\sin\theta$; also $z^n = \cos n\theta+i\sin n\theta$,

$$\frac{1}{z^n} = \cos n\theta - i\sin n\theta.$$

30. $\dfrac{u}{v} = \dfrac{\operatorname{cis}\theta}{\operatorname{cis}\phi} = \operatorname{cis}(\theta-\phi)$; $\dfrac{v}{u} = \operatorname{cis}(\phi-\theta) = \cos(\theta-\phi)-i\sin(\theta-\phi)$.

31. $(\sin\theta+i\cos\theta)^n = \left\{\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right\}^n$, use eqn. (14);

also as in No. 21,

$$\text{fraction} = \operatorname{cis}\left\{n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-n\left(-\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$$

$$= \operatorname{cis}\left(\frac{n\pi}{2}-n\theta\right).$$

Or $(\sin\theta+i\cos\theta)(1+\sin\theta-i\cos\theta)$, expanding,

$$= 1+\sin\theta+i\cos\theta.$$

32. (i) Take Q on circle, $|z|=1$, so that arc AQ $=2$ arc AP where A is (1, 0), then Q represents z^2; produce OQ to R so that OR $=2$OQ; R represents $2z^2$;

(ii) Take Q on circle, $|z|=1$, so that arc AQ $=3$ arc AP, then Q represents z^3;

(iii) The image of P in Ox, see Example 8, p. 153.

33. (i) If in Fig. 72, p. 151, Q coincides with P, K represents z^2;

(ii) Move the point K obtained in (i) parallel to OX, 3 units;

(iii) Construct R, representing $z+1$ by moving P 1 unit parallel to OX; then, as in (i), construct S so that $\triangle$ORS is directly similar to $\triangle$OAR;

(iv) In Fig. 75, R represents $1/z$; produce RO to R′ so that RO $=$ OR′, then R′ represents $-1/z$.

34. The points that correspond to the position $(x_1, 0)$ of P are, in the four cases, $(a+x_1, b)$; $(ax_1, 0)$; $(0, x_1)$; and (ax_1, bx_1). As x_1 changes from -1 to $+1$, these describe segments of st. lines as in answers.

35. The position of P at time t after passing the origin is given by $x=ut\left(\text{for } -\frac{1}{u}\leqslant t\leqslant\frac{1}{u}\right)$. Positions of Q, R representing z^2, $1/z$, are given by $x'=u^2t^2$, $x''=\frac{1}{ut}$. Q moves from (1, 0) to the origin and returns to (1, 0) with initial velocity $\dot{x}'=-2u^2$ and acceleration $\ddot{x}'=2u^2$. R moves from $(-1, 0)$ with initial velocity $\dot{x}''=-\frac{1}{u}$ and decreasing acceleration $\frac{2}{ut^3}$ for time $\frac{1}{u}$ receding to an indefinite distance along the neg. x-axis and returns along the pos. x-axis from an indefinite distance to (1, 0).

36. The point representing $1-z_1$ moves along the x-axis from (2, 0) to infinity $(+\infty)$, and then moves along the x-axis, from $-\infty$ to the origin. Hence z_2 moves from $(\frac{1}{2}, 0)$ in a negative direction to $-\infty$.
$z_3=1-z$, and this decreases from 2 to 0.

37. (i) By No. 32 (i), point moves anticlockwise round the circle $|z|=2$ with twice the angular velocity of P.
(ii) By No. 32 (iii), point moves round the circle $|z|=1$ at the same speed as P, but clockwise.
(iii) $z-2$ describes the circle, centre $(-2, 0)$, rad. 1, anticlockwise; as in Ex. 8, p. 153, $\frac{1}{z-2}$ describes the image in Ox of the inverse w.r.t. the circle $|z|=1$, of the circle traversed by $z-2$. The inverse is a circle, centre $(-\frac{2}{3}, 0)$, rad. $\frac{1}{3}$. The image of this circle in Ox is the circle itself. The inverse of $z-2$ moves clockwise; $\therefore$ the image of the inverse of $z-2$ moves anticlockwise.
(iv) The point P′ representing $z-1$ moves anticlockwise round the circle, centre $(-1, 0)$, rad. 1. If (r, θ) are polar coords. of corresponding position of $(z-1)^2$, the polar coords. of P′ are $(\sqrt{r}, \frac{1}{2}\theta)$; but P′ lies on circle, centre $(-1, 0)$, rad. 1;

$$\therefore\ \sqrt{r}=2\cos(\pi-\tfrac{1}{2}\theta)=-2\cos\frac{\theta}{2};$$
$$\therefore\ r=4\cos^2\frac{\theta}{2}=2(1+\cos\theta);$$

the same cardioid as in Ex. 7, p. 152. Point describes cardioid anticlockwise.

38. (i) $z-1$ describes circle, centre $(-1, 0)$, rad. 1, anticlockwise;
(ii) As in Ex. 8, p. 153, $\frac{1}{z-1}$ describes the image in Ox of the inverse of the circle $z-1$ traverses, w.r.t. the circle $|z|=1$; the inverse is the line $x=-\frac{1}{2}$; $\therefore$ the image is the same line; the inverse point travels down the line from $+\infty$ to $-\infty$; $\therefore$ the image travels up the line from $-\infty$ to $+\infty$;
(iii) From (ii), $\frac{2}{z-1}$ describes the line $x=-1$, upwards;
(iv) $\frac{z+1}{z-1}\equiv\frac{2}{z-1}+1$; $\therefore$ from (iii) $\frac{z+1}{z-1}$ describes the line $x=0$ upwards.

39. (i) $z+2$ describes circle, centre (2, 0), rad. 1;
(ii) As in No. 37 (iii), $\frac{1}{z+2}$ describes the image of the inverse circle w.r.t. $|z|=1$, i.e. the circle centre $(\frac{2}{3}, 0)$, rad. $\frac{1}{3}$, anticlockwise; $\therefore$ $\frac{3}{z+2}$ describes homothetic circle, centre (2, 0), rad. 1, anticlockwise; this is the same circle as in (i), but the points are always diametrically opposite one another;
(iii) $\frac{2z+1}{z+2}\equiv 2-\frac{3}{z+2}$; by (ii), $-\frac{3}{z+2}$ describes circle, centre $(-2, 0)$, rad. 1, clockwise; $\therefore$ $\frac{2z+1}{z+2}$ describes circle, centre (0, 0), rad. 1, clockwise;
(iv) $\frac{az+b}{cz+d}\equiv\frac{a}{c}+\frac{k}{z+f}$ where $f=\frac{d}{c}$, $k=\frac{bc-ad}{c^2}$; the loci described by $z+f$, $\frac{1}{z+f}$, $\frac{k}{z+f}$, $\frac{a}{c}+\frac{k}{z+f}$ are found in succession as in (i)-(iii).

40. If $f(z)$ describes a locus σ, $if(z)$ describes the locus obtained by rotating σ through angle $\frac{\pi}{2}$ about O, anticlockwise,
(i) iz describes circle $|z|=1$, $\frac{\pi}{2}$ ahead of P;
(ii) $z+1$ describes circle, centre (1, 0), rad. 1; $\therefore$ $i(z+1)$ describes the circle, centre (0, 1), rad. 1, anticlockwise;

(iii) $1/(z+1)$ by method of Ex. 8, p. 153, moves down $x=\frac{1}{2}$ from $y=0$ to $-\infty$ and $+\infty$ to 0; $\therefore$ $i/(z+1)$ moves on $y=\frac{1}{2}$ from $x=0$ to ∞ and $-\infty$ to 0.

(iv) $z+i$ describes the circle, centre $(0, 1)$, rad. 1, anti-clockwise; $\therefore$ $\frac{1}{z+i}$ describes the image in Ox of the inverse of this circle w.r.t. $|z|=1$, i.e. the line $y=-\frac{1}{2}$, from right to left; $\therefore$ $\frac{i}{z+i}$ describes $x=+\frac{1}{2}$, downwards.

Or $$\frac{i}{z+i}=\frac{i}{\cos\theta+i\sin\theta+i}=\frac{i(\cos\theta-i\sin\theta-i)}{\cos^2\theta+(\sin\theta+1)^2}$$
$$=\tfrac{1}{2}+i\cdot\frac{\cos\theta}{2+2\sin\theta}\equiv x+iy;$$

$\therefore$ point moves on $x=\frac{1}{2}$, and $y>0$ for $0<\theta<\frac{\pi}{2}$;

$$y<0 \text{ for } \frac{\pi}{2}<\theta<\frac{3\pi}{2};\quad y>0 \text{ for } \frac{3\pi}{2}<\theta<2\pi.$$

41. $z_1=-i+\frac{2}{z-i}$. The point $z-i$ describes the st. line from $(-1, -1)$ to $(+1, -1)$; $\therefore$ by p. 153, the point $\frac{1}{z-i}$ describes the image in OX of the inverse of the segment, w.r.t. $|z|=1$. The inverse is the lower half of the circle on $(0, 0)$ $(0, -1)$ as diameter; its image is the upper half of that on $(0, 0)$ $(0, 1)$. Thus $\frac{2}{z-i}$ describes the upper half of the circle on $(0, 0)$, $(0, 2)$ as diameter. z_1 is 1 unit below $\frac{2}{z-i}$ and describes the upper half of $|z|=1$.

Or, Put $z=\tan\frac{\theta}{2}$, then θ increases from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ and

$$z_1=\frac{1-i\tan\frac{\theta}{2}}{\tan\frac{\theta}{2}-i}=\frac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}{-i\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}=i(\cos\theta-i\sin\theta)$$
$$=\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right);$$

$\therefore$ the point z_1 moves along the upper half of $|z|=1$ from cis π to cis 0.

42. $z_2=\frac{1-iz_1}{z_1-i}=\frac{(z-i)-i(1-iz)}{(1-iz)-i(z-i)}=\frac{1}{z}$ moves as in No. 35. z_3, similarly, $=\frac{1}{z_1}$. By No. 41, $\frac{1}{z_1}$ describes the image of the inverse of the upper half of $|z|=1$, i.e. the lower half of the same circle from $(-1, 0)$ to $(+1, 0)$.

43. In Fig. 75 take Q on the bisector of $\angle x$OP so that OQ^2 = OP.OA. Then if the polar coordinates of Q are (r, θ) those of P are $(r^2, 2\theta)$; but OP = OB . cos BOP; $\therefore$ $r^2=2\cos 2\theta$; $\therefore$ locus of Q is the lemniscate, $r^2=2\cos 2\theta$.

EXERCISE VIII. f. (p. 156.)

1. For $0\leqslant x\leqslant 1$, $\sin^{-1}x$ has a value between 0 and $\frac{\pi}{2}$ inclusive; for $-1\leqslant x\leqslant 0$, it has a value between $-\frac{\pi}{2}$ and 0 inclusive; these are the numerically least values; $\sin^{-1}x$ is undefined if $|x|>1$.

2. For $0\leqslant x\leqslant 1$, $\cos^{-1}x$ has a value between $\frac{\pi}{2}$ and 0 inclusive, and a numerically equal value between $-\frac{\pi}{2}$ and 0, but the positive value is defined as the principal value. For $-1<x<0$, $\cos^{-1}x$ has no value numerically less than the value between π and $\frac{\pi}{2}$ $\left(\text{and that between } -\pi \text{ and } -\frac{\pi}{2}\right)$; the positive value is chosen, by definition. For $x=-1$, in the same way, the p.v. is π.

3. For all values of x, $\tan^{-1}x$ has *one* value between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$; there cannot be any smaller numerical value.

4. By definition. See p. 146.

5. The graphs are the reflections in the line $y=x$ of the graphs of $\sin x$, $\cos x$, $\tan x$.

6. (i) For $0\leqslant x\leqslant 1$, let $\sin^{-1}x=\theta$ where $0\leqslant\theta\leqslant\frac{\pi}{2}$,

$$\text{then } x=\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right).$$
$$\therefore\ \cos^{-1}x=\frac{\pi}{2}-\theta \quad\text{where } 0\leqslant\frac{\pi}{2}-\theta\leqslant\frac{\pi}{2},$$

so that $\frac{\pi}{2}-\theta$ is the principal value of $\cos^{-1}x$;

$$\therefore\ \sin^{-1}x+\cos^{-1}x=\theta+\left(\frac{\pi}{2}-\theta\right).$$

For $-1 \leqslant x < 0$, let $\sin^{-1}x = \phi$ where $-\frac{\pi}{2} \leqslant \phi < 0$, then $x = \sin\phi = \cos\left(\frac{\pi}{2} - \phi\right)$;

$$\therefore \cos^{-1}x = \frac{\pi}{2} - \phi \text{ where } \frac{\pi}{2} \leqslant \frac{\pi}{2} - \phi < \pi,$$

so that $\frac{\pi}{2} - \phi$ is the principal value of $\cos^{-1}x$;

$$\therefore \sin^{-1}x + \cos^{-1}x = \phi + \left(\frac{\pi}{2} - \phi\right).$$

(ii) For $x > 0$, as in (i), if $\tan^{-1}x = \theta$, where $0 < \theta < \frac{\pi}{2}$, then $\cot^{-1}x = \frac{\pi}{2} - \theta$. For $x < 0$, if $\tan^{-1}x = \phi$ where $-\frac{\pi}{2} < \phi < 0$, then $\cot^{-1}x = \left(\frac{\pi}{2} - \phi\right) - \pi = -\frac{\pi}{2} - \phi$, since $0 > -\frac{\pi}{2} - \phi > -\frac{\pi}{2}$; $\therefore \tan^{-1}x + \cot^{-1}x = \phi + \left(-\frac{\pi}{2} - \phi\right)$.

7. (i) For $0 \leqslant x \leqslant 1$, equal since each is between 0 and $\frac{\pi}{2}$ inclusive; for $-1 < x < 0$, unequal since $-\frac{\pi}{2} < \sin^{-1}x < 0$, but $\frac{\pi}{2} < \cos^{-1}\sqrt{(1-x^2)} < \pi$.

(ii) If $x^2 \neq 1$, $\frac{\pi}{2} < \cos^{-1}\{-\sqrt{(1-x^2)}\} < \pi$; $\therefore$ unequal, by No. 1. If $x = +1$, each is $\frac{\pi}{2}$; if $x = -1$,

$$\sin^{-1}x = -\frac{\pi}{2} \text{ and } \cos^{-1}\{-\sqrt{(1-x^2)}\} = +\frac{\pi}{2}.$$

(iii) As in (i), equal for $0 \geqslant x \geqslant -1$, since each lies between 0 and $-\frac{\pi}{2}$ inclusive.

8. (i) Each lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$;

(ii) $\frac{\pi}{2} < \text{am}(x+yi) < \pi$, $-\frac{\pi}{2} < \tan^{-1}\frac{y}{x} < 0$;

(iii) $-\pi < \text{am}(x+yi) < -\frac{\pi}{2}$, $0 < \tan^{-1}\frac{x}{y} < \frac{\pi}{2}$.

9. Graph of am(x) is the positive x-axis and the part of st. line $y = \pi$ for which $x < 0$. If $x > 0$, $\text{am}(ix) = \frac{\pi}{2}$; if $x < 0$, $\text{am}(ix) = -\frac{\pi}{2}$; $\therefore$ graph is two half-lines, $y = \frac{\pi}{2}$ for $x > 0$ and $y = -\frac{\pi}{2}$ for $x < 0$. am$(x+yi)$ is undefined for $x = y = 0$.

10. (i) $\text{am}(x+ix) = \frac{\pi}{4}$ for $x > 0$ and $\text{am}(x+ix) = -\frac{3\pi}{4}$ for $x<0$; graph is two half-lines $y = \frac{\pi}{4}$ for $x > 0$ and $y = -\frac{3\pi}{4}$ for $x < 0$;

(ii) $\text{am}(x-ix) = -\frac{\pi}{4}$ for $x > 0$ and $\text{am}(x-ix) = \frac{3\pi}{4}$ for $x < 0$.

11. For $m > 0$, $\tan^{-1}m$ lies between 0 and $\frac{\pi}{2}$; by No. 6 (ii), $\tan^{-1}m + \tan^{-1}\frac{1}{m} = \frac{\pi}{2}$; $\therefore n > \frac{1}{m}$. For $m<0$, $\tan^{-1}m$ lies between 0 and $-\frac{\pi}{2}$; $\therefore \tan^{-1}m + \tan^{-1}n$ is always $< \frac{\pi}{2}$. Also by No. 6 (ii) for $m < 0$, $\tan^{-1}m + \tan^{-1}\frac{1}{m} = -\frac{\pi}{2}$; $\therefore$ for $n < \frac{1}{m} < 0$, $\tan^{-1}m + \tan^{-1}n < -\frac{\pi}{2}$.

12. Put $\tan^{-1}\frac{m+n}{1-mn} = \text{T}$. If $mn < 1$,

$$|\tan^{-1}m + \tan^{-1}n| \leqslant |\tan^{-1}m| + |\tan^{-1}n|$$
$$< |\tan^{-1}m| + |\cot^{-1}m| = \frac{\pi}{2};$$

$\therefore$ left side lies between $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$; $\therefore k = 0$. If $n > \frac{1}{m} > 0$, by No. 11 (i), $\pi > \tan^{-1}m + \tan^{-1}n > \frac{\pi}{2}$; but $\frac{m+n}{1-mn} < 0$; $\therefore -\frac{\pi}{2} < \text{T} < 0$; $\therefore k = +1$. If $n < \frac{1}{m} < 0$ (and so $mn > 1$), by No. 11 (iii), $-\pi < \tan^{-1}m + \tan^{-1}n < -\frac{\pi}{2}$; but $m+n < 0$ and $1-mn<0$;

$$\therefore \frac{m+n}{1-mn} > 0; \quad \therefore 0 < \text{T} < \frac{\pi}{2}; \quad \therefore k = -1.$$

13. $\theta = \theta_1 + \theta_2 + 2k\pi$ where the integer k must be so chosen that $-\pi < \theta_1 + \theta_2 + 2k\pi \leqslant \pi$; also since θ_1 and θ_2 are principal values, $-\pi < \theta_1 \leqslant \pi$ and $-\pi < \theta_2 \leqslant \pi$;

$$\therefore -2\pi < \theta_1 + \theta_2 \leqslant 2\pi.$$

(i) $-2\pi < \theta_1 + \theta_2 \leqslant -\pi$; $\therefore\ 0 < \theta_1 + \theta_2 + 2\pi \leqslant \pi$; $\therefore\ k = 1$;
(ii) $-\pi < \theta_1 + \theta_2 \leqslant \pi$; $\therefore\ k = 0$;
(iii) $\pi < \theta_1 + \theta_2 < 2\pi$; $\therefore\ -\pi < \theta_1 + \theta_2 - 2\pi < 0$; $\therefore\ k = -1$.

14. By No. 12, putting $m = n = x$, $2\tan^{-1}x = k\pi + \tan^{-1}\dfrac{2x}{1-x^2}$, where $k = 0$ if $-1 < x < 1$, $k = 1$ if $x > 1$, $k = -1$ if $x < -1$; $\therefore$ graph is $y = -\pi$ for $x < -1$, $y = 0$ for $-1 < x < 1$, $y = \pi$ for $x > 1$; function is undefined for $x = \pm 1$.

15. Put $\cos^{-1}x = \theta$ where $0 \leqslant \theta \leqslant \pi$, so that $x = \cos\theta$; then

$2x^2 - 1 = 2\cos^2\theta - 1 = \cos 2\theta$, where $0 \leqslant 2\theta \leqslant 2\pi$;

$\therefore\ \cos^{-1}(2x^2-1) = 2\theta$ if $0 \leqslant 2\theta \leqslant \pi$, i.e. if $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$, i.e. for $1 \geqslant x \geqslant 0$; and $\cos^{-1}(2x^2-1) = 2\pi - 2\theta$ if $0 \leqslant 2\pi - 2\theta < \pi$, i.e. if $\dfrac{\pi}{2} < \theta \leqslant \pi$, i.e. for $0 > x \geqslant -1$;

$\therefore\ \cos^{-1}(2x^2-1) = 2\cos^{-1}x$ for $1 \geqslant x > 0$, and

$\cos^{-1}(2x^2-1) = 2\pi - 2\cos^{-1}x$, for $0 > x \geqslant -1$;

$\therefore$ function $= 0$ for $1 \geqslant x \geqslant 0$; and function $= 4\cos^{-1}x - 2\pi$, for $0 > x \geqslant -1$, $= -4\left(\dfrac{\pi}{2} - \cos^{-1}x\right) = -4\sin^{-1}x$, by No. 6 (i).

16. (i) $\text{Sin}^{-1}x$ increases as x increases if

$$2n\pi - \frac{\pi}{2} < \text{Sin}^{-1}x < 2n\pi + \frac{\pi}{2},$$

and $\therefore\ \dfrac{d}{dx}(\text{Sin}^{-1}x) = +\dfrac{1}{\sqrt{(1-x^2)}}$; but if

$$2n\pi + \frac{\pi}{2} < \text{Sin}^{-1}x < 2n\pi + \frac{3\pi}{2},$$

$$\frac{d}{dx}(\text{Sin}^{-1}x) = -\frac{1}{\sqrt{(1-x^2)}};$$

(ii) $\text{Cos}^{-1}x$ decreases as x increases if

$$2n\pi < \text{Cos}^{-1}x < (2n+1)\pi,$$

and $\therefore\ \dfrac{d}{dx}(\text{Cos}^{-1}x) = -\dfrac{1}{\sqrt{(1-x^2)}}$; but if

$$(2n-1)\pi < \text{Cos}^{-1}x < 2n\pi,$$

$$\frac{d}{dx}(\text{Cos}^{-1}x) = +\frac{1}{\sqrt{(1-x^2)}};$$

(iii) $\text{Tan}^{-1}x$ increases as x increases for any range of values over which the function is continuous, i.e. excluding

$\text{Tan}^{-1}x = k\pi + \dfrac{\pi}{2}$.

EXERCISE VIII. g. (p. 157.)

1. (i) $\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}$;

(ii) $1 + i = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$;

$$\therefore\ (1+i)^n = 2^{\frac{n}{2}} \cdot \left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right).$$

2. $1 + i\sqrt{3} = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$;

$$\text{expression} = 2^8\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right) + 2^8\left(\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right)$$

$$= 2^8 \,.\, 2\cos\frac{8\pi}{3} = 2^8 \,.\, 2 \,.\, (-\tfrac{1}{2}) = -2^8.$$

3. By VIII. c, No. 30, $1 + z = 2\cos\dfrac{\theta}{2} \cdot \text{cis}\,\dfrac{\theta}{2}$;

$$\therefore\ \frac{1}{1+z} = \frac{1}{2\cos\frac{\theta}{2}} \cdot \left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right).$$

Also by Ex. 5, p. 146,

$$1 - z = 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right) = -2i\sin\frac{\theta}{2} \cdot \text{cis}\,\frac{\theta}{2};$$

$$\therefore\ \frac{1+z}{1-z} = \frac{2\cos\frac{\theta}{2} \cdot \text{cis}\,\frac{\theta}{2}}{-2i\sin\frac{\theta}{2} \cdot \text{cis}\,\frac{\theta}{2}} = -\frac{1}{i}\cot\frac{\theta}{2}; \quad -\frac{1}{i} = \frac{i^2}{i} = i.$$

4. From 1 (ii), $(1+i)^n = 2^{\frac{n}{2}}\left(\cos\dfrac{n\pi}{4} + i\sin\dfrac{n\pi}{4}\right)$;

$\therefore\ (1-i)^n = 2^{\frac{n}{2}}\left(\cos\dfrac{n\pi}{4} - i\sin\dfrac{n\pi}{4}\right)$; add; $2 \,.\, 2^{\frac{n}{2}}\cos\dfrac{n\pi}{4}$.

5. Special cases of Ex. VIII. d, Nos. 19, 20;
(i) Put $m_1 = m_2 = 1$;
(ii) Put $m_1 = m_2 = m_3 = 1$;
(iii) Put $m_1 = k$, $m_2 = 1 - k$.

6. As in No. 5, E is $\frac{1}{2}(\alpha + \beta)$, G is $\frac{1}{2}(\gamma + \delta)$; $\therefore$ mid point of EG is $\frac{1}{2}\{\frac{1}{2}(\alpha+\beta) + \frac{1}{2}(\gamma+\delta)\} = \frac{1}{4}(\alpha+\beta+\gamma+\delta)$; similarly the mid point of FH is $\frac{1}{4}(\alpha+\beta+\gamma+\delta)$ and the mid point of PQ is $\frac{1}{4}(\alpha+\beta+\gamma+\delta)$; $\therefore$ the mid points of EG, FH, PQ coincide.

7. If A, B, C, D represent $\alpha, \beta, \gamma, \delta$, then $\frac{1}{2}(\alpha+\gamma)$ and $\frac{1}{2}(\beta+\delta)$ represent mid points of AC and BD; $\therefore$ these points coincide, i.e. AC and BD bisect each other.

8. (i) Length of AP $=|z-\alpha|=|\beta|=$constant;
(ii) Length of AP $=|z-\alpha|=|z-\beta|=$length of BP;
(iii) Length of AP $=3$ times length of BP.

9. $\{\frac{1}{2}\sqrt{2}(\pm 1 \pm i)\}^4=\{\frac{1}{2}[1+(-1)\pm 2i]\}^2=\{\pm i\}^2=-1$; $\therefore$ the roots of equation $x^4=-1$ are

$$x=\pm\frac{1}{\sqrt{2}}\pm\frac{i}{\sqrt{2}};$$

$$\therefore\ x^4+1\equiv\left(x-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\left(x-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\times$$
$$\left(x+\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$$
$$\equiv\left\{\left(x-\frac{1}{\sqrt{2}}\right)^2+\tfrac{1}{2}\right\}\left\{\left(x+\frac{1}{\sqrt{2}}\right)^2+\tfrac{1}{2}\right\}.$$

10. (i) $a+b=(\cos 2\alpha+\cos 2\beta)+i(\sin 2\alpha+\sin 2\beta)$; then as in Ex. VIII. c, No. 33;
(ii) Write $\frac{\pi}{2}+\beta$ for β in (i), $(\cos 2\alpha-\cos 2\beta)+i(\sin 2\alpha-\sin 2\beta)$
$$=2\cos\left(\alpha-\beta-\frac{\pi}{2}\right).\operatorname{cis}\left(\alpha+\beta+\frac{\pi}{2}\right);$$
(iii) By (i) and (ii), $(a-c)(b+d)$
$$=2\sin(\alpha-\gamma).\operatorname{cis}\left(\alpha+\gamma+\frac{\pi}{2}\right).\,2\cos(\beta-\delta).\operatorname{cis}(\beta+\delta).$$

11. $ab=\cos(\theta+\phi)+i\sin(\theta+\phi)$; $\frac{1}{ab}=\cos(\theta+\phi)-i\sin(\theta+\phi)$; add.

12. As in No. 5 (iii), the point $z=\alpha+t(\beta-\alpha)\equiv t\beta+(1-t)\alpha$ divides the join of α to β in the ratio $t:1-t$ and $\therefore$ represents any point on the line joining α to β. Or we may say the point $\alpha+t(\beta-\alpha)$ is reached by moving from α in the direction $\beta-\alpha$, a distance $t.|\beta-\alpha|$, i.e. along the line from α to β.

13. $z=\cos\theta+i\sin\theta$, where it is given that $-\pi<\theta\leqslant\pi$;
$$\therefore\ 1-z^2=1-\cos 2\theta-i\sin 2\theta=2\sin^2\theta-2i\sin\theta\cos\theta$$
$$=2\sin\theta(\sin\theta-i\cos\theta);$$
$$\therefore\ \frac{2}{1-z^2}=\frac{1}{\sin\theta(\sin\theta-i\cos\theta)}=\frac{\sin\theta+i\cos\theta}{\sin\theta(\sin^2\theta+\cos^2\theta)}$$
$$=\frac{1}{\sin\theta}.\operatorname{cis}\left(\frac{\pi}{2}-\theta\right)=\left(-\frac{1}{\sin\theta}\right).\operatorname{cis}\left(-\frac{\pi}{2}-\theta\right).$$

For $0<\theta<\pi$, $\operatorname{cosec}\theta>0$; $\therefore\left|\frac{2}{1-z^2}\right|=\operatorname{cosec}\theta$; also in this case $\frac{\pi}{2}>\frac{\pi}{2}-\theta>-\frac{\pi}{2}$; $\therefore$ am$\left(\frac{2}{1-z^2}\right)=\frac{\pi}{2}-\theta$.

For $-\pi<\theta<0$, $\operatorname{cosec}\theta<0$; $\therefore\left|\frac{2}{1-z^2}\right|=-\operatorname{cosec}\theta$; also in this case $\frac{\pi}{2}>-\frac{\pi}{2}-\theta>-\frac{\pi}{2}$; $\therefore$ am$\left(\frac{2}{1-z^2}\right)=-\frac{\pi}{2}-\theta$.

14. In Fig. 69, OR $=|z_1+z_2|=|z_1-z_2|=$OS; $\therefore$ $\triangle$OPR $\equiv$ $\triangle$OPS, 3 sides; $\therefore$ $\angle$OPR $=\frac{\pi}{2}$; $\therefore$ OP and OQ are at right-angles.

15. (i) am$(z-a)$ equals angle AP makes with Ox; $\therefore$ P lies on the half-line from A parallel to OB;
(ii) The angle AP makes with Ox exceeds the angle BP makes with Ox by $\frac{\pi}{6}$; $\therefore$ P is a point such that PB rotated anticlockwise about P through an angle $\frac{\pi}{6}$ takes up the position PA; $\therefore$ P lies on the major arc of a circle through A and B, centre K, such that an anticlockwise rotation $\frac{\pi}{3}$ transforms KB into KA.
Compare Ex. VIII. h, No. 15.

16. (i) Congruent curve found by displacing Σ an amount $\overline{OA}$;
(ii) If $\alpha\equiv a.\operatorname{cis}\alpha_1$ and $z=r.\operatorname{cis}\theta$, then $\alpha z=ar.\operatorname{cis}(\theta+\alpha_1)$; $\therefore$ if $\beta=\alpha z$, $|\beta|=ar$ and am$(\beta)=\theta+\alpha_1$; $\therefore$ β moves on curve obtained by magnifying Σ in ratio $|\alpha|:1$ and rotating it about O anticlockwise through angle am(α);
(iii) As in Ex. 8, p. 153, $\frac{1}{z}$ describes the reflection in Ox of the inverse of Σ w.r.t. $|z|=1$; $\therefore$ $|\alpha|\div z=|\alpha|.\frac{1}{z}$ describes this curve, magnified in ratio $|\alpha|:1$.

17. $$z_2=\frac{a(c+dz)+bz_1(c+dz)}{c(c+dz)+dz_1(c+dz)}=\frac{a(c+dz)+b(a+bz)}{c(c+dz)+d(a+bz)}$$
$$=\frac{a(b+c)+z(ad+b^2)}{ad+c^2+d(b+c)z},\ =\frac{z-1}{z}\text{ if}$$
$ad+c^2=0$ and $-a(b+c)=ad+b^2=d(b+c)\neq 0$.
As $b+c\neq 0$, $a=-d$; $\therefore$ $c^2=-ad=d^2$; also
$$d(b+c)=ad+b^2=-d^2+b^2;\ \therefore\text{ if }c=d,$$

$d=b-d$ or $b+d=0$, and if $c=-d$, $d=b+d$ or $b=d$.
Thus $a:b:c:d=-1:2:1:1$ or $-1:-1:1:1$ or $-1:0:-1:1$ or $-1:1;\ -1:1$,
but the second and fourth give no solution.

18. As in No. 17, $z_2=\dfrac{a(b+c)+z(ad+b^2)}{ad+c^2+zd(b+c)}$, $=\dfrac{1}{z}$ if $ad+b^2=ad+c^2=0$ and $a(b+c)=d(b+c)\neq 0$. Take $d=1$; $\therefore\ a=1$; $\therefore\ b^2=c^2=-1$, thus $b=c=i$ or $b=c=-i$ to avoid $b+c=0$.

19. Perpendicular from (0, 0) to $3x+4y=p$ is $\pm\dfrac{p}{5}$; no real solution if $\left|\dfrac{p}{5}\right|>|c|$ or $\left(\dfrac{p}{5}\right)^2>c^2$.

20. (i) $w^3-1\equiv(w-1)(w^2+w+1)=0$, but $w-1\neq 0$;
$\therefore\ w^2+w+1=0$;
(ii) $(w^2)^3=w^6=(w^3)^2=1$;
$w^4+w^2+1=w^3\,.\,w+w^2+1=w+w^2+1=0$ by (i);
(iii) $(wa+w^2b)(w^2a+wb)=w^3a^2+w^3b^2+ab(w^4+w^2)$
$=a^2+b^2+ab(-1)$ from (ii).

21. (i) $1+w=-w^2$; $\therefore\ (1+w)^3=-w^6=-1$;
(ii) $2w+2w^2=-2$; $\therefore$ expression $=(-1+w^2)(-1+w)$
$=1-w-w^2+w^3=1+1+1$;
(iii) If $n=3p$, sum $=(1+w+w^2)+w^3(1+w+w^2)+\ldots$
$+w^{3p-3}(1+w+w^2)=0+0+\ldots+0$. If $n=3p+1$,
sum $=0+0+\ldots+0+w^{3p}=1$. If $n=3p-1$,
sum $=0+0+\ldots+0-w^{3p-1}=-w^{3p-3}\,.\,w^2=-w^2$.

22. (i) $(a-bw)(a-bw^2)=a^2+b^2w^3-ab(w+w^2)=a^2+b^3-ab(-1)$
$=a^2+ab+b^2$; expression $=(a-b)(a^2+ab+b^2)$. **Or**, $a^3-b^3=0$ if $a=b$ or $a=wb$ or $a=w^2b$; $\therefore\ a-b,\ a-wb,\ a-w^2b$ are factors; etc.;
(ii) $(a+bw+cw^2)(a+bw^2+cw)=a^2+b^2w^3+c^2w^3+ab(w+w^2)$
$+bc(w^2+w^4)+ca(w^2+w)=a^2+b^2+c^2-ab-bc-ca$,
from No. 20;
expression $=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$.

EXERCISE VIII. h. (p. 159.)

1. By VIII. g, No. 10 (ii),
1st bracket $=\{2\sin\frac{1}{2}(\alpha-\beta)\,.\,\text{cis}\,[\frac{1}{2}(\alpha+\beta+\pi)]\}^n$
$=2^n\sin^n\frac{1}{2}(\alpha-\beta)\,.\,\{\cos n\theta+i\sin n\theta\}$

where $\theta=\frac{1}{2}(\alpha+\beta+\pi)$; writing $-i$ for i,
2nd bracket $=2^n\sin^n\frac{1}{2}(\alpha-\beta)\,.\,\{\cos n\theta-i\sin n\theta\}$;
$\therefore$ expression $=2\,.\,2^n\sin^n\frac{1}{2}(\alpha-\beta)\,.\,\cos\dfrac{n}{2}(\alpha+\beta+\pi)$;
also
$\cos\dfrac{n}{2}(\alpha+\beta+\pi)=\cos\dfrac{n}{2}(\alpha+\beta)\cos\dfrac{n\pi}{2}-\sin\dfrac{n}{2}(\alpha+\beta)\sin\dfrac{n\pi}{2}$;
$\therefore$ for n even,
$\cos\dfrac{n}{2}(\alpha+\beta+\pi)=\cos\dfrac{n}{2}(\alpha+\beta)\cos\dfrac{n\pi}{2}=(-1)^{\frac{n}{2}}\cos\dfrac{n}{2}(\alpha+\beta)$;
for n odd,
$$\cos\frac{n}{2}(\alpha+\beta+\pi)=-\sin\frac{n}{2}(\alpha+\beta)\sin\frac{n\pi}{2}$$
$$=-(-1)^{\frac{n-1}{2}}\,.\,\sin\frac{n}{2}(\alpha+\beta).$$

2. $\text{cis}\,2\theta+\text{cis}^{-1}2\phi=\text{cis}\,2\theta+\text{cis}\,(-2\phi)=$, by VIII. g, No. 10 (i), $2\cos(\theta+\phi)\,.\,\text{cis}\,(\theta-\phi)$;
$\therefore$ left side $=2\cos(\theta+\phi)\,.\,\text{cis}\,(\theta-\phi)\,.\,\text{cis}\,\phi$
$=2\cos(\theta+\phi)\,.\,\text{cis}\,\{\theta-\phi+\phi\}$.

3. Expression $=\dfrac{3(2+\cos\theta-i\sin\theta)}{(2+\cos\theta)^2+\sin^2\theta}=\dfrac{3(2+\cos\theta)+i(-3\sin\theta)}{5+4\cos\theta}$. In VIII. e, No. 39 (ii), put $z=\cos\theta+i\sin\theta$, the circle $|z-2|=1$ in the answers is the same as
$$(x-2)^2+y^2=1 \text{ or } x^2+y^2=4x-3.$$

4. $p+iq=x\,\text{cis}\,(\alpha+\theta)+y\,\text{cis}\,(\beta+\theta)$
$=\{x\cos(\alpha+\theta)+y\cos(\beta+\theta)\}$
$+i\{x\sin(\alpha+\theta)+y\sin(\beta+\theta)\}$;
$\therefore$ as on p. 143,
$p^2+q^2=\{x\cos(\alpha+\theta)+y\cos(\beta+\theta)\}^2$
$+\{x\sin(\alpha+\theta)+y\sin(\beta+\theta)\}^2$
$=x^2+y^2+2xy\{\cos(\alpha+\theta)\cos(\beta+\theta)+\sin(\alpha+\theta)\sin(\beta+\theta)\}$
$=x^2+y^2+2xy\cos\{(\alpha+\theta)-(\beta+\theta)\}$.

5. $\dfrac{a}{b}=\text{cis}\,(\alpha-\beta)=\cos(\alpha-\beta)+i\sin(\alpha-\beta)$,
$$\frac{b}{a}=\frac{1}{\text{cis}\,(\alpha-\beta)}=\cos(\alpha-\beta)-i\sin(\alpha-\beta);$$
$$\therefore\ \frac{a}{b}-\frac{b}{a}=2i\sin(\alpha-\beta);\quad \therefore\ \sin(\alpha-\beta)=\frac{a^2-b^2}{2i\,ab}=\frac{i(b^2-a^2)}{2ab}.$$

6. (i) $a^2-b^2=\text{cis}\,4\alpha-\text{cis}\,4\beta=$, by VIII. g, No. 10 (ii),

$$2\sin(2\alpha-2\beta)\,.\,\text{cis}\left(2\alpha+2\beta+\frac{\pi}{2}\right)$$

or $$=\{-2\sin(2\alpha-2\beta)\}\,.\,\text{cis}\left(2\alpha+2\beta-\frac{\pi}{2}\right).$$

The second form must be taken if $\sin(2\alpha-2\beta)<0$, that is, if $(n-\frac{1}{2})\pi<(\alpha-\beta)<n\pi$.

(ii) $ab-cd=\text{cis}\,(2\alpha+2\beta)-\text{cis}\,(2\gamma+2\delta)=$, as in (i),

$$2\sin(\alpha+\beta-\gamma-\delta)\,.\,\text{cis}\left(\alpha+\beta+\gamma+\delta+\frac{\pi}{2}\right)$$

or $$=2\sin(\gamma+\delta-\alpha-\beta)\,.\,\text{cis}\left(\alpha+\beta+\gamma+\delta-\frac{\pi}{2}\right);$$

(iii) $abcd=\text{cis}\,2(\alpha+\beta+\gamma+\delta)=\cos\theta+i\sin\theta$, where

$$\theta=\alpha+\beta+\gamma+\delta;\quad\therefore\ \frac{1}{abcd}=\frac{1}{\cos\theta+i\sin\theta}=\cos\theta-i\sin\theta;$$

$$\therefore\ abcd-\frac{1}{abcd}=2i\sin\theta=2\sin\theta\,.\,\text{cis}\,\frac{\pi}{2}$$

or $$=(-2\sin\theta)\,.\,\text{cis}\left(-\frac{\pi}{2}\right).$$

7. By VIII. c, No. 30,

$$1+\text{cis}\,\theta=2\cos\frac{\theta}{2}\cdot\text{cis}\,\frac{\theta}{2}\ \text{ and }\ 1+\text{cis}\,2\theta=2\cos\theta\,.\,\text{cis}\,\theta;$$

$$\therefore\ u+iv=2\cos\frac{\theta}{2}\cdot\text{cis}\,\frac{\theta}{2}\cdot2\cos\theta\,.\,\text{cis}\,\theta$$

$$=4\cos\theta\cos\frac{\theta}{2}\cdot\text{cis}\left(\frac{\theta}{2}+\theta\right);$$

$$\therefore\ u=4\cos\theta\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\ \text{ and }\ v=4\cos\theta\cos\frac{\theta}{2}\sin\frac{3\theta}{2}.$$

(i) Divide; (ii) Square and add.

8. $A+Bi=1+i\,.\,\Sigma(x)+i^2\Sigma(x_1x_2)+i^3x_1x_2x_3$

$$=1-\Sigma(x_1x_2)+i\{\Sigma(x)-x_1x_2x_3\};$$

$\therefore\ A=1-\Sigma(x_1x_2);\ B=\Sigma(x)-x_1x_2x_3;$

$\therefore\ \dfrac{B}{A}=\dfrac{\Sigma(x)-x_1x_2x_3}{1-\Sigma(x_1x_2)}=\tan\{\Sigma(\tan^{-1}x)\}$. See p. 173 or *E.T.*, p. 220.

Or, put $x_1=\tan\theta_1$, $x_2=\tan\theta_2$, etc.; then

$$A+Bi=(1+i\tan\theta_1)(1+i\tan\theta_2)(1+i\tan\theta_3)$$
$$=\sec\theta_1\,.\,\sec\theta_2\,.\,\sec\theta_3\,.\,\text{cis}\,\theta_1\,.\,\text{cis}\,\theta_2\,.\,\text{cis}\,\theta_3$$
$$=\sec\theta_1\sec\theta_2\sec\theta_3\,.\,\text{cis}\,(\theta_1+\theta_2+\theta_3);$$

$\therefore\ A=\sec\theta_1\sec\theta_2\sec\theta_3\,.\,\cos(\theta_1+\theta_2+\theta_3);$
$B=\sec\theta_1\sec\theta_2\sec\theta_3\,.\,\sin(\theta_1+\theta_2+\theta_3)$; divide.

9. $\overline{PQ}=-a=\overline{AO}$; $\therefore$ Q describes a congruent curve obtained by applying the displacement $\overline{AO}$ to the given curve.

10. az describes the circle, centre $(a, 0)$, rad. a; $\therefore\ az+b+ci$ describes the circle obtained by applying to the former circle the displacement $b+ci$, that is, the circle, centre $(a+b, c)$, rad. a; $\therefore$ as in VIII. e, No. 37 (ii), $\dfrac{1}{az+b+ci}$ describes the image in Ox of the inverse w.r.t. $|z|=1$ of the circle, centre $(a+b, c)$, rad. a.

11. If A, B are $(1, 0)$, $(-1, 0)$, the lengths of PA, PB are $|z-1|$, $|z+1|$; $\therefore$ PA . PB $=|z-1|\,.\,|z+1|=|z^2-1|=2$.

12. $2z+z^2=(z+1)^2-1$. $z+1$ describes the circle, centre $(1, 0)$, rad. 1; $\therefore$ by Ex. 7, p. 152, $(z+1)^2$ describes the cardioid $r=2(1+\cos\theta)$; $\therefore\ 2z+z^2$ describes the cardioid obtained by moving $r=2(1+\cos\theta)$ one unit in direction xO.

13. If P, C, D represent z, i, $-i$, that is C is $(0, 1)$, D is $(0, -1)$, $\overline{CP}=z-i$, $\overline{DP}=z+i$; $\therefore\ |Z|\equiv\left|\dfrac{z-i}{z+i}\right|=\dfrac{CP}{DP}$; but P is above O$x$ since $y>0$; $\therefore$ CP < DP; $\therefore$ $|Z|<1$.

14. If A, B are $(1, 0)$, $(-1, 0)$ and if Q represents $e^x\,.\,\text{cis}\,y$, then Q lies on the right of Oy, since $e^x>0$ and $-\dfrac{\pi}{2}<y<\dfrac{\pi}{2}$; also $|Z|=\dfrac{QA}{QB}$, but QA < QB; $\therefore$ $|Z|<1$.

15. If $\text{am}\left(\dfrac{z-\beta}{z-\alpha}\right)=\theta$ where $0<\theta<\pi$, and if P, A, B represent z, α, β, the lines PA, PB are so situated that an anticlockwise rotation about P through angle θ moves PA into the position PB.

(i) P lies on the major arc of a circle through A, B, containing an angle $\dfrac{\pi}{3}$, so drawn that an anticlockwise rotation of $\dfrac{\pi}{3}$ converts PA into PB, i.e. the centre K of the circle is situated so that an anticlockwise rotation of $\dfrac{2\pi}{3}$ converts KA into KB.

(ii) If $\text{am}\left(\dfrac{z-\beta}{z-\alpha}\right)=-\dfrac{\pi}{3}$, $\text{am}\left(\dfrac{z-\alpha}{z-\beta}\right)=+\dfrac{\pi}{3}$; $\therefore$ in (i) interchange everywhere A and B; the major arcs in (i) and (ii) are equal and lie on opposite sides of AB.

(iii) If $\text{am}\left(\frac{z-\beta}{z-\alpha}\right) = -\frac{2\pi}{3}$, $\text{am}\left(\frac{z-\alpha}{z-\beta}\right)=\frac{2\pi}{3}$; $\therefore$ as in (i), P lies on the minor arc of a circle through A, B containing an angle $\frac{2\pi}{3}$, so drawn that an anticlockwise rotation of $\frac{2\pi}{3}$ converts PB into PA, i.e. this arc and the arc of (i) form a complete circle.

Similarly the arc for $\text{am}\left(\frac{z-\beta}{z-\alpha}\right)=\frac{2\pi}{3}$ forms with the arc of (ii) a complete circle.

16. $z_3 = -\frac{z_1(a_2-a_3)+z_2(a_3-a_1)}{a_1-a_2} = \frac{z_1(a_2-a_3)+z_2(a_3-a_1)}{(a_2-a_3)+(a_3-a_1)}$;

$\therefore$ by VIII. d, No. 19, z_3 divides the join of z_1 to z_2 in the ratio $(a_3-a_1):(a_2-a_3)$.

17. $|\beta-\gamma|$ is the length of BC, etc.; $\therefore$ a, b, c are the lengths of the sides of $\triangle$ABC; but the in-centre of $\triangle$ABC is the mass-centre of masses a, b, c at A, B, C, see Ex. I. c, No. 20. Hence by VIII. d, No. 20. Also the e-centre opposite A is the mass-centre of $-a, b, c$ at A, B, C, see Ex. I. c, No. 21. Hence by VIII. d, No. 20.

18. If A, B, C represent α, β, γ, $|\alpha-\gamma|$ and $|\beta-\gamma|$ are the lengths of AC, BC; but

$$\left|\frac{\alpha-\gamma}{\beta-\gamma}\right|=1 \text{ and } \text{am}\left(\frac{\alpha-\gamma}{\beta-\gamma}\right)=\frac{\pi}{3};$$

$\therefore$ AC = BC and, as in No. 15, AC, BC are so placed that an anticlockwise rotation of $\frac{\pi}{3}$ converts CB into CA; $\therefore$ $\triangle$ABC is equilateral and the sense ABC round the triangle is clockwise.

If $w = \text{cis}\,\frac{2\pi}{3}$,

$$w^2 = \text{cis}\,\frac{4\pi}{3} \text{ and } 1+w+w^2=0 \text{ and } 1+w^2+w^4=0;$$

but $\frac{\alpha-\gamma}{\beta-\gamma} = \text{cis}\,\frac{\pi}{3} = -\text{cis}\,\frac{4\pi}{3} = -w^2$;

$$\therefore \frac{\beta-\gamma}{1}=\frac{\gamma-\alpha}{w^2}=\frac{\beta-\alpha}{1+w^2}=\frac{\alpha-\beta}{w};$$

$$\therefore (\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2=0.$$

19. If A, B, C, L, M, N represent $\alpha, \beta, \gamma, \lambda, \mu, \nu$, as in No. 18, $\frac{\text{CA}}{\text{BA}}=\frac{\text{NL}}{\text{ML}}$ and equal rotations convert AB into AC and LM into LN; $\therefore$ $\triangle$s ABC, LMN are directly similar:

$$(\alpha-\gamma)(\lambda-\mu)=(\alpha-\beta)(\lambda-\nu);$$

$$\therefore \alpha(\mu-\nu)+\beta(\nu-\lambda)+\gamma(\lambda-\mu)=0.$$

20. Let C, D be the points $(a, 0)$, (a, b) so that D represents $a+ib$; rotate $\triangle$OCD about O through angle $\frac{2\pi}{n}$ anticlockwise into position $\triangle$OC′D′, then D′ represents

$$\left(a\cos\frac{2\pi}{n}-b\sin\frac{2\pi}{n}\right)+i\left(a\sin\frac{2\pi}{n}+b\cos\frac{2\pi}{n}\right);$$

$\therefore$ this number and $(a+ib)$ have equal moduli and their amplitudes differ by $\frac{2\pi}{n}$; $\therefore$ their nth powers are equal, since their amplitudes differ by $n\times\frac{2\pi}{n}=2\pi$.

21. Using eccentric angles, if $z_1 = a\cos\phi + i\,.\,b\sin\phi$, then
$z_2 = \pm(a\sin\phi - i\,.\,b\cos\phi)$; $\therefore$ $z_1{}^2+z_2{}^2=a^2-b^2=d^2$.

22. Take the reflections of $y=\text{cosec}\,x$, $y=\sec x$, $y=\cot x$ in the line $y=x$

(i) undefined for $|x|<1$; as x increases from $-\infty$ to -1, y decreases from 0 to $-\frac{\pi}{2}$; as x increases from $+1$ to $+\infty$, y decreases from $\frac{\pi}{2}$ to 0;

(ii) undefined for $|x|<1$; as x increases from $-\infty$ to -1, y increases from $\frac{\pi}{2}$ to π; as x increases from $+1$ to $+\infty$, y increases from 0 to $\frac{\pi}{2}$;

(iii) as x increases from $-\infty$ to 0, y decreases from 0 to $-\frac{\pi}{2}$; as x increases from 0 to $+\infty$, y decreases from $\frac{\pi}{2}$ to 0.

23. Equation is $2\cos x\cos y+\cos x+\cos y=0$ or

$$(\cos x+\tfrac{1}{2})(\cos y+\tfrac{1}{2})=\tfrac{1}{4};$$

Draw the curve $(X+\frac{1}{2})(Y+\frac{1}{2})=\frac{1}{4}$. This is a rectangular hyperbola, centre $(-\frac{1}{2}, -\frac{1}{2})$, asymptotes $X=-\frac{1}{2}$, $Y=-\frac{1}{2}$; curve passes through origin. Since

$$|\cos x|\leqslant 1 \text{ and } |\cos y|\leqslant 1,$$

we want only that part of the hyperbola for which

$$|X| \leq 1 \text{ and } |Y| \leq 1;$$

the lower left branch of hyperbola contributes the single point $X = -1$, $Y = -1$; the upper right branch is cut by $X = 1$ at $Y = -\frac{1}{3}$ and is cut by $Y = 1$ at $X = -\frac{1}{3}$; $\therefore$ all points on the arc from $(-\frac{1}{3}, 1)$ to $(1, -\frac{1}{3})$ give possible pairs of values for $\cos x$, $\cos y$.

If $\cos x = -\frac{1}{3}$, $x \simeq 109° 28' \simeq 1{\cdot}91$ radians. When $\cos x$ increases from $-\frac{1}{3}$ to $+1$, i.e. when x decreases from $1{\cdot}91$ to 0, $\cos y$ decreases from 1 to $-\frac{1}{3}$, i.e. y increases from 0 to $1{\cdot}91$. But $\cos x = \cos(-x)$ and $\cos y = \cos(-y)$; $\therefore$ we must take also the reflection of this arc in Ox and in Oy and through the origin. The result is an oval curve, symmetrical about each axis. (Differentiation shows that it cuts Ox and Oy at right angles.) There is also the isolated point given by $\cos x = -1$, $\cos y = -1$, i.e. (π, π). The oval is inscribed in the square formed by $x = \pm 1{\cdot}91$, $y = \pm 1{\cdot}91$. There is a congruent oval in every square formed by $x = \pm 1{\cdot}91 + 2m\pi$, $y = \pm 1{\cdot}91 + 2n\pi$, also isolated points $(2m\pi + \pi, 2n\pi + \pi)$, for all integral values of m, n. The points $(m\pi + \frac{1}{2}\pi, n\pi + \frac{1}{2}\pi)$ do not belong to the curve since $\sec \frac{1}{2}\pi$ is meaningless.

24. If $2k\pi - \pi < x \leq 2k\pi + \pi$, $\text{am}(\cos x + i \sin x) = x - 2k\pi$;

$$\therefore \frac{x - \text{am}(\text{cis } x)}{2\pi} = \frac{x - (x - 2k\pi)}{2\pi} = k;$$

but $x \neq (2n+1)\pi$; $\therefore 2k\pi < x + \pi < 2k\pi + 2\pi$;

$$\therefore k < \frac{x+\pi}{2\pi} < k+1; \quad \therefore k = \left[\frac{x+\pi}{2\pi}\right].$$

25. Put $z = \tan \frac{1}{2}\theta$; then θ increases from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

$$(Z + i) \sin \tfrac{1}{2}\theta = (1 + Zi) \cos \tfrac{1}{2}\theta;$$

$$\therefore Z(\sin \tfrac{1}{2}\theta - i \cos \tfrac{1}{2}\theta) = \cos \tfrac{1}{2}\theta - i \sin \tfrac{1}{2}\theta;$$

$$\therefore Z = \text{cis}(-\tfrac{1}{2}\theta) \div \text{cis}\left(\tfrac{1}{2}\theta - \frac{\pi}{2}\right) = \text{cis}\left(\frac{\pi}{2} - \theta\right);$$

$\therefore$ Z describes the upper semicircle from $\text{cis } \pi$ through $\text{cis } \frac{\pi}{2}$ to cis 0.

$z = (1 + Zi)/(Z + i)$ and $Z = (1 + z_1 i)/(z_1 + i)$ give

$$z = \{z_1 + i + i(1 + z_1 i)\}/\{(1 + z_1 i) + i(z_1 + i)\} = 2i/(2iz_1) = \frac{1}{z_1};$$

$\therefore$ z_1 describes the part of the x-axis outside $(\pm 1, 0)$. Also $z_1 = (1 + z_2 i)/(z_2 + i)$ gives

$$z_2 = \frac{z - i}{1 - iz} = \frac{\sin \frac{1}{2}\theta - i \cos \frac{1}{2}\theta}{\cos \frac{1}{2}\theta - i \sin \frac{1}{2}\theta} = \text{cis}\left(\theta - \frac{\pi}{2}\right),$$

so z_2 describes the lower semicircle from $\text{cis}(-\pi)$ through $\text{cis}\left(-\frac{\pi}{2}\right)$ to cis 0. $z_3 = \frac{1}{z_1} = z$, $z_4 = \frac{1}{z_2} = Z$, etc.

26. If $n = 3p$, $w^n = 1 = w^{2n}$; if $n = 3p+1$, $w^n = (w^3)^p \cdot w = w$, $w^{2n} = w^2$; if $n = 3p+2$, $w^n = (w^3)^p \cdot w^2 = w^2$, $w^{2n} = w^4 = w$; now use $w^2 + w = -1$.

27. If $x = yw$, $(x+y)^n - x^n - y^n = y^n(1+w)^n - y^n w^n - y^n$

$$= y^n\{(-w^2)^n - w^n - 1\} = y^n\{-w^{2n} - w^n - 1\},$$

since n is odd, $= 0$, since by No. 26, $w^{2n} + w^n = -1$ if n is not a multiple of 3; $\therefore x - yw$ is a factor; similarly $x - yw^2$ is a factor. But

$$(x - yw)(x - yw^2) \equiv x^2 - xy(w + w^2) + y^2 \equiv x^2 + xy + y^2.$$

28. If $x = -yw - zw^2$,

$$(z - x)^n + (x - y)^n = (z + yw + zw^2)^n + (-yw - zw^2 - y)^n$$

$$= (yw - zw)^n + (yw^2 - zw^2)^n,$$

since $1 + w + w^2 = 0$, $= (y - z)^n\{w^n + w^{2n}\} = -(y - z)^n$, by No. 26, since n is not a multiple of 3; $\therefore x + yw + zw^2$ is a factor of $(z-x)^n + (x-y)^n + (y-z)^n$; similarly $x + yw^2 + zw$ is a factor. But

$$(x + yw + zw^2)(x + yw^2 + zw) = \Sigma(x^2) - \Sigma(yz),$$

see VIII. g, No. 22 (ii).

29. $\frac{d}{dx}\{(y-z)^n + (z-x)^n + (x-y)^n\} = -n(z-x)^{n-1} + n(x-y)^{n-1}$,

which when $x = -yw - zw^2$, becomes

$$-n(z + yw + zw^2)^{n-1} + n(-yw - zw^2 - y)^{n-1}$$

$$= -n(yw - zw)^{n-1} + n(yw^2 - zw^2)^{n-1},$$

since $1 + w + w^2 = 0$, $= -n(y-z)^{n-1}\{w^{n-1} - w^{2n-2}\}$

$$= -n(y-z)^{n-1}\{w^{3p} - w^{6p}\}, \text{ since } n = 3p+1,$$

$$= -n(y-z)^{n-1}\{1 - 1\} = 0;$$

$\therefore x + yw + zw^2$ is a factor of $\frac{d}{dx}\{\Sigma(y-z)^n\}$; but by No. 28, it is a factor of $\Sigma(y-z)^n$; $\therefore (x + yw + zw^2)^2$ must be a factor of $\Sigma(y-z)^n$; similarly $(x + yw^2 + zw)^2$ must be a factor; then as in No. 28.

30. By VIII. g, No. 22 (ii), $(x+yw+zw^2)(x+yw^2+zw) \equiv \Sigma(x^2-yz)$;

but $x^3+y^3+z^3-3xyz \equiv (x+y+z)\{\Sigma x^2 - \Sigma yz\}$. Let

$$\mathrm{A} \equiv x^2-yz,\ \mathrm{B} \equiv y^2-zx,\ \mathrm{C} = z^2-xy;$$

then expression $\equiv \mathrm{A}^3+\mathrm{B}^3+\mathrm{C}^3-3\mathrm{ABC}$

$$\equiv (\mathrm{A}+\mathrm{B}+\mathrm{C})(\mathrm{A}+\mathrm{B}w+\mathrm{C}w^2)(\mathrm{A}+\mathrm{B}w^2+\mathrm{C}w);$$

but $\mathrm{A}+\mathrm{B}+\mathrm{C} \equiv \Sigma(x^2-yz) = (x+yw+zw^2)(x+yw^2+zw)$;

and $\mathrm{A}+\mathrm{B}w+\mathrm{C}w^2$

$$\equiv x^2-yz+(y^2-zx)w+(z^2-xy)w^2$$
$$\equiv (x+y+z)(x+yw+zw^2);$$

and $\mathrm{A}+\mathrm{B}w^2+\mathrm{C}w$

$$\equiv (x+y+z)(x+yw^2+zw);$$

$\therefore\ \mathrm{A}^3+\mathrm{B}^3+\mathrm{C}^3-3\mathrm{ABC}$

$$\equiv (x+y+z)^2 . (x+yw+zw^2)^2 . (x+yw^2+zw)^2.$$

31. (i) If $r=3p$, $a_r x^r + a_r(wx)^r + a_r(w^2x)^r$

$$= a_r x^r\{1+w^{3p}+w^{6p}\} = a_r x^r(1+1+1) = 3a_r x^r;$$

if $r=3p \pm 1$, coefficient of x^r is

$$a_r\{1+w^{3p\pm1}+w^{6p\pm2}\} = a_r\{1+w^{\pm1}+w^{\pm2}\} = 0,$$

$$\text{since } 1+w+w^2 = 0 = 1+\frac{1}{w}+\frac{1}{w^2};$$

$$\therefore \text{ general term} = 3a_{3p}x^{3p};$$

(ii) If $r=3p-1$, coefficient of x^r is

$$a_r\{1+w . w^{3p-1}+w^2 . w^{6p-2}\}$$
$$= a_r\{1+w^{3p}+w^{6p}\} = a_r(1+1+1) = 3a_r;$$

if $r=3p$, coefficient of x^r is

$$a_r\{1+w . w^{3p}+w^2 . w^{6p}\} = a_r\{1+w+w^2\} = 0;$$

similarly, if $r=3p+1$, coefficient of x^r is 0;

$$\therefore \text{ general term} = 3a_{3p-1}\, x^{3p-1}.$$

CHAPTER IX

EXERCISE IX. a. (p. 168.)

1. The square roots of cis α are cis $\dfrac{\alpha}{2}$, cis $\left(\dfrac{\alpha}{2}+\pi\right)$, i.e. $\pm$ cis $\left(\dfrac{\alpha}{2}\right)$.

And the given expressions are

(i) cis 2θ; (ii) cis (-3θ); (iii) cis $\left(\dfrac{\pi}{2}-\theta\right)$;

(iv) cis $\dfrac{\pi}{2}$; (v) cis $\left(-\dfrac{\pi}{2}\right)$.

2. The cube roots of cis α are

$$\text{cis}\,\frac{\alpha}{3},\ \text{cis}\,\frac{\alpha+2\pi}{3},\ \text{cis}\,\frac{\alpha+4\pi}{3} \equiv \text{cis}\,\frac{\alpha-2\pi}{3};$$

and the given expressions are

(i) cis 3θ; (ii) cis 0; (iii) cis $\dfrac{\pi}{2}$;

(iv) cis $\left(-\dfrac{\pi}{2}\right)$; (v) cis $(-\theta)$; (vi) cis $\left(\theta-\dfrac{\pi}{2}\right)$.

3. (i) $(\text{cis}\,\pi)^{\frac{2}{3}} = \text{cis}\,\dfrac{2\pi}{3},\ \text{cis}\,\dfrac{4\pi}{3},\ \text{cis}\,\dfrac{6\pi}{3}$;

(ii) $\left[\text{cis}\left(-\dfrac{\pi}{2}\right)\right]^{\frac{3}{4}} = \left[\text{cis}\left(-\dfrac{3\pi}{2}\right)\right]^{\frac{1}{4}} = \text{cis}\left(-\dfrac{3\pi}{8}+\dfrac{2r\pi}{4}\right)$;

(iii) $1+i = \sqrt{2} . \text{cis}\,\dfrac{\pi}{4}$;

(iv) $1-i\sqrt{3} = 2\,\text{cis}\left(-\dfrac{\pi}{3}\right)$;

$$(1-i\sqrt{3})^{\frac{2}{5}} = \left[2^2\,\text{cis}\left(-\frac{2\pi}{3}\right)\right]^{\frac{1}{5}} = \sqrt[5]{4} . \text{cis}\left(-\frac{2\pi}{15}+\frac{2r\pi}{5}\right);$$

(v) $[2^7 . \text{cis}\,(2r\pi)]^{\frac{1}{7}}$.

4. (i) $x = (\text{cis}\,2r\pi)^{\frac{1}{5}}$;

(ii) $x^4 = -1 = \text{cis}\,\pi$, $x = \text{cis}\left(\dfrac{\pi}{4}+\dfrac{2r\pi}{4}\right) = \pm\,\text{cis}\,\dfrac{\pi}{4},\ \pm\,\text{cis}\,\dfrac{3\pi}{4}$.

5. See Fig. 77, p. 166, vertices of regular polygons;

(i) Points on $|z|=1$ for which $\theta = \dfrac{\pi}{3},\ \pi,\ -\dfrac{\pi}{3}$;

(ii) Points on $|z|=1$ for which $\theta = \dfrac{\pi}{8},\ \dfrac{5\pi}{8},\ -\dfrac{7\pi}{8},\ -\dfrac{3\pi}{8}$;

(iii) Points on $|z|=2$ for which $\theta = \dfrac{2r\pi}{5}$, $r=0$ to 4;

(iv) $-5-12i = 13$ cis α, where $\cos\alpha = -\frac{5}{13}$, $\sin\alpha = -\frac{12}{13}$;

$$\therefore\ -\pi < \alpha < -\frac{\pi}{2};\ (-5-12i)^3 = 13^3\,\text{cis}\,3\alpha.$$

6. Expression $=\text{cis}\frac{4\pi}{8}\div\text{cis}\left(-\frac{2\pi}{4}\right)=\text{cis}\left(\frac{4\pi}{8}+\frac{2\pi}{4}\right)=\text{cis}\,\pi$.

7. By p. 165, the principal value of the square root of $\left[\text{cis}\left(-\frac{\pi}{6}\right)\right]^{11}$ is the p.v. of $\left[\text{cis}\left(-\frac{\pi}{6}\right)\right]^{\frac{11}{2}}$, and this is $\text{cis}\left(-\frac{11\pi}{12}\right)$ by p. 164; it is not the same as the p.v. of $\sqrt{\left[\text{cis}\left(-\frac{11\pi}{6}\right)\right]}$, this would be $\sqrt{\left[\text{cis}\frac{\pi}{6}\right]}=\text{cis}\frac{\pi}{12}$. Expression $=\text{cis}\left(-\frac{11\pi}{12}\right)\div\text{cis}\frac{\pi}{12}=\text{cis}\left(-\frac{11\pi}{12}-\frac{\pi}{12}\right)=\text{cis}(-\pi)$.

8. (i) $\text{cis}\frac{\pi}{3}$, $\text{cis}\frac{4\pi}{3}$;

(ii) $\left[\text{cis}\frac{2\pi}{3}\right]^{\frac{4}{8}}=\left[\text{cis}\frac{8\pi}{3}\right]^{\frac{1}{8}}=\text{cis}\left(\frac{\pi}{3}+\frac{2r\pi}{8}\right)$; the p.v. is

$$\text{cis}\left(\frac{2\pi}{3}\times\tfrac{4}{8}\right) \text{ because } -\pi<\frac{2\pi}{3}<\pi.$$

9. (i) p.v. is $\text{cis}\left(\frac{3\pi}{4}\times\tfrac{7}{3}\right)$;

(ii) p.v. is $\text{cis}\left(-\frac{3\pi}{4}\times\tfrac{1}{3}\right)$, because

$$-\pi<\left(\frac{5\pi}{4}-2\pi\right)=-\frac{3\pi}{4}<\pi.$$

10. (i) $-\pi<\left(\frac{4\pi}{3}-2\pi\right)=-\frac{2\pi}{3}<\pi$; $\therefore$ p.v. is $\text{cis}\left(-\frac{2\pi}{3}\times\tfrac{1}{8}\right)$;

(ii) p.v. is $\text{cis}\left(\frac{\pi}{3}\times\tfrac{1}{2}\right)$.

11. $(\text{cis}\,\pi)^{\frac{2}{5}}=(\text{cis}\,2\pi)^{\frac{1}{5}}$; $\therefore$ the values are $\text{cis}\frac{2r\pi}{5}$, $r=0$ to 4;

$$\text{product}=\text{cis}\left[\frac{2\pi}{5}(0+1+2+3+4)\right]=\text{cis}\,4\pi=1;$$

$$\Sigma\cos\frac{2r\pi}{5}=0=\Sigma\sin\frac{2r\pi}{5},\ r=0 \text{ to } 4,$$

see Ex. VII. b, No. 9;

$\therefore$ sum of roots $=0$. If $z=(\text{cis}\,2\pi)^{\frac{1}{5}}$, $z^5=1$, $z^5-1=0$; hence product of roots $=1$, sum of roots $=-$coefficient of $z^4=0$.

12. $x^5=1$ if $x=\text{cis}\frac{2r\pi}{5}$;

$$\therefore\ x^5-1\equiv\Pi\left(x-\text{cis}\frac{2r\pi}{5}\right),\ r=0,\ \pm1,\ \pm2;$$

but $\left[x-\text{cis}\frac{2\pi}{5}\right]\left[x-\text{cis}\left(-\frac{2\pi}{5}\right)\right]$

$$\equiv x^2-2x\cos\frac{2\pi}{5}+1;\ \cos\frac{2\pi}{5}=\tfrac{1}{4}(\sqrt{5}-1),$$

see *E.T.*, p. 263;

similarly $\left[x-\text{cis}\frac{4\pi}{5}\right]\left[x-\text{cis}\left(-\frac{4\pi}{5}\right)\right]$

$$=x^2-2x\cos\frac{4\pi}{5}+1;\ \cos\frac{4\pi}{5}=-\tfrac{1}{4}(\sqrt{5}+1).$$

13. $x=(\text{cis}\,2r\pi)^{\frac{1}{n}}$; then as in No. 11;
$(x^n-1)\equiv(x-1)(x^{n-1}+x^{n-2}+\ldots+x+1)=0$ is satisfied by $x=\alpha$, where $\alpha\neq1$. **Or** the roots are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ and their sum $=0$.

14. $x^2+x+1\equiv\frac{x^3-1}{x-1}$ for $x\neq1$; $\therefore$ roots of $x^2+x+1=0$ are $\text{cis}\frac{2\pi}{3}$, $\text{cis}\frac{4\pi}{3}$; but roots of $x^6=1$ are $\text{cis}\frac{2r\pi}{6}$, $r=0$ to 5; exclude $\text{cis}\frac{4\pi}{6}$, $\text{cis}\frac{8\pi}{6}$.

15. $(x+i)^6=(x-i)^6\,.\,\text{cis}\,\pi$; $\therefore$ $(x+i)=(x-i)\,.\,\text{cis}\,\alpha$, where $\alpha=\frac{(2r-1)\pi}{6}$, $r=1$ to 6; $\therefore$ $x(1-\cos\alpha-i\sin\alpha)$

$$=\sin\alpha-i(1+\cos\alpha);\ x\,.\,2\sin\frac{\alpha}{2}\left(\sin\frac{\alpha}{2}-i\cos\frac{\alpha}{2}\right)$$

$$=2\cos\frac{\alpha}{2}\left(\sin\frac{\alpha}{2}-i\cos\frac{\alpha}{2}\right);\ \therefore\ x=\cot\frac{\alpha}{2}.$$

16. $1+x=(1-x)\,.\,\text{cis}\,\alpha$ where $\alpha=\frac{2r\pi}{n}$, $r=0$ to $n-1$;

$$\therefore\ x(1+\cos\alpha+i\sin\alpha)=\cos\alpha-1+i\sin\alpha;$$

$$\therefore\ x\,.\,2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)=-2\sin\frac{\alpha}{2}\left(\sin\frac{\alpha}{2}-i\cos\frac{\alpha}{2}\right);$$

$$\therefore\ x\,.\,2\cos\frac{\alpha}{2}\cdot\text{cis}\frac{\alpha}{2}=-2\sin\frac{\alpha}{2}\cdot\text{cis}\frac{\alpha-\pi}{2};$$

$$\therefore\ x=-\tan\frac{\alpha}{2}\cdot\text{cis}\left[\frac{\alpha-\pi}{2}-\frac{\alpha}{2}\right]=-\tan\frac{\alpha}{2}\,\text{cis}\left(-\frac{\pi}{2}\right).$$

17. $(x^n - \cos n\alpha)^2 = -\sin^2 n\alpha$;

$$\therefore\ x^n = \cos n\alpha \pm i \sin n\alpha = \text{cis}\,(\pm n\alpha);$$

$$\therefore\ x = \text{cis}\left(\pm\alpha + \frac{2r\pi}{n}\right),\ r = 0 \text{ to } n-1.$$

18. $(x^5-1)(x^4+1)=0$; then as in No. 4.

19. $$\frac{1+z}{1-z} = \frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} = \frac{2\cos\frac{\theta}{2}\cdot\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\cdot\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)} = i\cot\frac{\theta}{2};$$

(i) For $0<\theta<\pi$, $\cot\frac{\theta}{2}>0$;

$$\therefore\ \sqrt{\left(\frac{1+z}{1-z}\right)} = \sqrt{\left(\cot\frac{\theta}{2}\right)}\cdot\sqrt{\left(\text{cis}\frac{\pi}{2}\right)} = \sqrt{\left(\cot\frac{\theta}{2}\right)}\cdot\text{cis}\frac{\pi}{4};$$

(ii) For $\pi<\theta<2\pi$, $\cot\frac{\theta}{2}<0$;

$$\therefore\ \sqrt{\left(\frac{1+z}{1-z}\right)} = \sqrt{\left(-\cot\frac{\theta}{2}\right)}\cdot\sqrt{\left[\text{cis}\left(-\frac{\pi}{2}\right)\right]}$$
$$= \sqrt{\left(-\cot\frac{\theta}{2}\right)}\cdot\text{cis}\left(-\frac{\pi}{4}\right).$$

20. $i^{\frac{1}{3}} = \text{cis}\frac{\pi}{6}\cdot\text{cis}\frac{2p\pi}{3}$, $p=0, 1, 2$; $(-i)^{\frac{1}{3}} = \text{cis}\frac{-\pi}{6}\cdot\text{cis}\frac{2q\pi}{3}$, $q=0$, 1, 2. There are 9 combinations of p, q, but 3 of them give the same sum, zero. The product $=\text{cis}\frac{2(p+q)\pi}{3}$ has only 3 distinct values given by $p+q=0, 1, 2$.

21. $1+\cos\theta+i\sin\theta = 2\cos\frac{\theta}{2}\cdot\text{cis}\frac{\theta}{2}$;

(i) For $-\pi<\theta<\pi$, $\cos\frac{\theta}{2}>0$ and $-\frac{\pi}{2}<\frac{\theta}{2}<\frac{\pi}{2}$;

$$\therefore\ \text{p.v.} = \left(2\cos\frac{\theta}{2}\right)^{\frac{3}{4}}\cdot\text{cis}\left(\frac{\theta}{2}\times\frac{3}{4}\right);$$

for $\pi<\theta<3\pi$, $\cos\frac{\theta}{2}<0$,

$$1+\cos\theta+i\sin\theta = \left(-2\cos\frac{\theta}{2}\right)\cdot\text{cis}\left(\frac{\theta}{2}-\pi\right);$$

also for $\pi<\theta<3\pi$, $-\frac{\pi}{2}<\left(\frac{\theta}{2}-\pi\right)<\frac{\pi}{2}$;

$$\therefore\ \text{p.v.} = \left(-2\cos\frac{\theta}{2}\right)^{\frac{3}{4}}\cdot\text{cis}\left[\left(\frac{\theta}{2}-\pi\right)\times\frac{3}{4}\right].$$

22. As on p. 166, see Fig. 77.

23. (i) $2z$ describes circle, centre O, rad. 2, anticlockwise from $(-2, 0)$ round to $(-2, 0)$ again; $\therefore\ w=2z+3$ describes circle, centre $(3, 0)$, rad. 2, from $(3-2, 0)$ anticlockwise;

(ii) iz describes circle $|z|=1$, keeping $\frac{1}{4}$ of the circle ahead of P; $\therefore\ iz+2$ describes circle, centre $(2, 0)$, rad. 1, from $(2, -1)$ anticlockwise;

(iii) $z^2 \equiv (\cos 2\theta + i\sin 2\theta)$ describes circle $|z|=1$, twice anticlockwise, starting from $(1, 0)$; $\therefore\ 3z^2$ describes circle $|z|=3$, twice, starting from $(3, 0)$.

24. (i) $z^3\equiv(\cos 3\theta + i\sin 3\theta)$ describes circle $|z|=1$, three times anticlockwise, starting from $(-1, 0)$;

(ii) From (i), z^3+1 describes circle, centre $(1, 0)$, rad. 1, three times, anticlockwise, starting from $(0, 0)$;

(iii) $z^{\frac{1}{2}} \equiv \left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$ or $\left[\cos\left(\frac{\theta}{2}+\pi\right)+i\sin\left(\frac{\theta}{2}+\pi\right)\right]$; $\therefore$ as in Example 3, there are two positions of Q, say Q_1, Q_2, for each position of P; Q_1 describes the half circle $|z|=1$ from $\theta=-\frac{\pi}{2}$ to $\theta=+\frac{\pi}{2}$, Q_2 describes the other half circle, each moving anticlockwise.

25. (i) $\frac{1}{\sqrt{z}}$ is the principal square root of $\frac{1}{z}$; $\frac{1}{z}$ describes the circle $|z|=1$ from $\theta=\pi$ to $\theta=-\pi$ clockwise; $\therefore\ \frac{1}{\sqrt{z}}$ describes the half circle $|z|=1$ from $\theta=\frac{\pi}{2}$ to $\theta=-\frac{\pi}{2}$ clockwise.

(ii) $z+1$ describes the circle, centre $(1, 0)$, rad. 1. If $(z+1)^2\equiv r\,\text{cis}\,\theta$, $\left(\sqrt{r}, \frac{\theta}{2}\right)$ lies on the circle, centre $(1, 0)$, rad. 1; $\therefore\ \sqrt{r}=2\cos\frac{\theta}{2}$. (Cf. Ex. 7, p. 152); $\therefore\ r=4\cos^2\frac{\theta}{2}=2(1+\cos\theta)$, a cardioid.

When P moves round the circle,

$$\frac{\theta}{2} \text{ varies from } -\frac{\pi}{2} \text{ to } +\frac{\pi}{2};$$

$\therefore$ Q moves on the cardioid so that θ varies from $-\pi$ to $+\pi$;

(iii) $z^2+2z=(z+1)^2-1$; $\therefore$ Q moves on the cardioid obtained by moving the path in (ii) one unit to the left.

26. (i) As in No. 25 (ii), if $\sqrt{(z+1)} \equiv r \operatorname{cis}\theta$, then $(r^2, 2\theta)$ is a point on circle, centre $(1, 0)$, rad. 1; $\therefore\ r^2 = 2\cos 2\theta$, a lemniscate; $\sqrt{(z+1)}$ describes the loop obtained by making θ increase from $-\frac{\pi}{4}$ to $+\frac{\pi}{4}$;

(ii) Since $\frac{1}{r(\cos\theta + i\sin\theta)} = \frac{1}{r}\cdot\operatorname{cis}(-\theta)$, when $\sqrt{(z+1)}$ describes one loop of $r^2 = 2\cos 2\theta$, anticlockwise, $\frac{1}{\sqrt{(z+1)}}$ describes the image in Ox of the inverse w.r.t. $|z|=1$, i.e. one branch of the rect. hyperbola, $2r^2\cos 2\theta = 1$, i.e. $x^2 - y^2 = \frac{1}{2}$, downwards;

(iii) As in Example 3, $(z+1)^{\frac{1}{2}}$ represents either square root of $(z+1)$; $\therefore$ there are two positions of Q for each position of P; $\therefore$ both loops of the lemniscate in (i) are described simultaneously.

27. $\operatorname{cis} 2\alpha - \operatorname{cis} 2\beta = (\cos 2\alpha - \cos 2\beta) + i(\sin 2\alpha - \sin 2\beta)$

$= 2\sin(\alpha+\beta)\,.\,\sin(\beta-\alpha) + 2i\cos(\alpha+\beta)\,.\,\sin(\alpha-\beta)$

$= 2\sin(\alpha-\beta)\,.\,\operatorname{cis}\left[\frac{\pi}{2} + (\alpha+\beta)\right]$; $\therefore\ (a_1-a_2)(a_3-a_4)$

$= 4\sin(\alpha_1-\alpha_2)\sin(\alpha_3-\alpha_4)\,.\,\operatorname{cis}[\pi + (\alpha_1+\alpha_2+\alpha_3+\alpha_4)]$.

EXERCISE IX. b. (p. 171.)

1. Use eqn. (3). **2.** Use eqn. (3).

3. As in Example 4,

$$(2\cos\theta)^3 = \left(z+\frac{1}{z}\right)^3 = \left(z^3+\frac{1}{z^3}\right) + 3\left(z+\frac{1}{z}\right)$$
$$= (2\cos 3\theta) + 3(2\cos\theta).$$

4. $(2\cos\theta)^4 = \left(z+\frac{1}{z}\right)^4 = \left(z^4+\frac{1}{z^4}\right) + 4\left(z^2+\frac{1}{z^2}\right) + 6$

$= 2\cos 4\theta + 4(2\cos\theta) + 6.$

5. $(2\cos\theta)^7 = \left(z+\frac{1}{z}\right)^7$

$$= \left(z^7+\frac{1}{z^7}\right) + 7\left(z^5+\frac{1}{z^5}\right) + 21\left(z^3+\frac{1}{z^3}\right) + 35\left(z+\frac{1}{z}\right) = \text{etc.}$$

6. $(2i\sin\theta)^4 = \left(z-\frac{1}{z}\right)^4 = \left(z^4+\frac{1}{z^4}\right) - 4\left(z^2+\frac{1}{z^2}\right) + 6 = \text{etc.}$

7. $(2i\sin\theta)^7 = \left(z-\frac{1}{z}\right)^7$

$$= \left(z^7-\frac{1}{z^7}\right) - 7\left(z^5-\frac{1}{z^5}\right) + 21\left(z^3-\frac{1}{z^3}\right) - 35\left(z-\frac{1}{z}\right) = \text{etc.}$$

8. $(2i\sin\theta)^3\,.\,(2\cos\theta) = \left(z-\frac{1}{z}\right)^3\cdot\left(z+\frac{1}{z}\right) = \left(z-\frac{1}{z}\right)^2\cdot\left(z^2-\frac{1}{z^2}\right)$

$$= \left(z^4-\frac{1}{z^4}\right) - 2\left(z^2-\frac{1}{z^2}\right) = \text{etc.}$$

9. $(2\cos\theta)^4\,.\,(2i\sin\theta)^3 = \left(z+\frac{1}{z}\right)^4\cdot\left(z-\frac{1}{z}\right)^3 = \left(z+\frac{1}{z}\right)\cdot\left(z^2-\frac{1}{z^2}\right)^3$

$$= \left(z^7-\frac{1}{z^7}\right) + \left(z^5-\frac{1}{z^5}\right) - 3\left(z^3-\frac{1}{z^3}\right) - 3\left(z-\frac{1}{z}\right) = \text{etc.}$$

10. $(2\cos\theta)^5\,.\,(2i\sin\theta)^4 = \left(z+\frac{1}{z}\right)^5\cdot\left(z-\frac{1}{z}\right)^4 = \left(z+\frac{1}{z}\right)\left(z^2-\frac{1}{z^2}\right)^4$

$$= \left(z^9+\frac{1}{z^9}\right) + \left(z^7+\frac{1}{z^7}\right) - 4\left(z^5+\frac{1}{z^5}\right) - 4\left(z^3+\frac{1}{z^3}\right) + 6\left(z+\frac{1}{z}\right) = \text{etc.}$$

11. If $z = \operatorname{cis}\theta$,

$$(2\cos\theta)^5 = \left(z+\frac{1}{z}\right)^5 = \left(z^5+\frac{1}{z^5}\right) + 5\left(z^3+\frac{1}{z^3}\right) + 10\left(z+\frac{1}{z}\right),$$

$$\therefore\ (2\cos\theta)^5 - 2\cos 5\theta = 5\left(z+\frac{1}{z}\right)\left\{\left(z^2-1+\frac{1}{z^2}\right)+2\right\}$$
$$= 5\,.\,2\cos\theta\,.\,(2\cos 2\theta + 1).$$

12. (i) From Example 5, expression

$$= \tfrac{1}{16}\int(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\,d\theta;$$

(ii) From No. 5, expression

$$= \tfrac{1}{64}\int(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)\,d\theta;$$

(iii) From No. 10, expression

$$= \tfrac{1}{256}\int(\cos 9\theta + \cos 7\theta - 4\cos 5\theta - 4\cos 3\theta + 6\cos\theta)\,d\theta.$$

Or, in (i) put $\cos\theta = x$, integral $= -\int(1-x^2)^2dx$; similarly in (ii) and (iii) put $\sin\theta = x$.

13. $(2i\sin\theta)^4\,.\,(2\cos\theta)^6 = \left(z-\frac{1}{z}\right)^4\cdot\left(z+\frac{1}{z}\right)^6 = \left(z^2-\frac{1}{z^2}\right)^4\cdot\left(z+\frac{1}{z}\right)^2$

$$= \left(z^{10}+\frac{1}{z^{10}}\right) + 2\left(z^8+\frac{1}{z^8}\right) - 3\left(z^6+\frac{1}{z^6}\right)$$
$$- 8\left(z^4+\frac{1}{z^4}\right) + 2\left(z^2+\frac{1}{z^2}\right) + 12;$$

$\therefore \sin^4\theta\cos^6\theta$

$$=\frac{1}{2^9}(\cos 10\theta + 2\cos 8\theta - 3\cos 6\theta - 8\cos 4\theta + 2\cos 2\theta + 6);$$

then as in No. 12.

14. By No. 13,

the general term $=\sin^4 r\mathrm{A}\cos^6 r\mathrm{A} = \frac{1}{2^9}[\cos(10r\mathrm{A}) + 2\cos(8r\mathrm{A})$

$- 3\cos(6r\mathrm{A}) - 8\cos(4r\mathrm{A}) + 2\cos(2r\mathrm{A}) + 6]$;

but $\Sigma\cos(r\theta)$, for $r=1$ to n,

$=\cos\frac{1}{2}(n+1)\theta \,.\, \sin\frac{1}{2}n\theta \,.\, \operatorname{cosec}\frac{1}{2}\theta$, see p. 128, eqn. (13).

15. See note on p. 171,

$$(2\cos\theta)^n = \left(z+\frac{1}{z}\right)^n = z^n + \binom{n}{1}z^{n-2} + \dots + \binom{n}{r}z^{n-2r} + \dots + \frac{1}{z^n};$$

the number of terms is $n+1$ which is odd; there is a middle term and $\frac{n}{2}$ pairs of terms such as the first and last which $=z^n+\frac{1}{z^n}=2\cos n\theta$ and the $(r+1)$th and $(n-r+1)$th which

$$=\binom{n}{r}z^{n-2r}+\binom{n}{n-r}z^{n-2n+2r}$$

$$=\binom{n}{r}\left(z^{n-2r}+\frac{1}{z^{n-2r}}\right)=\binom{n}{r}\,.\,2\cos(n-2r)\theta.$$

The values 1, 2, ... $\frac{n}{2}-1$ of r give the $\frac{n}{2}$ pairs except for $z^n+\frac{1}{z^n}$; taking $\binom{n}{0}=1$ this pair can be made a special case of the general term. The middle term $=\binom{n}{\frac{n}{2}}=\frac{n!}{\left(\frac{n}{2}!\right)^2}$.

16. Compare No. 15. The terms of $\left(z+\frac{1}{z}\right)^n$ now consist of $\frac{n+1}{2}$ pairs such as $\binom{n}{r}2\cos(n-2r)\theta$ with $r=1, 2, \dots \frac{n-1}{2}$, and, assuming $\binom{n}{0}=1$, $r=0$.

17. $2^{2n}(-1)^n\sin^{2n}\theta=(2i\sin\theta)^{2n}=\left(z-\frac{1}{z}\right)^{2n}$

$$=z^{2n}-\binom{2n}{1}z^{2n-2}+\dots+(-1)^r\binom{2n}{r}z^{2n-2r}+\dots+\frac{1}{z^{2n}};$$

the number of terms is $2n+1$; there is a middle term and n pairs of terms such as

$$(-1)^r\binom{2n}{r}z^{2n-2r}+(-1)^{2n-r}\binom{2n}{2n-r}z^{2n-4n+2r}$$

$$=(-1)^r\binom{2n}{r}\left(z^{2n-2r}+\frac{1}{z^{2n-2r}}\right)=(-1)^r\binom{2n}{r}2\cos(2n-2r)\theta.$$

The values 0, 1, 2, ... $(n-1)$ of r give the n pairs, assuming $\binom{2n}{0}=1$.

The middle term $=(-1)^n\binom{2n}{n}=(-1)^n\frac{(2n)!}{(n!)^2}$.

18. As in No. 17,

$$2^{2n}\,.\,2i\,.\,(-1)^n\sin^{2n+1}\theta=(2i\sin\theta)^{2n+1}=\left(z-\frac{1}{z}\right)^{2n+1}$$

$$=\Sigma(-1)^r\binom{2n+1}{r}z^{2n+1-2r}$$

for $r=0, 1, 2, \dots 2n+1$, assuming $\binom{2n+1}{0}=1$. There are $n+1$ pairs of terms such as

$$(-1)^k\binom{2n+1}{k}z^{2n+1-2k}+(-1)^{2n+1-k}\binom{2n+1}{2n+1-k}z^{2n+1-2(2n+1-k)}$$

$$=(-1)^k.\binom{2n+1}{k}.\left(z^{2n+1-2k}-\frac{1}{z^{2n+1-2k}}\right)$$

$$=(-1)^k.\binom{2n+1}{k}.\,2i\sin(2n+1-2k)\theta;$$

the first term $(k=0)$ is $2i\sin(2n+1)\theta$, and the last $(k=n)$ is $(-1)^n\binom{2n+1}{n}.\,2i\sin\theta$.

19. $\frac{d^2}{d\theta^2}(\cos n\theta)=-n^2\cos n\theta$;

$\therefore \mathrm{A}_1\cos\theta+9\mathrm{A}_3\cos 3\theta+25\mathrm{A}_5\cos 5\theta+49\mathrm{A}_7\cos 7\theta$

$$=-\frac{d^2}{d\theta^2}(\cos^3\theta\sin^4\theta)=\frac{d}{d\theta}(3\cos^2\theta\sin^5\theta-4\cos^4\theta\sin^3\theta)$$

$=$ an expression with a factor $\sin^2\theta$. Put $\theta=0$.

Differentiating twice again,

$$\mathrm{A}_1\cos\theta+3^4\mathrm{A}_3\cos 3\theta+5^4\mathrm{A}_5\cos 5\theta+7^4.\,\mathrm{A}_7\cos 7\theta$$

$$=-\frac{d^3}{d\theta^3}(3\cos^2\theta\sin^5\theta-4\cos^4\theta\sin^3\theta),$$

which, for $\theta=0$, is the same as $\frac{d^3}{d\theta^3}(4\cos^4\theta\sin^3\theta)$ and so is the same as $\frac{d^3}{d\theta^3}(4\theta^3)$, and this is 24.

Ex. 6 gives $\mathrm{A}_1=\frac{3}{64}$, $\mathrm{A}_3=-\frac{3}{64}$, $\mathrm{A}_5=-\frac{1}{64}$, $\mathrm{A}_7=\frac{1}{64}$, and the expression

$$=\tfrac{1}{64}(3-3^5-5^4+7^4)$$
$$=\tfrac{1}{64}\{-3(3^2+1)(3^2-1)+(7^2+5^2)(7^2-5^2)\}$$
$$=\tfrac{3}{8}(-10+74).$$

EXERCISE IX. c. (p. 173.)

1. $\sin 5\theta = 5c^4 s - 10c^2 s^3 + s^5 = 5s(1-s^2)^2 - 10s^3(1-s^2) + s^5$.

2. $\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4 = c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2$.

3. $\sin 6\theta = 6c^5 s - 20c^3 s^3 + 6cs^5 = s\{6c^5 - 20c^3(1-c^2) + 6c(1-c^2)^2\}$.

4. $\cos 6\theta = c^6 - 15c^4 s^2 + 15c^2 s^4 - s^6$; put $c^2 = 1 - s^2$, etc.

5. Use eqn. (6). 6. From eqn. (6), $6t - 20t^3 + 6t^5 = 0$.

7. If $7\theta = \frac{\pi}{2}$, $\cot 7\theta = 0$; $\therefore\ 1 - \binom{7}{2}t^2 + \binom{7}{4}t^4 - \binom{7}{6}t^6 = 0$.

8. In eqn. (7), write $-\theta_3$ for θ_3, and use $\tan(-\theta_3) = -\tan\theta_3$.

9. (i) $\tan(\theta_1 + \theta_2 + \theta_3) = 0$, use eqn. (7);
 (ii) $\tan(\Sigma\theta) = 0$, use eqn. (7);
 (iii) If $\theta_1 + \theta_2 + \theta_3 = \frac{3\pi}{2}$, $\cot(\theta_1 + \theta_2 + \theta_3) = 0$.

10. $(\sin\theta + i\cos\theta)^n = s^n + \binom{n}{1}s^{n-1} \,.\, ic + \binom{n}{2}s^{n-2} \,.\, (ic)^2 + \dots$
 $= \{s^n - \binom{n}{2}s^{n-2}c^2 + \dots\} + i\{n \,.\, s^{n-1}c - \binom{n}{3}s^{n-3}c^3 + \dots\}$;
 also it $= \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)$; equate first parts and second parts.

13. Coefficient of $c^n = 1 + \binom{n}{2} + \binom{n}{4} + \dots = \frac{1}{2}\{(1+1)^n + (1-1)^n\}$ by binomial theorem for positive integral index $= \frac{1}{2} \,.\, 2^n$.

14. $\frac{1}{2}\{(1+z)^n + (1-z)^n\} = 1 + \binom{n}{2}z^2 + \binom{n}{4}z^4 + \dots$; put $z = i$, then given series $= \frac{1}{2}\{(1+i)^n + (1-i)^n\}$; but

$$(1+i)^n = \left(\sqrt{2} \,.\, \text{cis}\frac{\pi}{4}\right)^n = 2^{\frac{n}{2}} \,.\, \left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right);$$

similarly $$(1-i)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right);$$ add.

Or, in eqn. (4), put $\theta = \frac{\pi}{4}$, so that $s = c = \frac{1}{\sqrt{2}}$.

15. From eqn. (4), $\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$;
 $\therefore\ \cos 5\theta \,.\, \sec\theta = c^4 - 10c^2 s^2 + 5s^4$
 $= (1-s^2)^2 - 10s^2(1-s^2) + 5s^4$.

16. $a^3 \,\text{cis}(-3B) + 3a^2 b \,\text{cis}(A - 2B) + 3ab^2 \,\text{cis}(2A - B) + b^3 \,\text{cis}\, 3A$
 $= [a \,\text{cis}(-B) + b \,\text{cis}\, A]^3$
 $= [a(\cos B - i\sin B) + b(\cos A + i\sin A)]^3$
 $= [(a\cos B + b\cos A) - i(a\sin B - b\sin A)]^3$
 $= [c + i \,.\, 0]^3 = c^3$; equate first parts.

17. As in No. 6, $\binom{7}{1}t - \binom{7}{3}t^3 + \binom{7}{5}t^5 - \binom{7}{7}t^7 = 0$ is satisfied by $t = \tan\theta$ if $\tan 7\theta = 0$, hence by

$$t = \tan\theta \text{ for } \theta = \pm\frac{\pi}{7},\ \pm\frac{2\pi}{7},\ \pm\frac{3\pi}{7},\ 0.$$

Removing the factor t, corresponding to $\theta = 0$, eqn. becomes $\binom{7}{1} - \binom{7}{3}t^2 + \binom{7}{5}t^4 - t^6 = 0$.

18. $\Sigma_1 = -b$, $\Sigma_2 = c$, $\Sigma_3 = -e$, $\Sigma_4 = f$; use eqn. 7.

EXERCISE IX. d. (p. 176.)

1. $\text{cis}\,\theta + \frac{1}{2}\text{cis}\,2\theta + \dots + \frac{1}{2^{n-1}}\text{cis}\,n\theta = z\left[1 + \frac{1}{2}z + \dots + \left(\frac{z}{2}\right)^{n-1}\right]$, where $z = \text{cis}\,\theta$,

$$= z\frac{\left(1 - \frac{z^n}{2^n}\right)\left(1 - \frac{1}{2z}\right)}{\left(1 - \frac{z}{2}\right)\left(1 - \frac{1}{2z}\right)} = \frac{z - \frac{1}{2} - \frac{1}{2^n}z^{n+1} + \frac{1}{2^{n+1}}z^n}{1 - \frac{1}{2}\left(z + \frac{1}{z}\right) + \frac{1}{4}};$$

but $z + \frac{1}{z} = 2\cos\theta$;

$$\therefore\ \text{expression} = \frac{\text{cis}\,\theta - \frac{1}{2} - \frac{1}{2^n}\text{cis}\,(n+1)\theta + \frac{1}{2^{n+1}}\text{cis}\,n\theta}{1\frac{1}{4} - \cos\theta};$$

equate first parts.

Also when $n \to \infty$, $\frac{1}{2^n} \to 0$ and $|\cos\phi| \leqslant 1$; $\therefore$ sum to infinity $= \frac{\cos\theta - \frac{1}{2}}{1\frac{1}{4} - \cos\theta}$.

2. Put $C = \cos\phi\cos\theta + \cos^2\phi\cos 2\theta + \dots$,
 $S = \cos\phi\sin\theta + \cos^2\phi\sin 2\theta + \dots$ to n terms, and
 $z = \text{cis}\,\theta$, then $C + iS = z\cos\phi + z^2\cos^2\phi + \dots$

$$= \frac{z\cos\phi - z^{n+1}\cos^{n+1}\phi}{1 - z\cos\phi} \,.\, \frac{1 - \frac{\cos\phi}{z}}{1 - \frac{\cos\phi}{z}}$$

$$= \frac{z\cos\phi - \cos^2\phi - z^{n+1}\cos^{n+1}\phi + z^n\cos^{n+2}\phi}{1 - \cos\phi\left(z + \frac{1}{z}\right) + \cos^2\phi}$$

$$= \frac{\text{cis}\,\theta\cos\phi - \cos^2\phi - \text{cis}\,(n+1)\theta \,.\, \cos^{n+1}\phi + \text{cis}\,n\theta \,.\, \cos^{n+2}\phi}{1 - 2\cos\phi\cos\theta + \cos^2\phi};$$

Put $\phi = \theta$ and equate the first parts, thus

$$C = \frac{\cos\theta\cos\theta - \cos^2\theta - \cos(n+1)\theta\cos^{n+1}\theta + \cos n\theta\cos^{n+2}\theta}{1 - 2\cos^2\theta + \cos^2\theta}$$

$$= \frac{\cos^{n+1}\theta\{\cos n\theta\cos\theta - \cos(n+1)\theta\}}{1-\cos^2\theta} = \frac{\cos^{n+1}\theta}{\sin^2\theta}(\sin n\theta\sin\theta).$$

If, however, $\theta = r\pi$, $1 - z\cos\theta = 0$ and the work is inapplicable; the series is $1+1+1+\ldots = n$. For $\theta \neq r\pi$ $|\cos\theta| < 1$; $\therefore \cos^{n+1}\theta \to 0$; $\therefore C_n \to 0$.

3. In solution of No. 2 put $\phi = \frac{\pi}{2} - \theta$ and equate the second parts; thus

$$S \equiv \cos\left(\frac{\pi}{2}-\theta\right)\sin\theta + \cos^2\left(\frac{\pi}{2}-\theta\right)\sin 2\theta + \ldots$$

$$= \left\{\sin\theta\cos\left(\frac{\pi}{2}-\theta\right) - \sin(n+1)\theta\cos^{n+1}\left(\frac{\pi}{2}-\theta\right) + \sin n\theta\cos^{n+2}\left(\frac{\pi}{2}-\theta\right)\right\} \Big/ (1 - 2\sin\theta\cos\theta + \sin^2\theta)$$

$$= \{\sin^2\theta - \sin^{n+1}\theta\sin(n+1)\theta + \sin^{n+2}\theta\sin n\theta\}/(1 - \sin 2\theta + \sin^2\theta).$$

This applies even when $\theta = (2r+1)\frac{\pi}{2}$;

if $\theta \neq (2r+1)\frac{\pi}{2}$, $|\sin\theta| < 1$; $\therefore \sin^{n+1}\theta \to 0$.

4. $1 - b^n = 1 - \cos n\beta - i\sin n\beta = 2\sin\frac{n\beta}{2}\operatorname{cis}\frac{1}{2}(n\beta - \pi)$;

$1 - b = 2\sin\frac{1}{2}\beta \,.\, \operatorname{cis}\frac{1}{2}(\beta - \pi)$;

$\therefore \frac{a(1-b^n)}{1-b} = \frac{\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}\cdot\operatorname{cis}\{\alpha + \frac{1}{2}(n\beta-\pi) - \frac{1}{2}(\beta-\pi)\}$;

also $ab^r = \operatorname{cis}\alpha \,.\, \operatorname{cis} r\beta = \operatorname{cis}(\alpha + r\beta)$;

equate first and second parts, e.g. $\Sigma\cos(\alpha + r\beta)$, $r = 0$ to $n-1$.

$= \operatorname{cosec}\frac{1}{2}\beta \,.\, \sin\frac{1}{2}n\beta \,.\, \cos\{\alpha + \frac{1}{2}(n-1)\beta\}$.

5. $1 - \binom{n}{1}\operatorname{cis}2\theta + \binom{n}{2}\operatorname{cis}4\theta \ldots - \operatorname{cis}2n\theta = (1 - \operatorname{cis}2\theta)^n$, n odd,

$$= [2\sin^2\theta - 2i\sin\theta\cos\theta]^n = \left[2\sin\theta \,.\, \operatorname{cis}\left(\theta - \frac{\pi}{2}\right)\right]^n$$

$$= (2\sin\theta)^n \,.\, \operatorname{cis}\left(n\theta - n\cdot\frac{\pi}{2}\right);$$

but $\cos\left(n\theta - n\cdot\frac{\pi}{2}\right)$, n odd, $= (-1)^{\frac{n-1}{2}}\cdot\sin n\theta$.

6. $\binom{n}{1}\operatorname{cis}2\theta - \binom{n}{2}\operatorname{cis}4\theta + \ldots = 1 - (1 - \operatorname{cis}2\theta)^n =$, as in No. 5,

$$1 - (2\sin\theta)^n \,.\, \operatorname{cis}\left(n\theta - \frac{n\pi}{2}\right);$$

$$\therefore \text{ series} = -(2\sin\theta)^n \,.\, \sin\left(n\theta - \frac{n}{2}\cdot\pi\right)$$

$=$, for n even, $-(2\sin\theta)^n \,.\, (-1)^{\frac{n}{2}}\sin n\theta$.

7. $(2\cos\theta)^n - \binom{n}{1} \,.\, (2\cos\theta)^{n-1} \,.\, \operatorname{cis}\theta + \binom{n}{2} \,.\, (2\cos\theta)^{n-2} \,.\, \operatorname{cis}2\theta - \ldots$

$= (2\cos\theta - \operatorname{cis}\theta)^n = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$.

8. $\sin^n\phi \,.\, \operatorname{cis} n\theta + \binom{n}{1}\sin^{n-1}\phi \,.\, \sin(\theta-\phi) \,.\, \operatorname{cis}(n-1)\theta$

$+ \binom{n}{2}\sin^{n-2}\phi \,.\, \sin^2(\theta-\phi) \,.\, \operatorname{cis}(n-2)\theta + \ldots$

$= \{\sin\phi\operatorname{cis}\theta + \sin(\theta-\phi)\}^n = \{\sin\theta\cos\phi + i\sin\theta\sin\phi\}^n$

$= \sin^n\theta(\cos n\phi + i\sin n\phi)$.

9. $\operatorname{cis}A + \frac{b}{c}\operatorname{cis}2A + \frac{b^2}{c^2}\operatorname{cis}3A + \ldots$, for $\left|\frac{b}{c}\right| < 1$, has sum to infinity equal to

$$\frac{\operatorname{cis}A}{1 - \frac{b}{c}\operatorname{cis}A} = \frac{c\operatorname{cis}A}{c - b\cos A - ib\sin A} = \frac{c\operatorname{cis}A}{a\cos B - ia\sin B}$$

$$= \frac{c}{a}\cdot\frac{\operatorname{cis}A}{\operatorname{cis}(-B)} = \frac{c}{a}\cdot\operatorname{cis}(A+B);$$

given series $= \frac{c}{a}\cdot\sin(A+B) = \frac{c\sin C}{a}$.

10. $\sin^n\theta\operatorname{cis}\left(n\phi - \frac{n\pi}{2}\right) = \sin^n\theta\left\{\cos\left(\phi - \frac{\pi}{2}\right) + i\sin\left(\phi - \frac{\pi}{2}\right)\right\}^n$

$= \sin^n\theta(\sin\phi - i\cos\phi)^n \equiv \{\cos(\theta-\phi) - \cos\phi\operatorname{cis}\theta\}^n$

$= \Sigma(-1)^r\binom{n}{r}\cos^{n-r}(\theta-\phi)\cos^r\phi\operatorname{cis}r\theta$;

equate first parts.

11. $a^n\operatorname{cis}nB + \binom{n}{1}a^{n-1}b\operatorname{cis}\{(n-1)B - A\} + \ldots$

$= \{a\operatorname{cis}B + b\operatorname{cis}(-A)\}^n$

$= \{(a\cos B + b\cos A) + i(a\sin B - b\sin A)\}^n = c^n$.

12. (i) Expression $\equiv \frac{1}{x}\left\{\frac{1 - x\cos\theta}{1 - 2x\cos\theta + x^2} - 1\right\} =$, by eqn. (10),

$\cos\theta + x\cos 2\theta + \ldots$;

(ii) $1 +$ expression $= \frac{2(1 - x\cos\theta)}{1 - 2x\cos\theta + x^2} =$, by eqn. 10,

$2 + 2x\cos\theta + 2x^2\cos 2\theta + \ldots$.

13. $\text{cis}\,\alpha + \cos\theta\,.\,\text{cis}\,(\alpha+\theta) + \cos^2\theta\,.\,\text{cis}\,(\alpha+2\theta) + \dots$, for $|\cos\theta| < 1$, has sum to infinity

$$\frac{\text{cis}\,\alpha}{1-\cos\theta\,\text{cis}\,\theta} = \frac{\text{cis}\,\alpha}{1-\cos^2\theta - i\sin\theta\cos\theta} = \frac{\text{cis}\,\alpha}{\sin\theta\,.\,\text{cis}\left(\theta-\frac{\pi}{2}\right)}$$
$$= \text{cosec}\,\theta\,.\,\text{cis}\left(\alpha-\theta+\frac{\pi}{2}\right);$$
$$\therefore \text{ given series} = \text{cosec}\,\theta\,.\,\sin\left(\alpha-\theta+\frac{\pi}{2}\right).$$

14. $1 + nx\,\text{cis}\,\theta + \binom{n}{2}x^2\,\text{cis}\,2\theta + \dots = (1 + x\,\text{cis}\,\theta)^n$
$$= (1 + x\cos\theta + ix\sin\theta)^n = r^n(\cos n\alpha + i\sin n\alpha),$$
where $r\cos\alpha = 1 + x\cos\theta$, $r\sin\alpha = x\sin\theta$;
$$\therefore\ r = +\sqrt{\{(1+x\cos\theta)^2 + x^2\sin^2\theta\}}.$$

15. $\text{cis}\,n\theta + nx^2\,\text{cis}\,(n-2)\theta + \binom{n}{2}x^4\,\text{cis}\,(n-4)\theta + \dots$
$$= \{\text{cis}\,\theta + x^2\,\text{cis}\,(-\theta)\}^n = \{\cos\theta\,.\,(1+x^2) + i\sin\theta\,.\,(1-x^2)\}^n$$
$= r^n(\cos n\alpha + i\sin n\alpha)$, where $r\cos\alpha = \cos\theta\,.\,(1+x^2)$,
$r\sin\alpha = \sin\theta\,.\,(1-x^2)$;
$$\therefore\ r = +\sqrt{\{\cos^2\theta\,.\,(1+x^2)^2 + \sin^2\theta\,.\,(1-x^2)^2\}}.$$

16. $\Sigma(r+1)z^r = \Sigma(r+1)\,\text{cis}\,r\theta$; $(1-z)^2 = (1-\cos\theta - i\sin\theta)^2$
$$= \left(2\sin\frac{\theta}{2}\,.\,\text{cis}\,\frac{\theta-\pi}{2}\right)^2 = -4\sin^2\frac{\theta}{2}\,.\,\text{cis}\,\theta;$$
$$\therefore\ \text{r.h.s.} = \frac{1-(n+1)\,\text{cis}\,n\theta + n\,\text{cis}\,(n+1)\theta}{-4\sin^2\frac{1}{2}\theta\,.\,\text{cis}\,\theta}$$
$$= -\tfrac{1}{4}\,\text{cosec}^2\,\tfrac{1}{2}\theta\,.\,\{\text{cis}\,(-\theta) - (n+1)\,\text{cis}\,(n-1)\theta + n\,\text{cis}\,n\theta\}.$$

17. $\binom{2n}{1}\text{cis}\,\theta + \binom{2n}{3}\text{cis}\,3\theta + \dots = \frac{1}{2}\{(1+\text{cis}\,\theta)^{2n} - (1-\text{cis}\,\theta)^{2n}\}$
$$= \tfrac{1}{2}\left\{\left[2\cos\frac{\theta}{2}\,.\,\text{cis}\,\frac{\theta}{2}\right]^{2n} - \left[2\sin\frac{\theta}{2}\,.\,\text{cis}\,\frac{\theta-\pi}{2}\right]^{2n}\right\}$$
$$= \tfrac{1}{2}\,.\,2^{2n}\left\{\cos^{2n}\frac{\theta}{2}\,.\,\text{cis}\,n\theta - \sin^{2n}\frac{\theta}{2}\,.\,\text{cis}\,(n\theta - n\pi)\right\}.$$

18. $\text{cis}\,\theta + x\,\text{cis}\,3\theta + x^2\,\text{cis}\,5\theta + \dots$, for $|x| < 1$, has sum to infinity
$$\frac{\text{cis}\,\theta}{1-x\,\text{cis}\,2\theta} = \frac{\text{cis}\,\theta\,[1-x\,\text{cis}\,(-2\theta)]}{[1-x\,\text{cis}\,2\theta]\,[1-x\,\text{cis}\,(-2\theta)]}$$
$$= \frac{\text{cis}\,\theta - x\,\text{cis}\,(-\theta)}{1-2x\cos 2\theta + x^2};$$
$$\therefore\ \text{given series} = \frac{\cos\theta - x\cos\theta}{1-2x\cos 2\theta + x^2};$$

$$1 - 2x\cos 2\theta + x^2 = (1-x)^2 + 4x\sin^2\theta;$$
$$\therefore\ \frac{\cos\theta\,(1-x)}{1-2x\cos 2\theta + x^2} = \frac{\cos\theta}{1-x}\left\{\frac{1}{1-\frac{x}{(1-x)^2}(-4\sin^2\theta)}\right\}$$

for values of x for which $\left|\frac{4x}{(1-x)^2}\right| < 1$, e.g. for $|x| < \frac{1}{6}$, this expression is the sum of the G.P.,
$$\frac{\cos\theta}{1-x}\left\{1 + \sum\frac{x^r(-4\sin^2\theta)^r}{(1-x)^{2r}}\right\}$$
$$= \frac{\cos\theta}{1-x} + \sum\frac{x^r\cos\theta(-4\sin^2\theta)^r}{(1-x)^{2r+1}};$$
for $|x| < \frac{1}{6}$, this may be expanded into an absolutely convergent power series; the coefficient of x^n in the general term $= \cos\theta\,.\,(-4\sin^2\theta)^r\,.\,\{\text{coeff. of } x^{n-r} \text{ in } (1-x)^{-2r-1}\}$
$$= \cos\theta\,.\,(-4\sin^2\theta)^r\,.\,\frac{(2r+1)(2r+2)\dots(2r+n-r)}{(n-r)!}$$
$$= \cos\theta\,.\,(-4\sin^2\theta)^r\,.\,\frac{(n+r)!}{(n-r)!\,(2r)!};$$
but coeff. of x^n in $\frac{\cos\theta\,(1-x)}{1-2x\cos 2\theta + x^2}$ is $\cos(2n+1)\theta$.

19. $$\frac{a}{ax^2 - 2bx + c} = \frac{\frac{a}{c}}{1 - 2x\cos\theta\,.\,\sqrt{\left(\frac{a}{c}\right)} + \frac{a}{c}x^2},$$
where $b = \cos\theta\,.\,\sqrt{(ac)}$,
$$= \frac{a}{c}\,.\,\frac{1}{1-2y\cos\theta + y^2}, \text{ where } y = x\sqrt{\left(\frac{a}{c}\right)};$$
$$\frac{1}{1-2y\cos\theta + y^2} = \frac{1}{[1-y\,\text{cis}\,\theta]\,[1-y\,\text{cis}\,(-\theta)]}$$
$$= \frac{1}{2iy\sin\theta}\left\{\frac{1}{1-y\,\text{cis}\,\theta} - \frac{1}{1-y\,\text{cis}\,(-\theta)}\right\}$$
$$= \frac{1}{2i\,y\sin\theta}\,\Sigma y^r\{\text{cis}\,r\theta - \text{cis}\,(-r\theta)\},$$
in which the coefficient of y^n is
$$\frac{1}{2i\sin\theta}\left\{\text{cis}\,(n+1)\theta - \text{cis}\,(-n-1)\theta\right\} = \frac{\sin(n+1)\theta}{\sin\theta};$$
$$\therefore\ \text{general term is } \left\{x\sqrt{\left(\frac{a}{c}\right)}\right\}^n\,.\,\frac{a}{c}\,\frac{\sin(n+1)\theta}{\sin\theta}.$$

20. $\{\text{cis}(\frac{1}{2}x)+\text{cis}(-\frac{1}{2}x)\}^{2n}$

$=\text{cis}\, nx+2n\,.\,\text{cis}\,(n-1)x+\ldots+\binom{2n}{n}+\ldots+\text{cis}\,(-nx)$;

$\therefore\ \left(2\cos\frac{x}{2}\right)^{2n}=[\text{cis}\, nx+\text{cis}\,(-nx)]+\ldots$

$+\binom{2n}{r}\,.\,[\text{cis}\,(n-r)x+\text{cis}\,(-n+r)x]+\ldots+\binom{2n}{n}$

$=(2\cos nx)+\ldots+\binom{2n}{r}\,.\,[2\cos(n-r)x]+\ldots+\binom{2n}{n}$;

$\therefore\ 2^n(1+\cos x)^n$

$=2\{\cos nx+\ldots+\binom{2n}{r}\cos(n-r)x+\ldots+\frac{1}{2}\,.\,\binom{2n}{n}\}$;

differentiate w.r.t. x, and divide each side by $2\,.\,(2n)!$; result follows.

EXERCISE IX. e. (p. 182.)

1. $\sin n\theta=p_n s^n+\ldots+p_1 s$, then $p_1=\lim\limits_{\theta\to 0}\dfrac{\sin n\theta}{\sin\theta}=n$. In

$\cos n\theta=2^{n-1}\cos^n\theta+\ldots$, write $\dfrac{\pi}{2}-\theta$ for θ;

$$\therefore\ \cos\left(n\,.\,\frac{\pi}{2}-n\theta\right)=2^{n-1}\sin^n\theta+\ldots;$$

but $\cos\left(n\,.\,\dfrac{\pi}{2}-n\theta\right)$, for n odd, $=(-1)^{\frac{1}{2}(n-1)}\,.\,\sin n\theta$.

2. (i) $\cos n\theta=2^{n-1}\cos^n\theta+\ldots+a_0$; put

$$\theta=\frac{\pi}{2},\ a_0=\cos\left(\frac{n}{2}\pi\right)=(-1)^{\frac{1}{2}n}.$$

(ii) Write $\dfrac{\pi}{2}-\theta$ for θ, then

$$\cos\left(\frac{n}{2}\pi-n\theta\right)=2^{n-1}\sin^n\theta+\ldots+(-1)^{\frac{1}{2}n};$$

but $\cos\left(\dfrac{n}{2}\pi-n\theta\right)=(-1)^{\frac{1}{2}n}\cos n\theta$.

3. As in eqn. (13), $\dfrac{\sin n\theta}{\sin\theta}=2^{n-1}\cos^{n-1}\theta+\ldots$; write $\dfrac{\pi}{2}-\theta$ for θ;

then $\sin\left(\dfrac{n}{2}\pi-n\theta\right)=(-1)^{\frac{1}{2}n-1}\sin n\theta$;

$$\therefore\ \frac{\sin n\theta}{\cos\theta}=(-1)^{\frac{1}{2}n-1}\,.\,2^{n-1}\sin^{n-1}\theta+\ldots+p_1\sin\theta;$$

$$\therefore\ p_1=\lim_{\theta\to 0}\frac{\sin n\theta}{\cos\theta\sin\theta}=\lim\frac{\sin n\theta}{\sin\theta}=n.$$

4. (i) $\cos n\theta=2^{n-1}\cos^n\theta+\ldots+p_1\cos\theta$;

$$\therefore\ p_1=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos n\theta}{\cos\theta}=\lim\frac{-n\sin n\theta}{-\sin\theta}$$

$$=n\sin\left(n\,.\,\frac{\pi}{2}\right)=n\,.\,(-1)^{\frac{1}{2}(n-1)}$$

(ii) As in No. 3; for n odd,

$$\sin\left(n\,.\,\frac{\pi}{2}-n\theta\right)=(-1)^{\frac{1}{2}(n-1)}\cos n\theta;$$

$\therefore$ by eqn. (13) $\dfrac{\cos n\theta}{\cos\theta}=(-1)^{\frac{1}{2}(n-1)}\,.\,2^{n-1}\sin^{n-1}\theta+\ldots+q_0$;

for q_0, put $\theta=0$.

5. (i) As in No. 1, $\dfrac{\sin n\theta}{\sin\theta}=(-1)^{\frac{1}{2}(n-1)}\,.\,2^{n-1}\,.\,\sin^{n-1}\theta+\ldots+n$;

(ii) From eqn. (13),

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\cos^{n-1}\theta+\ldots+b_0\text{ where }b_0=\frac{\sin\left(n\,.\,\frac{\pi}{2}\right)}{\sin\frac{\pi}{2}}$$

6. (i) In powers of $\sin\theta$ from No. 3;

(ii) In powers of $\cos\theta$, from eqn. (13),

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\cos^{n-1}\theta+\ldots+p_1\cos\theta;$$

$$p_1=\lim_{\theta\to\frac{\pi}{2}}\frac{\sin n\theta}{\sin\theta\cos\theta}=\lim\frac{\sin n\theta}{\cos\theta}$$

$$=\lim\frac{n\cos n\theta}{-\sin\theta}=-n\cos\left(\frac{n}{2}\pi\right)=-n(-1)^{\frac{1}{2}n}.$$

7. Put $\theta=0$, $a_0=1$; as on p. 181, $-n\sin n\theta=\Sigma r a_r s^{r-1}c$;

$$\therefore\ -n^2\Sigma(a_r s^r)=-n^2\cos n\theta=\Sigma\{r(r-1)a_r s^{r-2}c^2-ra_r s^r\}$$

$$=\Sigma\{r(r-1)a_r s^{r-2}-r^2 a_r s^r\};$$

$$\therefore\ -n^2 a_r=(r+2)(r+1)a_{r+2}-r^2 a_r;$$

$\therefore\ a_{r+2}=-\dfrac{n^2-r^2}{(r+1)(r+2)}\,a_r$; hence result in No. 22.

8. $a_1=\lim\limits_{\theta\to 0}\dfrac{\sin n\theta}{\sin\theta}=n$; $n\cos n\theta=\Sigma\, r a_r s^{r-1}c$; $\therefore$ as in No. 7,

$$-n^2\Sigma a_r s^r=\Sigma\{r(r-1)a_r s^{r-2}-r^2 a_r s^r\};$$

$$\therefore\ a_{r+2}=-\frac{n^2-r^2}{(r+1)(r+2)}\,a_r;$$

hence result in No. 25.

9. See Example 9, p. 181.

10. $\dfrac{\sin 7\theta}{\sin\theta}=\dfrac{2\sin 7\theta\cos\theta}{2\sin\theta\cos\theta}=\dfrac{\sin 8\theta+\sin 6\theta}{\sin 2\theta}$; put $2\theta=\alpha$; $\therefore\ y=2\cos\alpha$;

$$\frac{\sin 4\alpha}{\sin\alpha}=\frac{2\sin 2\alpha\cos 2\alpha}{\sin\alpha}=4\cos\alpha(2\cos^2\alpha-1)=y^3-2y;$$

$$\frac{\sin 3\alpha}{\sin\alpha}=3-4\sin^2\alpha=y^2-1;\ \text{add.}$$

11. $\dfrac{\sin 9\theta}{\sin\theta}=\dfrac{3\sin 3\theta-4\sin^3 3\theta}{\sin\theta}=\dfrac{\sin 3\theta}{\sin\theta}\cdot\{3-4\sin^2 3\theta\}$

$$=(3-4s^2).\{3-4(3s-4s^3)^2\}$$
$$=(4c^2-1)\{3-4(1-c^2)(4c^2-1)^2\}$$
$$=(x^2-1)\{3-(4-x^2)(x^2-1)^2\}.$$

12. As on p. 179, $a_n=2^{n-1}$; by eqn. (18),

$$a_{n-2}=-\frac{(n-1).n}{n^2-(n-2)^2}\cdot a_n=-n\cdot 2^{n-3};$$

$$a_{n-4}=-\frac{(n-3)(n-2)}{n^2-(n-4)^2}a_{n-2}=\text{etc.}$$

13. From No. 12, by differentiation. **Or** by method of Example 9.

14. As on p. 180, for n even, $a_0=\cos\left(\dfrac{n}{2}\pi\right)=(-1)^{\frac{1}{2}n}$; by eqn. (18),

$$a_2=-\frac{n^2}{2}a_0;\quad a_4=-\frac{n^2-2^2}{3.4}a_2=\text{etc.}$$

15. As on p. 180, for n odd, $a_1=n(-1)^{\frac{1}{2}(n-1)}$; by eqn. (18),

$$a_3=-\frac{n^2-1^2}{2.3}a_1;\ \text{etc.}$$

16. From No. 14, by differentiation. **Or** by method of Example 9.

17. From No. 15, by differentiation. **Or** by method of Example 9.

18. From No. 12, writing $\dfrac{\pi}{2}-\theta$ for θ; for n even,

$$\cos\left(\frac{n}{2}\pi-n\theta\right)=(-1)^{\frac{1}{2}n}\cos n\theta.$$

19. From No. 12, writing $\dfrac{\pi}{2}-\theta$ for θ; for n odd,

$$\cos\left(n\cdot\frac{\pi}{2}-n\theta\right)=(-1)^{\frac{1}{2}(n-1)}\sin n\theta.$$

20. From No. 13, writing $\dfrac{\pi}{2}-\theta$ for θ; for n even,

$$\sin\left(\frac{n}{2}\pi-n\theta\right)=(-1)^{\frac{1}{2}n-1}.\sin n\theta.$$

21. From No. 13, writing $\dfrac{\pi}{2}-\theta$ for θ; for n odd,

$$\sin\left(n\cdot\frac{\pi}{2}-n\theta\right)=(-1)^{\frac{1}{2}(n-1)}.\cos n\theta.$$

22. From No. 14, writing $\dfrac{\pi}{2}-\theta$ for θ; for n even,

$$\cos\left(\frac{n}{2}\pi-n\theta\right)=(-1)^{\frac{1}{2}n}\cos n\theta.$$

23. From No. 17, writing $\dfrac{\pi}{2}-\theta$ for θ; for n odd,

$$\sin\left(n\cdot\frac{\pi}{2}-n\theta\right)=(-1)^{\frac{1}{2}(n-1)}\cos n\theta.$$

24. From No. 16, writing $\dfrac{\pi}{2}-\theta$ for θ; for n even,

$$\sin\left(\frac{n}{2}\pi-n\theta\right)=(-1)^{\frac{1}{2}n-1}\sin n\theta.$$

25. From No. 15, writing $\dfrac{\pi}{2}-\theta$ for θ; for n odd,

$$\cos\left(n\cdot\frac{\pi}{2}-n\theta\right)=(-1)^{\frac{1}{2}(n-1)}.\sin n\theta.$$

26. See No. 12.

27. l.h.s. $=\dfrac{\sin\dfrac{2p\pi}{4}}{\cos\dfrac{\pi}{4}}=\sqrt{2}\sin\dfrac{p\pi}{2}$;

$$\text{r.h.s.}=\frac{1}{\sqrt{2}}\left\{2p-\frac{2p(2^2p^2-2^2)}{3!}\cdot\frac{1}{2}+\frac{2p(2^2p^2-2^2)(2^2p^2-4^2)}{5!}\cdot\frac{1}{2^2}-\dots\right\}$$

$$=\frac{1}{\sqrt{2}}\left\{2p-\frac{p(p^2-1^2)}{3!}\cdot 2^2+\frac{p(p^2-1^2)(p^2-2^2)}{5!}\cdot 2^3-\dots\right\}.$$

28. From No. 13, putting $\theta=\frac{\pi}{4}$, and writing $n+1$ for n, $\sqrt{2}\sin(n+1)\frac{\pi}{4}$

$$=(\sqrt{2})^n-(n-1)(\sqrt{2})^{n-2}+\frac{(n-2)(n-3)}{2!}(\sqrt{2})^{n-4}-\dots;$$

multiply each side by $(\sqrt{2})^n$; result follows.

29. In No. 22, write $2n$ for n;

$$\cos 2n\theta=1-\frac{2^2n^2}{2!}s^2+\frac{2^2n^2(2^2n^2-2^2)}{4!}s^4-\dots$$

$$=1-\frac{n^2}{2!}(2s)^2+\frac{n^2(n^2-1^2)}{4!}(2s)^4-\dots;$$

put $\theta=\frac{\pi}{6}$, so that $2s=1$.

30. In No. 24, write $2n$ for n, then, as in No. 29,

$$\frac{\sin 2n\theta}{\cos\theta}=n\left\{2s-\frac{n^2-1^2}{3!}(2s)^3+\frac{(n^2-1^2)(n^2-2^2)}{5!}(2s)^5-\dots\right\};$$

put $\theta=\frac{\pi}{6}$, so that $2s=1$.

31. $\frac{1}{x}=\cos\theta-i\sin\theta$; $\therefore\ x+\frac{1}{x}=2\cos\theta$; $x^n+\frac{1}{x^n}=2\cos n\theta$; use No. 12.

32. As in No. 31; if $x+\frac{1}{x}=y$,

$$x^9+\frac{1}{x^9}=y^9-9y^7+\frac{9\,.\,6}{2!}y^5-\frac{9\,.\,5\,.\,4}{3!}y^3+\frac{9\,.\,4\,.\,3\,.\,2}{4!}y.$$

33. $x=\cos\theta+i\sin\theta$, $\frac{1}{x}=\cos\theta-i\sin\theta$, $x-\frac{1}{x}=2i\sin\theta$; but, from No. 19,

$$2\sin 7\theta=-\left\{(2s)^7-7\,.\,(2s)^5+\frac{7\,.\,4}{2}(2s)^3-\frac{7\,.\,3\,.\,2}{3!}(2s)\right\};$$

$$\therefore\ \frac{1}{i}\left(x^7-\frac{1}{x^7}\right)=$$

$$-\frac{1}{i}\left(x-\frac{1}{x}\right)\left\{-\left(x-\frac{1}{x}\right)^6-7\left(x-\frac{1}{x}\right)^4-14\left(x-\frac{1}{x}\right)^2-7\right\}.$$

34. In No. 15, coeff. of c^{2r+1} is

$$(-1)^{\frac{1}{2}(n-1)}\,.\,(-1)^r\,.\,\frac{n(n^2-1^2)(n^2-3^2)\dots[n^2-(2r-1)^2]}{(2r+1)!}$$

$$=(-1)^{\frac{1}{2}(n-1)+r}$$

$$\times\frac{n(n-1)(n+1)(n-3)(n+3)\dots(n-2r+1)(n+2r-1)}{(2r+1)!};$$

this fraction

$$=\frac{\frac{1}{2}(n+2r-1)\,.\,\frac{1}{2}(n+2r-3)\dots\frac{1}{2}(n-2r+1)\,.\,2^{2r}\,.\,n}{(2r+1)!}$$

$$=\frac{\{\frac{1}{2}(n+2r-1)\}!\,2^{2r}\,.\,n}{(2r+1)!\{\frac{1}{2}(n-2r-1)\}!};$$

in No. 12, coeff. of c^{2r+1}

$$=(-1)^{\frac{1}{2}(n-2r-1)}\,.\,\frac{n\left\{\frac{n+2r-1}{2}\dots(2r+2)\right\}}{\{\frac{1}{2}(n-2r-1)\}!}\,.\,2^{2r}$$

$$=(-1)^{\frac{1}{2}(n-1)-r}\,.\,\frac{n\{\frac{1}{2}(n+2r-1)\}!}{\{\frac{1}{2}(n-2r-1)\}!\,(2r+1)!}\,.\,2^{2r}.$$

35. In No. 14, coeff. of c^{2r} is

$$(-1)^{\frac{n}{2}}\,.\,(-1)^r\,.\,\frac{n^2(n^2-2^2)\dots[n^2-(2r-2)^2]}{(2r)!}$$

=, as in No. 34,

$$(-1)^{\frac{n}{2}+r}\,.\,\frac{\frac{1}{2}(n+2r-2)\,.\,\frac{1}{2}(n+2r-4)\dots\frac{1}{2}(n-2r+2)}{(2r)!}\,.\,n\,.\,2^{2r-1}$$

$$=(-1)^{\frac{n}{2}+r}\,.\,\frac{\{\frac{1}{2}(n+2r-2)\}!}{(2r)!\{\frac{1}{2}(n-2r)\}!}\,.\,n\,.\,2^{2r-1};$$

in No. 12, coeff. of c^{2r} is

$$(-1)^{\frac{1}{2}(n-2r)}\,.\,2^{2r-1}\,.\,\frac{n\left\{\frac{n+2r-2}{2}\dots(2r+1)\right\}}{\{\frac{1}{2}(n-2r)\}!}$$

$$=2^{2r-1}\,.\,n\,.\,\frac{\{\frac{1}{2}(n+2r-2)\}!\,(-1)^{\frac{1}{2}n-r}}{\{\frac{1}{2}(n-2r)\}!\,(2r)!}.$$

36. $x=\sin\theta$; $\therefore\ \frac{dx}{d\theta}=\cos\theta$;

$$\frac{dy}{dx}\cdot\frac{dx}{d\theta}=n\cos n\theta;\quad\therefore\ \cos\theta\frac{dy}{dx}=n\cos n\theta;$$

$$\therefore\ \cos\theta\cdot\frac{d^2y}{dx^2}\cdot\frac{dx}{d\theta}-\sin\theta\frac{dy}{dx}=-n^2\sin n\theta;$$

$$\therefore\ (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-n^2y;$$

$$\therefore\ (1-x^2)y_{k+2}+k(-2x)y_{k+1}$$

$$+\frac{k(k-1)}{2}\,.\,(-2)y_k-xy_{k+1}-ky_k=-n^2y_k;$$

$$\therefore\ (1-x^2)y_{k+2}-(2k+1)xy_{k+1}=(k^2-n^2)y_k;$$

28. From No. 13, putting $\theta=\frac{\pi}{4}$, and writing

$n+1$ for n, $\sqrt{2}\sin(n+1)\frac{\pi}{4}$

$$=(\sqrt{2})^n-(n-1)(\sqrt{2})^{n-2}+\frac{(n-2)(n-3)}{2!}(\sqrt{2})^{n-4}-\dots;$$

multiply each side by $(\sqrt{2})^n$; result follows.

29. In No. 22, write $2n$ for n;

$$\cos 2n\theta=1-\frac{2^2n^2}{2!}s^2+\frac{2^2n^2(2^2n^2-2^2)}{4!}s^4-\dots$$

$$=1-\frac{n^2}{2!}(2s)^2+\frac{n^2(n^2-1^2)}{4!}(2s)^4-\dots;$$

put $\theta=\frac{\pi}{6}$, so that $2s=1$.

30. In No. 24, write $2n$ for n, then, as in No. 29,

$$\frac{\sin 2n\theta}{\cos\theta}=n\left\{2s-\frac{n^2-1^2}{3!}(2s)^3+\frac{(n^2-1^2)(n^2-2^2)}{5!}(2s)^5-\dots\right\};$$

put $\theta=\frac{\pi}{6}$, so that $2s=1$.

31. $\frac{1}{x}=\cos\theta-i\sin\theta$; $\therefore\ x+\frac{1}{x}=2\cos\theta$; $x^n+\frac{1}{x^n}=2\cos n\theta$; use No. 12.

32. As in No. 31; if $x+\frac{1}{x}=y$,

$$x^9+\frac{1}{x^9}=y^9-9y^7+\frac{9\,.\,6}{2!}y^5-\frac{9\,.\,5\,.\,4}{3!}y^3+\frac{9\,.\,4\,.\,3\,.\,2}{4!}y.$$

33. $x=\cos\theta+i\sin\theta$, $\frac{1}{x}=\cos\theta-i\sin\theta$, $x-\frac{1}{x}=2i\sin\theta$; but, from No. 19,

$$2\sin 7\theta=-\left\{(2s)^7-7\,.\,(2s)^5+\frac{7\,.\,4}{2}(2s)^3-\frac{7\,.\,3\,.\,2}{3!}(2s)\right\};$$

$$\therefore\ \frac{1}{i}\left(x^7-\frac{1}{x^7}\right)=$$

$$-\frac{1}{i}\left(x-\frac{1}{x}\right)\left\{-\left(x-\frac{1}{x}\right)^6-7\left(x-\frac{1}{x}\right)^4-14\left(x-\frac{1}{x}\right)^2-7\right\}.$$

34. In No. 15, coeff. of c^{2r+1} is

$$(-1)^{\frac{1}{2}(n-1)}\,.\,(-1)^r\,.\,\frac{n(n^2-1^2)(n^2-3^2)\dots[n^2-(2r-1)^2]}{(2r+1)!}$$

$$=(-1)^{\frac{1}{2}(n-1)+r}$$

$$\times\frac{n(n-1)(n+1)(n-3)(n+3)\dots(n-2r+1)(n+2r-1)}{(2r+1)!};$$

this fraction

$$=\frac{\frac{1}{2}(n+2r-1)\,.\,\frac{1}{2}(n+2r-3)\dots\frac{1}{2}(n-2r+1)\,.\,2^{2r}\,.\,n}{(2r+1)!}$$

$$=\frac{\{\frac{1}{2}(n+2r-1)\}!\,2^{2r}\,.\,n}{(2r+1)!\,\{\frac{1}{2}(n-2r-1)\}!};$$

in No. 12, coeff. of c^{2r+1}

$$=(-1)^{\frac{1}{2}(n-2r-1)}\,.\,\frac{n\left\{\frac{n+2r-1}{2}\dots(2r+2)\right\}}{\{\frac{1}{2}(n-2r-1)\}!}\,.\,2^{2r}$$

$$=(-1)^{\frac{1}{2}(n-1)-r}\,.\,\frac{n\{\frac{1}{2}(n+2r-1)\}!}{\{\frac{1}{2}(n-2r-1)\}!\,(2r+1)!}\,.\,2^{2r}.$$

35. In No. 14, coeff. of c^{2r} is

$$(-1)^{\frac{n}{2}}\,.\,(-1)^r\,.\,\frac{n^2(n^2-2^2)\dots[n^2-(2r-2)^2]}{(2r)!}$$

=, as in No. 34,

$$(-1)^{\frac{n}{2}+r}\,.\,\frac{\frac{1}{2}(n+2r-2)\,.\,\frac{1}{2}(n+2r-4)\dots\frac{1}{2}(n-2r+2)}{(2r)!}\,.\,n\,.\,2^{2r-1}$$

$$=(-1)^{\frac{n}{2}+r}\,.\,\frac{\{\frac{1}{2}(n+2r-2)\}!}{(2r)!\,\{\frac{1}{2}(n-2r)\}!}\,.\,n\,.\,2^{2r-1};$$

in No. 12, coeff. of c^{2r} is

$$(-1)^{\frac{1}{2}(n-2r)}\,.\,2^{2r-1}\,.\,\frac{n\left\{\frac{n+2r-2}{2}\dots(2r+1)\right\}}{\{\frac{1}{2}(n-2r)\}!}$$

$$=2^{2r-1}\,.\,n\,.\,\frac{\{\frac{1}{2}(n+2r-2)\}!\,(-1)^{\frac{1}{2}n-r}}{\{\frac{1}{2}(n-2r)\}!\,(2r)!}.$$

36. $x=\sin\theta$; $\therefore\ \frac{dx}{d\theta}=\cos\theta$;

$$\frac{dy}{dx}\,.\,\frac{dx}{d\theta}=n\cos n\theta;\quad\therefore\ \cos\theta\frac{dy}{dx}=n\cos n\theta;$$

$$\therefore\ \cos\theta\,.\,\frac{d^2y}{dx^2}\,.\,\frac{dx}{d\theta}-\sin\theta\frac{dy}{dx}=-n^2\sin n\theta;$$

$$\therefore\ (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-n^2y;$$

$$\therefore\ (1-x^2)y_{k+2}+k(-2x)y_{k+1}$$

$$+\frac{k(k-1)}{2}\,.\,(-2)y_k-xy_{k+1}-ky_k=-n^2y_k;$$

$$\therefore\ (1-x^2)y_{k+2}-(2k+1)xy_{k+1}=(k^2-n^2)y_k;$$

$\therefore$ for $x=0$, $y_{k+2}=-(n^2-k^2)y_k$; also for $x=0$, $\theta=0$;
$\therefore$ $y_0=0$, $y_1=n$, $y_2=0$, $y_3=-(n^2-1^2)y_1=-(n^2-1^2)n$; etc.; if k is even, $y_k=0$;

$$\therefore \sin n\theta = y_1 x + \frac{y_3}{3!}x^3 + \frac{y_5}{5!}x^5 + \ldots \text{ gives No. 25.}$$

37. As in No. 36, $\dfrac{dx}{d\theta}=\cos\theta$;

$$\frac{dy}{dx}\cdot\frac{dx}{d\theta}=-n\sin n\theta;\quad \therefore \cos\theta\cdot\frac{dy}{dx}=-n\sin n\theta;$$

$$\therefore \cos\theta\cdot\frac{d^2y}{dx^2}\cdot\frac{dx}{d\theta}-\sin\theta\frac{dy}{dx}=-n^2\cos n\theta;$$

hence $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=-n^2y$ and

for $x=0$, $y_{k+2}=-(n^2-k^2)y_k$;

but $y_0=1$, $y_1=0$, $y_2=-n^2$, etc.; if k is odd, $y_k=0$;

$$\therefore \cos n\theta = y_0+\frac{y_2}{2!}x^2+\frac{y_4}{4!}x^4+\ldots \text{ gives No. 22.}$$

EXERCISE IX. f. (p. 185.)

1. $\dfrac{x^7-1}{x-1}=0$, $x\neq 1$; $\therefore$ $x^7=\text{cis}\,2r\pi$, $x=\text{cis}\,\dfrac{2r\pi}{7}$, $r\neq 0$, $r=1$ to 6.

2. $\dfrac{x^{18}+1}{x^6+1}=0$, $x^6\neq -1$; $\therefore$ $x^{18}=\text{cis}\,(2r+1)\pi$;

$\therefore$ $x=\text{cis}\,\dfrac{(2r+1)\pi}{18}$, excluding $x=\text{cis}\,\dfrac{(2k+1)\pi}{6}=\dfrac{(6k+3)\pi}{18}$,
i.e. excluding $r=3k+1$.

3. $ax-b=(a-bx)\,.\,\text{cis}\,\dfrac{2r\pi}{n}$;

$$\therefore x\left(a+b\cos\frac{2r\pi}{n}+ib\sin\frac{2r\pi}{n}\right)$$
$$=b+a\cos\frac{2r\pi}{n}+i\,.\,a\sin\frac{2r\pi}{n};$$

multiply each side by $a+b\cos\dfrac{2r\pi}{n}-ib\sin\dfrac{2r\pi}{n}$.

4. Put $x=\tan\theta$; $\therefore$ $(1-i\tan\theta)^n+i(1+i\tan\theta)^n=0$;
$\therefore$ $(\cos\theta-i\sin\theta)^n+i(\cos\theta+i\sin\theta)^n=0$, $\cos\theta\neq 0$;
$\therefore$ $\cos n\theta-i\sin n\theta+i(\cos n\theta+i\sin n\theta)=0$;
$\therefore$ $(1+i)\cos n\theta=(1+i)\sin n\theta$; $\therefore$ $\tan n\theta=1$;
$\therefore$ $n\theta=r\pi+\dfrac{\pi}{4}$.

5. $x^4+x^3+x^2+x+1\equiv\dfrac{x^5-1}{x-1}$, $x\neq 1$; $\therefore$ roots are $x=\text{cis}\,\dfrac{2r\pi}{5}$, for $r=1, 2, 3, 4$, that is $\text{cis}\,\dfrac{2k\pi}{10}$ for $k=2, 4, 6, 8$.

6. $(x-1)(x^4+x^3+x^2+x+1)=0$;

$\therefore$ $x=1$ or $x^4+x^3+x^2+x+1=0$;

if $x=1$, $x^4+x-x^3-x^2=0$;
if $x^4+x^3+x^2+x+1=0$, $(x^2+\frac{1}{2}x+1)^2=\frac{5}{4}x^2$;
$\therefore$ $x^4-x^3-x^2+x=(x^4-x^3-x^2+x)+(x^4+x^3+x^2+x+1)$

$$=2x^4+2x+1=\frac{2}{x}(x^5+x^2+\tfrac{1}{2}x)$$
$$=\frac{2}{x}(1+x^2+\tfrac{1}{2}x)=\pm\frac{2}{x}\cdot\frac{\sqrt{5}}{2}x.$$

7. $\tan 3\alpha=\dfrac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}=\dfrac{6-8}{1-12}=\frac{2}{11}$. Let α be the acute angle whose tangent is 2; since

$$\tan\frac{\pi}{3}=\sqrt{3},\ \frac{\pi}{3}<\alpha<\frac{\pi}{2};\quad \therefore \pi<3\alpha<\frac{3\pi}{2};$$

$$\therefore \sin 3\alpha=-\frac{2}{\sqrt{(4+121)}}=-\frac{2}{\sqrt{125}},$$
$$\cos 3\alpha=-\frac{11}{\sqrt{125}}.$$

$\sqrt[3]{(88+16i)}$
$=\sqrt[3]{\{-8\sqrt{125}\,.\,(\cos 3\alpha+i\sin 3\alpha)\}}$
$=-2\sqrt{5}\,.\,(\cos\alpha+i\sin\alpha)=-2(1+2i)$;

the other cube roots are

$$-2(1+2i)\,.\left(\cos\frac{2\pi}{3}\pm i\sin\frac{2\pi}{3}\right)=(1+2i)(1\mp i\sqrt{3}).$$

8. Coeff. of x^{2r} is $\frac{1}{2}(1-i)\,.\binom{n}{2r}.\{i^{2r}+i\,.\,i^{2r}\}$
$=\frac{1}{2}\binom{n}{2r}.\,i^{2r}(1-i)(1+i)=\binom{n}{2r}.(-1)^r$;
coeff. of x^{2r+1} is $\frac{1}{2}(1-i)\,.\binom{n}{2r+1}.\{-i^{2r+1}+i\,.\,i^{2r+1}\}$
$=\frac{1}{2}\,.\binom{n}{2r+1}.(1-i)^2.(-i^{2r+1})$
$=\frac{1}{2}\,.\binom{n}{2r+1}(-2i)(-i)(-1)^r=\binom{n}{2r+1}.(-1)^{r+1}$.

9. $w^3=\text{cis}\,2\pi=1$, $w^2+w+1=\dfrac{w^3-1}{w-1}=0$, $w\neq 1$.

(i) Coeff. of x^{3k} is $\binom{3n}{3k}.(1+w^{3k}+w^{6k})=\binom{3n}{3k}.3$; coeff. of x^{3k+1} has factor $1+w^{3k+1}+w^{6k+2}=1+w+w^2=0$; coeff. of x^{3k+2} has factor

$$1+w^{3k+2}+w^{6k+4}=1+w^2+w^4=1+w^2+w=0.$$

(ii) Coeff. of x^{3k+2} is $\binom{3n}{3k+2}\cdot(1+w\cdot w^{3k+2}+w^2\cdot w^{6k+4})$; but $w\cdot w^{3k+2}=w^{3k+3}=1$ and $w^2\cdot w^{6k+4}=w^{6k+6}=1$. As in (i), coeffs. of x^{3k} and x^{3k+1} are zero.

(iii) As in (ii), coeff. of x^{3k+1} is $\binom{3n}{3k+1}\cdot(1+w^2\cdot w^{3k+1}+w\cdot w^{6k+2})$ $=\binom{3n}{3k+1}\cdot 3$; coeffs. of x^{3k} and x^{3k+2} are zero.

10. (i) $y_1{}^2=(x+x^3+x^9)^2=x^2+x^6+x^{18}+2x^4+2x^{10}+2x^{12}$
$=x^2+x^6+x^5+2(x^4+x^{10}+x^{12})=y_2+2y_3$;

(ii) $y_1y_2=x^3+x^6+x^7+x^5+x^8+x^9+x^{11}+x^{14}+x^{15}$, but $x^{14}=x$ and $x^{15}=x^2$;

(iii) Similarly, $y_1y_4=x+x^3+x^4+x^7+x^8+x^9+x^{10}$
$+x^{11}+x^{12}=-1-(x^2+x^5+x^6)$, since

$\dfrac{x^{13}-1}{x-1}\equiv\Sigma x^r$ for $r=0$ to 12, so that $\Sigma x^r=0$, for $x\neq 1$.

11. $r\theta=s\theta+2k\pi$.

12. If $(1-z)^n=z^n$, then $|1-z|=|z|$; also if P represents z in the Argand Diagram and A is $(1, 0)$, OP $=|z|=|1-z|=$ AP; $\therefore$ P lies on $x=\frac{1}{2}$.

13. As in No. 12, if B is $(-1, 0)$, OP $=$ BP.

14. If $w=\dfrac{1+z}{1-z}$, $z=\dfrac{w-1}{w+1}$, but $|z|=1$; $\therefore$ $|w-1|=|w+1|$; $\therefore$ the point w in the Argand Diagram lies on the y-axis; $\therefore$ $\sqrt{w}$ is on one of the lines $y=\pm x$.

15. In the Argand Diagram let A, B, P, Q represent the given numbers and let R represent $-a+bi$. By 153, Q is the image in OX of the inverse of P w.r.t. $|z|=1$; $\therefore$ Q, O, R are collinear, and QO . OR $=$ QO . OP $=1=$ AO . OB; $\therefore$ Q, R, A, B are concyclic; P also lies on the circle since the centre is on the y-axis.

16. (i) $\sqrt{(2z-3)}=r(\cos\theta+i\sin\theta)$;

$\therefore$ $2z-3=r^2(\cos 2\theta+i\sin 2\theta)$;

$\therefore$ $|r^2\cos 2\theta+3+i\,r^2\sin 2\theta|=2$;

$\therefore$ $(r^2\cos 2\theta+3)^2+r^4\sin^2 2\theta=4$;

(ii) $r(\cos\theta+i\sin\theta)=(\cos\phi+i\sin\phi+1)^3$

$$=\left[2\cos\frac{\phi}{2}\cdot\operatorname{cis}\frac{\phi}{2}\right]^3=8\cos^3\frac{\phi}{2}\cdot\operatorname{cis}\frac{3\phi}{2};$$

$\therefore$ $r=8\cos^3\dfrac{\phi}{2}$, $\theta=\dfrac{3\phi}{2}$; $\therefore$ $r=8\cos^3\dfrac{\theta}{3}$.

17. $(t-1)^2=-1$; $\therefore$ $t=1\pm i$; expression $=\dfrac{(x+1+i)^n-(x+1-i)^n}{2i}$;
put $x+1=r\cos\phi$, $1=r\sin\phi$, so that $\cot\phi=(x+1)$;

$$\text{expression}=\frac{r^n\operatorname{cis}n\phi-r^n\operatorname{cis}(-n\phi)}{2i}=r^n\sin n\phi;$$

but $r=\operatorname{cosec}\phi$.

18. $1+\binom{n}{1}\operatorname{cis}\theta+\binom{n}{2}\operatorname{cis}2\theta+\ldots=(1+\operatorname{cis}\theta)^n$

$$=\left\{2\cos\frac{\theta}{2}\cdot\operatorname{cis}\frac{\theta}{2}\right\}^n=\left(2\cos\frac{\theta}{2}\right)^n\cdot\operatorname{cis}\frac{n\theta}{2};$$

equate first parts.

19. $$\tfrac{1}{2}\left\{\left(z+\frac{1}{z}\right)^{4n}+\left(z-\frac{1}{z}\right)^{4n}\right\}=z^{4n}+\binom{4n}{2}z^{4n-4}+\ldots+\binom{4n}{2n}+\ldots+\frac{1}{z^{4n}}$$

$$=\left(z^{4n}+\frac{1}{z^{4n}}\right)+\binom{4n}{2}\left(z^{4n-4}+\frac{1}{z^{4n-4}}\right)+\ldots+\binom{4n}{2n};$$

put $z=\operatorname{cis}\theta$; $\therefore$ $z^r+\dfrac{1}{z^r}=2\cos r\theta$;

$\therefore$ $\frac{1}{2}\{(2\cos\theta)^{4n}+(2i\sin\theta)^{4n}\}$
$=2\cos 4n\theta+\binom{4n}{2}\{2\cos(4n-4)\theta\}+\ldots$; hence result.

20. (i) If $x=\operatorname{cis}\theta$, $\dfrac{1}{x}=\operatorname{cis}(-\theta)$;

$$\therefore\ x-\frac{1}{x}=2i\sin\theta \text{ and } x^{2n+1}-\frac{1}{x^{2n+1}}=2i\sin(2n+1)\theta;$$

expression $=\dfrac{\sin(2n+1)\theta}{\sin\theta}=$, by IX. e, No. 19, $(-1)^n\{(2s)^{2n}-(2n+1)\cdot(2s)^{2n-2}+\ldots\}$ which is a polynomial P of degree n in $(2s)^2\equiv-\left(x-\dfrac{1}{x}\right)^2$; coeff. of $\left(x-\dfrac{1}{x}\right)^{2n}$ is $(-1)^n\cdot(-1)^n=1$,

$$\text{constant term}=\lim_{\theta\to 0}\frac{\sin(2n+1)\theta}{\sin\theta}=2n+1.$$

It has thus been proved that the equation, $x^{4n+2}-1=x^{2n}(x^2-1)$ P, holds for all values of x such that $|x|=1$. As it is true for more than $4n+2$ values of x, it must be an identity.

(ii) As above, $\sin(2n+1)\theta=(2n+1)\sin\theta+p_3\sin^3\theta+\ldots$;

$\therefore$ $(2n+1)\sin\theta-\sin(2n+1)\theta$
$=-\sin^3\theta(p_3+p_5\sin^2\theta+\ldots)$.

21. $u_n - u_{n-1} = \{(n+1)\sin n\theta - n\sin(n+1)\theta\}$
$$- \{n\sin(n-1)\theta - (n-1)\sin n\theta\}$$
$$= 2n\sin n\theta - n\{\sin(n+1)\theta + \sin(n-1)\theta\}$$
$$= 2n\sin n\theta - 2n\sin n\theta\cos\theta = 2n\sin n\theta\,(1-\cos\theta);$$
$\therefore$ if $1-\cos\theta$ is a factor of u_{n-1}, it is also a factor of u_n; but $1-\cos\theta$ is a factor of
$$u_1 \equiv 2\sin\theta - \sin 2\theta \equiv 2\sin\theta\,(1-\cos\theta);$$
hence by induction.

22. If $z = \text{cis}\,\theta$ is represented by the point P on the circle, centre O, rad. 1, and if A is the point (1, 0), $\sqrt[n]{z}$ is represented by point P_n on arc AP such that arc $AP_n = \frac{1}{n}$. arc AP; $\therefore\ \sqrt[n]{z} - 1$ is represented by $\overline{AP}_n$; when $n \to \infty$, $\overline{AP}_n \to$ the tangent at A; $\therefore$ for $0 < \theta < \pi$, am $(\overline{AP}_n) \to \frac{\pi}{2}$; for $-\pi < \theta < 0$ am $(\overline{AP}_n) \to -\frac{\pi}{2}$;

also $$|n\,.\,AP_n| = \left|n\,.\,2\sin\frac{\theta}{2n}\right| = \left|\theta\cdot\frac{\sin\frac{\theta}{2n}}{\frac{\theta}{2n}}\right| \to |\theta|;$$

$\therefore\ n\,.\,\overline{AP}_n \to \theta\,.\,i$ for $0 < \theta < \pi$, and $n\,.\,\overline{AP}_n \to |\theta|\,.\,(-i)$, for $-\pi < \theta < 0$, that is $\to \theta i$.

23. $$\int_0^{\frac{\pi}{2}} \cos^3\theta\sin^4\theta\,d\theta$$
$$= \left[A_1\sin\theta + \tfrac{1}{3}A_3\sin 3\theta + \tfrac{1}{5}A_5\sin 5\theta + \tfrac{1}{7}A_7\sin 7\theta\right]_0^{\frac{\pi}{2}}$$
$= A_1 - \frac{1}{3}A_3 + \frac{1}{5}A_5 - \frac{1}{7}A_7$; but integral
$$= \int_0^{\frac{\pi}{2}} (1-\sin^2\theta)\,.\,\sin^4\theta\,.\,d(\sin\theta)$$
$$= \left[\tfrac{1}{5}\sin^5\theta - \tfrac{1}{7}\sin^7\theta\right]_0^{\frac{\pi}{2}} = \tfrac{1}{5} - \tfrac{1}{7} = \tfrac{2}{35}.$$

24. If $x = \text{cis}\,\theta$, $a_1 = \text{cis}\,\alpha_1$, etc.,
$$x - a = \cos\theta - \cos\alpha + i(\sin\theta - \sin\alpha)$$
$$= 2\sin\tfrac{1}{2}(\theta-\alpha)\,[-\sin\tfrac{1}{2}(\theta+\alpha) + i\cos\tfrac{1}{2}(\theta+\alpha)]$$
$$= 2\sin\tfrac{1}{2}(\theta-\alpha)\,.\,\text{cis}\,\tfrac{1}{2}(\theta+\alpha+\pi);$$

$$\text{l.h.s.} = \frac{1}{4\sin\frac{1}{2}(\theta-\alpha_1)\,.\,\sin\frac{1}{2}(\theta-\alpha_2)} \times \text{cis}\,[-\tfrac{1}{2}(\pi+\theta+\alpha_1) - \tfrac{1}{2}(\pi+\theta+\alpha_2)]$$
$$= \frac{1}{4\sin\frac{1}{2}(\theta-\alpha_1)\,.\,\sin\frac{1}{2}(\theta-\alpha_2)} \times \text{cis}\,[\pi - \tfrac{1}{2}(2\theta+\alpha_1+\alpha_2)];\ \text{etc.}$$

25. If $a = \cos\alpha + i\sin\alpha$, etc.,
$$a + b = \cos\alpha + \cos\beta + i(\sin\alpha + \sin\beta)$$
$$= 2\cos\frac{\alpha-\beta}{2}\,.\,\text{cis}\,\frac{\alpha+\beta}{2};$$
$$\therefore\ \text{l.h.s.} = \text{cis}\,(\alpha+\beta+\gamma) + 8\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\,.\,\text{cis}\,(\alpha+\beta+\gamma);$$
$$\text{r.h.s.} = \{\Sigma\cos\alpha + i\Sigma\sin\alpha\} \times \{\Sigma\cos(\beta+\gamma) + i\Sigma\sin(\beta+\gamma)\}$$
$$= \Sigma\cos\alpha\,.\,\Sigma\cos(\beta+\gamma) - \Sigma\sin\alpha\,.\,\Sigma\sin(\beta+\gamma) + i\{\ldots\ldots\};$$
equate first parts.

26. $2\sin^2 n\theta = 1 - \cos 2n\theta =$, by IX. e, No. 22,
$$1 - \left\{1 - \frac{(2n)^2}{2!}s^2 + \frac{2^2n^2(2^2n^2-2^2)}{4!}s^4 - \ldots\right\} = \text{given series.}$$

EXERCISE IX. g. (p. 186.)

1. $x^7 + 1 = 0$ if $x^7 = \text{cis}\,(2r+1)\pi$, if
$$x = \text{cis}\,\frac{(2r+1)\pi}{7};\ r = 0,\ \pm 1,\ \pm 2,\ \pm 3;$$
$$\therefore\ x^7 + 1 \equiv \left(x - \text{cis}\,\frac{\pi}{7}\right)\left(x - \text{cis}\,\frac{3\pi}{7}\right)\left(x - \text{cis}\,\frac{5\pi}{7}\right) \times (x - \text{cis}\,\pi)\left(x - \text{cis}\,\frac{-\pi}{7}\right)\left(x - \text{cis}\,\frac{-3\pi}{7}\right)\left(x - \text{cis}\,\frac{-5\pi}{7}\right);$$
but $x - \text{cis}\,\pi = x + 1$ and
$$\left(x - \text{cis}\,\frac{\pi}{7}\right)\left(x - \text{cis}\,\frac{-\pi}{7}\right) = x^2 - x\left(\text{cis}\,\frac{\pi}{7} + \text{cis}\,\frac{-\pi}{7}\right) + 1$$
$$= x^2 - 2x\cos\frac{\pi}{7} + 1,\ \text{etc.}$$

2. Expression $=\frac{x^9-1}{x-1}$, $x\neq 1$; $x^9-1=0$ if

$$x=\text{cis}\,\frac{2r\pi}{9};\ r=\pm1,\ \pm2,\ \pm3,\ \pm4;$$

$\therefore$ as in No. 1, $\left(x-\text{cis}\,\frac{2\pi}{9}\right)\left(x-\text{cis}\,\frac{-2\pi}{9}\right)$

$$\equiv x^2-2x\cos\frac{2\pi}{9}+1 \text{ is a factor, etc.}$$

3. $(1-a^p)(1+a^p+a^{2p}+\dots+a^{(n-1)p})=1-a^{np}$

$$=1-\text{cis}\,2rp\pi=0 \text{ and } a^p\neq 1.$$

4. $(x+p)^2+q^2\equiv(x+p+iq)(x+p-iq)$; $\therefore$ $\alpha=p+iq$, $\beta=p-iq$;

$$\therefore\ \text{expression}=\frac{(x+p+iq)^n-(x+p-iq)^n}{2iq};$$

write $x+p=r\cos\theta$, $q=r\sin\theta$, so that $\tan\theta=\frac{q}{x+p}$;

$$\text{expression}=\frac{r^n\,\text{cis}\,(n\theta)-r^n\,\text{cis}\,(-n\theta)}{2iq}$$

$$=\frac{2ir^n\sin n\theta}{2iq}=\frac{q^n\,\text{cosec}^n\theta\,.\,\sin n\theta}{q}.$$

5. $$\frac{1+\cos 9\theta}{1+\cos\theta}=\frac{2\cos^2\frac{9\theta}{2}}{2\cos^2\frac{\theta}{2}}=\left\{\frac{\cos\frac{9\theta}{2}}{\cos\frac{\theta}{2}}\right\}^2;$$

$$\frac{\cos\frac{9\theta}{2}}{\cos\frac{\theta}{2}}=\frac{2\cos\frac{9\theta}{2}\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}=\frac{\sin 5\theta-\sin 4\theta}{\sin\theta}$$

=, from IX. e, No. 13, $\{x^4-3x^2+1\}-\{x^3-2x\}$.

6. If $u_k=p\sin k\theta+q\cos k\theta$ and if

$u_{k+1}=p\sin(k+1)\theta+q\cos(k+1)\theta$, then

$u_{k+2}=2\cos\theta\{p\sin(k+1)\theta+q\cos(k+1)\theta\}-\{p\sin k\theta+q\cos k\theta\}$

$=p\{2\cos\theta\sin(k+1)\theta-\sin k\theta\}+q\{2\cos\theta\cos(k+1)\theta-\cos k\theta\}$

$=p\sin(k+2)\theta+q\cos(k+2)\theta$;

but from the data these relations are true for $k=1$; $\therefore$ for $k=2, 3, 4$, etc.

7. If $a=\cos\alpha+i\sin\alpha$, etc., as in IX. f, No. 24,

$$a-b=2\sin\tfrac12(\alpha-\beta)\,\text{cis}\,\tfrac12(\alpha+\beta+\pi);$$

$\therefore$ l.h.s. $=\Sigma\{16\sin^4\frac12(\beta-\gamma)\,.\,\text{cis}\,(2\beta+2\gamma)\}$;

r.h.s. $=2\Sigma\{16\sin^2\frac12(\alpha-\beta)\times\sin^2\frac12(\alpha-\gamma)\,.\,\text{cis}\,(\alpha+\beta+\pi+\alpha+\gamma+\pi)\}$

$=32\Sigma\{\sin^2\frac12(\alpha-\beta)\,.\,\sin^2\frac12(\alpha-\gamma)\,.\,\text{cis}\,(2\alpha+\beta+\gamma)\}$.

8. $\frac12\{(1+x)^n+(1-x)^n\}=a_0+a_2x^2+a_4x^4+\dots$;

$\frac12\{(1+ix)^n+(1-ix)^n\}=a_0-a_2x^2+a_4x^4-\dots$;

add and halve and put $x=1$; then $a_0+a_4+a_8+\dots$

$=\frac14\{2^n+0+(1+i)^n+(1-i)^n\}$

$=\frac14\left\{2^n+\left(\sqrt2\,\text{cis}\,\frac{\pi}{4}\right)^n+\left(\sqrt2\,\text{cis}\,\frac{-\pi}{4}\right)^n\right\}$

$=2^{n-2}+\frac14\,.\,2^{\frac12 n}\left(\text{cis}\,\frac{n\pi}{4}+\text{cis}\,\frac{-n\pi}{4}\right)$

$=2^{n-2}+\frac14\,.\,2^{\frac12 n}\,.\left(2\cos\frac{n\pi}{4}\right)$.

9. (i) $\frac{1-z}{1+z}=\frac{r+a-b-ic}{r+a+b+ic}$; but $a+ic=\frac{a^2+c^2}{a-ic}=\frac{r^2-b^2}{a-ic}$;

$$\therefore\ \frac{1-z}{1+z}=\frac{r+a-b-ic}{r+b+\frac{r^2-b^2}{a-ic}}=\frac{(r+a-b-ic)(a-ic)}{(r+b)(a-ic+r-b)}$$

$$=\frac{a-ic}{r+b}=\frac{1}{i}\cdot\frac{c+ia}{r+b};$$

(ii) $\frac{1+iz}{1-iz}=\frac{r+a+ib-c}{r+a-ib+c}$; but $a-ib=\frac{a^2+b^2}{a+ib}=\frac{r^2-c^2}{a+ib}$;

$$\therefore\ \frac{1+iz}{1-iz}=\frac{r+a+ib-c}{r+c+\frac{r^2-c^2}{a+ib}}=\frac{(r+a+ib-c)(a+ib)}{(r+c)(a+ib+r-c)}=\frac{a+ib}{r+c}.$$

10. Put $z_r=\text{cis}\,\theta_r$, then $\Sigma z_r=\Sigma\cos\theta_r+i\Sigma\sin\theta_r=0$ and

$$\Sigma\frac{1}{z_r}=\Sigma\cos\theta_r-i\Sigma\sin\theta_r=0.$$

To prove $\Sigma z_r^4=2\Sigma\Sigma z_r^2z_s^2$. Let $z_1, z_2, \dots, z_5$ be the roots of $z^5+az^4+bz^3+cz^2+dz+e=0$; $-a=\Sigma z_r=0$, $-\frac{d}{e}=\Sigma\frac{1}{z_r}=0$;

$\therefore$ equation is $z^5+bz^3+cz^2+e=0$; $\therefore$ $z^4=-bz^2-cz-\frac{e}{z}$;

$\therefore$ $\Sigma z_r^4=-b\Sigma z_r^2=-b\{(\Sigma z_r)^2-2\Sigma(z_rz_s)\}=-b\{0-2b\}=2b^2$;

$\therefore$ $\Sigma(z_r^4-2z_r^2z_s^2)=2\Sigma z_r^4-(\Sigma z_r^2)^2$

$=4b^2-[(\Sigma z_r)^2-2\Sigma z_rz_s]^2=4b^2-(-2b)^2=0$.

11. $2ad + 2bc = ab + ac + bd + cd$;

$\therefore\ ad - bd - ac + bc = -ad + cd + ab - bc$;

$\therefore\ (a-b)(d-c) = -(a-c)(d-b)$; $\therefore\ \dfrac{a-b}{d-b} = -\dfrac{a-c}{d-c}$;

let A, B, C, D be the corresponding points. Since

$$\text{am}\left(\frac{a-b}{d-b}\right) = \text{am}\left(-\frac{a-c}{d-c}\right) = \text{am}\left(\frac{a-c}{d-c}\right) \pm \pi,$$

the anticlockwise rotation necessary to convert DB into AB = the anticlockwise rotation necessary to convert DC into AC, $\pm\pi$; this is the test that ACDB is a cyclic quadrilateral. Also from $\left|\dfrac{a-b}{d-b}\right| = \left|\dfrac{a-c}{d-c}\right|$ we have $\dfrac{AB}{DB} = \dfrac{AC}{DC}$; let tangents to circle at B, C cut AD produced at P, Q, then $\dfrac{AP}{DP} = \dfrac{\triangle ABP}{\triangle DBP} = \dfrac{AB^2}{BD^2} = \dfrac{AC^2}{DC^2} =$ similarly $\dfrac{AQ}{DQ}$; $\therefore$ P coincides with Q; $\therefore$ pole of BC lies on AD.

12. Let $ac - b^2 = k^2$, $AC - B^2 = K^2$; then points are $z = \dfrac{1}{a}(-b \pm ik)$, $z = \dfrac{1}{A}(-B \pm iK)$. If PN is perp. from P to Ox, diameter of circle through O, P with centre on Ox is $\dfrac{OP^2}{ON}$; $\therefore$ diameter of circle through O, $\dfrac{1}{a}(-b \pm ik)$ is $\dfrac{b^2+k^2}{a^2} \div \dfrac{b}{a} = \dfrac{b^2+k^2}{ab} = \dfrac{ac}{ab} = \dfrac{c}{b}$;

$\therefore$ condition is $\dfrac{c}{b} = \dfrac{C}{B}$.

13. $z = \text{cis}\,\theta$, $w = \dfrac{1}{1 - \text{cis}\,\theta + \text{cis}\,2\theta}$

$$= \frac{1}{1 - \cos\theta + \cos 2\theta + i(\sin 2\theta - \sin\theta)}$$

$$= \frac{1}{2\cos^2\theta - \cos\theta + i(2\sin\theta\cos\theta - \sin\theta)} = \frac{1}{2\cos\theta - 1}\cdot \text{cis}(-\theta);$$

$\therefore$ OQ, OP make angles θ, $-\theta$ with Ox, so that Ox bisects $\angle$ between OP and OQ. Also the point which divides the line QP in the ratio $(2\cos\theta - 1) : 1$ is

$$\frac{\text{cis}\,\theta + (2\cos\theta - 1)\cdot\dfrac{\text{cis}(-\theta)}{2\cos\theta - 1}}{1 + (2\cos\theta - 1)},$$ see p. 150, Ex. VIII. d, No. 19,

$$= \frac{\text{cis}\,\theta + \text{cis}(-\theta)}{2\cos\theta} = 1;$$ $\therefore$ the point (1, 0) lies on PQ.

Or, Let A be (1, 0); since $|z| = 1$, Q lies on the unit circle; also the point R representing z^2 is also on that circle, and arc AR = 2 . arc AQ; complete the parallelogram ROAS, then by p. 139, S represents $1 + z^2$. Take T on OS, so that OQ = TS, then $\overline{OT} = 1 + z^2 - z$; let T′ be the image of T in Ox, then since $w = \dfrac{1}{1 - z + z^2}$, the point P, which represents w is the inverse of T′ w.r.t. $|z| = 1$, see p. 153; $\therefore$ Ox bisects $\angle$POQ. Since $\triangle OTA \equiv \triangle SQA$,

$$\angle OT'A = \angle OTA = \angle SQA = 180° - \angle OQA;$$

$\therefore$ O, T′, A, Q are concyclic; $\therefore$ the inverses of T′, A, Q w.r.t. $|z| = 1$ are collinear, i.e. P, A, Q are collinear.

14. If $\tan^{-1}\left(\dfrac{1}{a_r + 1}\right) = \theta_r$, $t_r \equiv \tan\theta_r = \dfrac{1}{a_r + 1}$, where $a_r^5 = 1$;

$\therefore\ a_r t_r = 1 - t_r$; $\therefore\ t_r^5 = (1 - t_r)^5$;

$\therefore$ t_r, for $r = 1$ to 5, are the roots of $t^5 = (1-t)^5$, that is, $2t^5 - 5t^4 + 10t^3 - 10t^2 + 5t - 1 = 0$;

$$\tan(\theta_1 + \theta_2 + \ldots + \theta_5) = \frac{\Sigma t_1 - \Sigma t_1 t_2 t_3 + t_1 t_2 t_3 t_4 t_5}{1 - \Sigma t_1 t_2 + \Sigma t_1 t_2 t_3 t_4}$$

$$= \frac{\frac{5}{2} - \frac{10}{2} + \frac{1}{2}}{1 - \frac{10}{2} + \frac{5}{2}} = \tfrac{4}{3}.$$

$\therefore\ \Sigma(\theta_r) = n\pi + \tan^{-1}(\tfrac{4}{3})$.

15. nth differential coefficient

$$= \frac{i}{2}(-1)^n \,.\, n!\left\{\frac{1}{(x+i)^{n+1}} - \frac{1}{(x-i)^{n+1}}\right\};$$

put $x = r\cos\theta$, $1 = r\sin\theta$, so that $\cot\theta = x$; contents of bracket

$$= \frac{1}{r^{n+1}\,.\,\text{cis}(n+1)\theta} - \frac{1}{r^{n+1}\,.\,\text{cis}(-n-1)\theta}$$

$$= \frac{1}{r^{n+1}}\{\text{cis}(-n-1)\theta - \text{cis}(n+1)\theta\}$$

$$= \frac{1}{r^{n+1}}\cdot\{-2i\sin(n+1)\theta\};$$ but $\dfrac{1}{r^{n+1}} = \sin^{n+1}\theta$.

16. $\dfrac{x}{x^2+1} = \tfrac{1}{2}\left\{\dfrac{1}{x+i} + \dfrac{1}{x-i}\right\}$; then as in No. 15.

17. From IX. e, No. 13,

$$\frac{\sin(n+1)\theta}{\sin\theta} = (2c)^n - (n-1)\,.\,(2c)^{n-2} + \frac{(n-2)(n-3)}{2!}(2c)^{n-4} - \ldots;$$

put $\theta=\frac{2\pi}{3}$, then $2c=-1$; $\therefore\ \sin\frac{2(n+1)\pi}{3}\operatorname{cosec}\frac{2\pi}{3}$

$$=(-1)^n\left\{1-(n-1)+\frac{(n-2)(n-3)}{2!}+\dots\right\}.$$

18. From IX. e, No. 13,

$$\frac{\sin(3n+2)\theta}{\sin\theta}=(2c)^{3n+1}-3n\,.\,(2c)^{3n-1}$$

$$+\frac{(3n-1)(3n-2)}{2!}(2c)^{3n-3}-\dots;$$ put $\theta=\frac{\pi}{3}$, then $2c=1$;

$$\text{l.h.s.}=\frac{\sin\left(n\pi+\frac{2\pi}{3}\right)}{\sin\frac{\pi}{3}}=(-1)^n\cdot\frac{\sin\frac{2\pi}{3}}{\sin\frac{\pi}{3}}=(-1)^n.$$

19. In IX. e, No. 12, put $\theta=\frac{\pi}{3}$.

20. From IX. e, No. 14,

$$\cos(2n\theta)=(-1)^n\left\{1-\frac{2^2n^2}{2!}c^2+\frac{2^2n^2(2^2n^2-2^2)}{4!}c^4-\dots\right\};$$

$$\therefore\ (-1)^n\cos(2n\theta)=1-\frac{n^2}{2!}(2c)^2+\frac{n^2(n^2-1^2)}{4!}(2c)^4-\dots;$$

$$\therefore\ \text{r.h.s.}=\tfrac{1}{2}\{1+(-1)^n\}-\tfrac{1}{2}(-1)^n\{1-(-1)^n\cos 2n\theta\}$$
$$=\tfrac{1}{2}+\tfrac{1}{2}\cos 2n\theta=\cos^2 n\theta.$$

21. $(1+2x\cos\theta+x^2)^n=\{x(2\cos\theta+x)+1\}^n$

$$=x^n(2\cos\theta+x)^n+c_1x^{n-1}(2\cos\theta+x)^{n-1}+\dots$$
$$+c_rx^{n-r}(2\cos\theta+x)^{n-r}+\dots;$$

$\therefore$ coeff. of x^n is

$$(2\cos\theta)^n+c_1\cdot\binom{n-1}{1}\cdot(2\cos\theta)^{n-2}+c_2\cdot\binom{n-2}{2}(2\cos\theta)^{n-4}+$$
$$\dots+c_r\cdot\binom{n-r}{r}\cdot(2\cos\theta)^{n-2r}+\dots,$$

where general term $=\frac{n!}{r!(n-r)!}\cdot\frac{(n-r)!}{r!(n-2r)!}\cdot(2\cos\theta)^{n-2r}$

$$=\frac{n!}{(r!)^2\,.\,(n-2r)!}\cdot(2\cos\theta)^{n-2r};$$

also $(1+2x\cos\theta+x^2)^n=\{[1+x\operatorname{cis}\theta]\,.\,[1+x\operatorname{cis}(-\theta)]\}^n$

$$=\{1+c_1\,.\,x\operatorname{cis}\theta+c_2\,.\,x^2\operatorname{cis}2\theta+\dots+x^n\operatorname{cis}n\theta\}$$
$$\times\{1+c_1x\operatorname{cis}(-\theta)+c_2x^2\operatorname{cis}(-2\theta)+\dots\};$$

coeff. of x^n in this product is, since $c_{n-r}=c_r$,

$$\operatorname{cis}n\theta+c_1^2\operatorname{cis}(n-2)\theta+c_2^2\operatorname{cis}(n-4)\theta+\dots$$
$$\equiv\operatorname{cis}n\theta\{1+c_1^2\operatorname{cis}(-2\theta)+c_2^2\operatorname{cis}(-4\theta)+\dots\};$$

equating coefficients, and dividing by $\operatorname{cis}n\theta$, we have

$1+c_1^2\operatorname{cis}(-2\theta)+c_2^2\operatorname{cis}(-4\theta)+\dots=$

$\operatorname{cis}(-n\theta)\times$

$$\left\{(2\cos\theta)^n+\dots+\frac{n!}{(r!)^2\,.\,(n-2r)!}\cdot(2\cos\theta)^{n-2r}+\dots\right\};$$

equate first parts.

22. Since the expression is unaltered by writing $\frac{1}{x}$ for x, the coeff. of $\frac{1}{x^r}$ is also c_r;

$$\therefore\ (x^{-n}+\dots+1+\dots+x^n)^4=c_0+\Sigma c_r\left(x^r+\frac{1}{x^r}\right);$$

put $x=\operatorname{cis}2\theta$, then

$$x^r+\frac{1}{x^r}=\operatorname{cis}(2r\theta)+\operatorname{cis}(-2r\theta)=2\cos(2r\theta);$$

also $x^{-n}+\dots+1+\dots+x^n=1+2\cos2\theta+2\cos4\theta+\dots$

$$+2\cos 2n\theta=1+2\cdot\frac{\cos(n+1)\theta\sin n\theta}{\sin\theta}$$

$$=1+\frac{\sin(2n+1)\theta-\sin\theta}{\sin\theta}=\frac{\sin(2n+1)\theta}{\sin\theta};$$

hence first result.

Also $\int_0^{\frac{\pi}{2}}\cos 2r\theta\,d\theta$, for $r\neq0$, $=\left[\frac{1}{2r}\sin 2r\theta\right]_0^{\frac{\pi}{2}}=0$;

$\therefore$ given integral $=\int_0^{\frac{\pi}{2}}c_0\,d\theta=\frac{\pi}{2}\cdot c_0$, where $c_0=$ constant term in $\left(\frac{1+x+x^2+\dots+x^{2n}}{x^n}\right)^4$;

$\therefore$ $c_0=$ coeff. of x^{4n} in $\left(\frac{1-x^{2n+1}}{1-x}\right)^4$ or in $(1-4x^{2n+1})(1-x)^{-4}$;

$\therefore$ $c_0=$ coeff. of x^{4n} in $(1-x)^{-4}-4$. coeff. of x^{2n-1} in $(1-x)^{-4}$

$$=\frac{(4n+1)(4n+2)(4n+3)}{6}-4\,.\,\frac{2n(2n+1)(2n+2)}{6}$$

$$=\frac{2n+1}{3}(8n^2+8n+3).$$

23. $2\sin r\alpha\cos(n-r)\beta=\sin\{r(\alpha-\beta)+n\beta\}+\sin\{r(\alpha+\beta)-n\beta\}$. As in Ex. IX. d, No. 4, or from p. 128, $\Sigma\sin\{r(\alpha-\beta)+n\beta\}$, for

$$r=1\text{ to }n-1,=\frac{\sin\frac{n}{2}(\alpha+\beta)\,.\,\sin\frac{n-1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha-\beta)};$$

but $\sin\frac{n-1}{2}(\alpha-\beta)=\sin\frac{n}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha-\beta)$

$$-\cos\frac{n}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\alpha-\beta);$$

$$\therefore \text{ expression } =\sin\frac{n}{2}(\alpha+\beta)\sin\frac{n}{2}(\alpha-\beta)\cot\tfrac{1}{2}(\alpha-\beta)$$

$$-\sin\frac{n}{2}(\alpha+\beta)\cos\frac{n}{2}(\alpha-\beta).$$

Writing $-\beta$ for β and adding, given series

$$=\tfrac{1}{2}\left\{\sin\frac{n}{2}(\alpha+\beta)\sin\frac{n}{2}(\alpha-\beta)\left[\cot\tfrac{1}{2}(\alpha-\beta)+\cot\tfrac{1}{2}(\alpha+\beta)\right]\right.$$

$$\left.-\sin\left[\frac{n}{2}(\alpha+\beta)+\frac{n}{2}(\alpha-\beta)\right]\right\},$$

$$=\tfrac{1}{4}(\cos n\beta-\cos n\alpha)\frac{\sin\alpha}{\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\alpha+\beta)}-\tfrac{1}{2}\sin n\alpha$$

$$=\tfrac{1}{2}\sin\alpha\left(\frac{\cos n\beta-\cos n\alpha}{\cos\beta-\cos\alpha}-\frac{\sin n\alpha}{\sin\alpha}\right).$$

24. $1+t\operatorname{cis}\alpha+t^2\operatorname{cis}2\alpha+\ldots+t^{2n}\operatorname{cis}2n\alpha=\dfrac{1-t^{2n+1}\operatorname{cis}(2n+1)\alpha}{1-t\operatorname{cis}\alpha}$;

similarly for $1+\Sigma t\operatorname{cis}\beta$ and $1+\Sigma t\operatorname{cis}\gamma$; $\Sigma\operatorname{cis}(p\alpha+q\beta+r\gamma)$ is the coeff. of t^n in the product of these three expressions, i.e. in

$$\frac{\{1-t^{2n+1}\operatorname{cis}(2n+1)\alpha\}\{1-t^{2n+1}\operatorname{cis}(2n+1)\beta\}\{1-t^{2n+1}\operatorname{cis}(2n+1)\gamma\}}{(1-t\operatorname{cis}\alpha)(1-t\operatorname{cis}\beta)(1-t\operatorname{cis}\gamma)}$$

which is the same as the coeff. of t^n in

$$\frac{1}{(1-t\operatorname{cis}\alpha)(1-t\operatorname{cis}\beta)(1-t\operatorname{cis}\gamma)},$$

or, using partial fractions (see p. 231), in

$$\sum_{\alpha,\beta,\gamma}\left\{\frac{1}{1-t\operatorname{cis}\alpha}\cdot\frac{1}{\{1-\operatorname{cis}(\beta-\alpha)\}\{1-\operatorname{cis}(\gamma-\alpha)\}}\right\};$$

for $|t|<1$, this may be expanded in powers of t, and the coeff. of t^n is

$$\sum\frac{\operatorname{cis}n\alpha}{\{1-\operatorname{cis}(\beta-\alpha)\}\{1-\operatorname{cis}(\gamma-\alpha)\}}$$

$$=\sum\frac{\operatorname{cis}n\alpha}{2\sin\frac{1}{2}(\beta-\alpha)\operatorname{cis}\frac{1}{2}(\beta-\alpha-\pi)\,.\,2\sin\frac{1}{2}(\gamma-\alpha)\operatorname{cis}\frac{1}{2}(\gamma-\alpha-\pi)}$$

$$=\sum\frac{\operatorname{cis}\{n\alpha-\frac{1}{2}(\beta-\alpha-\pi)-\frac{1}{2}(\gamma-\alpha-\pi)\}}{4\sin\frac{1}{2}(\beta-\alpha)\sin\frac{1}{2}(\gamma-\alpha)}$$

$$=\sum\frac{-\operatorname{cis}\{(n+1)\alpha-\frac{1}{2}(\beta+\gamma)\}}{-4\sin\frac{1}{2}(\gamma-\alpha)\sin\frac{1}{2}(\alpha-\beta)}.$$

CHAPTER X

EXERCISE X. a. (p. 196.)

1. $\exp(1+i\pi)=\exp(1)\,.\,\exp(i\pi)=e\,.\,(\cos\pi+i\sin\pi)$.

2. In eqn. (11), put $y=1$; $\cos 1=\frac{1}{2}\{\exp(i)+\exp(-i)\}$.

3. $\exp(-1)\,.\,\exp\left(\frac{i\pi}{3}\right)=e^{-1}\,.\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=\frac{1}{e}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$.

4. $\exp(\cos\theta)\,.\,\exp(i\sin\theta)$.

5. $\exp\{(a+ib)+(a-ib)\}=\exp(2a)$.

6. $\exp(\log r)\,.\,\exp(i\theta)$; by definition, Ch. IV., $\exp(\log r)=r$.

7. $\exp\{\sec\alpha(\cos\alpha+i\sin\alpha)\}$
$=\exp\{1+i\tan\alpha\}=\exp(1)\,.\,\exp(i\tan\alpha)$.

8. $\exp\{x\operatorname{cis}\theta+y\operatorname{cis}\phi\}$
$=\exp\{(x\cos\theta+y\cos\phi)+i(x\sin\theta+y\sin\phi)\}$;
use eqn (10).

9. $\cos\pi+i\sin\pi$.

10. $\cos\pi-i\sin\pi$.

11. $\exp(\cos\theta+i\sin\theta)+\exp(\cos\theta-i\sin\theta)$
$=\exp(\cos\theta)\,.\,\exp(i\sin\theta)+\exp(\cos\theta)\,.\,\exp(-i\sin\theta)$
$=e^{\cos\theta}\,.\,\{\exp(i\sin\theta)+\exp(-i\sin\theta)\}$
$=e^{\cos\theta}\,.\,\{2\cos(\sin\theta)\}$ by eqn. (11).

12. $\exp\{\tan\theta(\cos\theta+i\sin\theta)\}=\exp\{\sin\theta+i\sin\theta\tan\theta\}$
$=\exp(\sin\theta)\,.\,\operatorname{cis}(\sin\theta\tan\theta)$.

13. $\exp\{i(\cos\theta+i\sin\theta)\}-\exp\{-i(\cos\theta-i\sin\theta)\}$
$=\exp\{-\sin\theta+i\cos\theta\}-\exp\{-\sin\theta-i\cos\theta\}$
$=\exp(-\sin\theta)\,.\,\{\exp(i\cos\theta)-\exp(-i\cos\theta)\}$
$=e^{-\sin\theta}\,.\,\{2i\sin(\cos\theta)\}$ by eqn. (12).

14. By No. 4, $\exp(\operatorname{cis}\theta)=e^{\cos\theta}\,.\,\operatorname{cis}(\sin\theta)$;
$\therefore\ \exp\{\exp(\operatorname{cis}\theta)\}=\exp\{e^{\cos\theta}\,.\,\cos(\sin\theta)+ie^{\cos\theta}\,.\,\sin(\sin\theta)\}$;
use eqn. (10).

15. By definition, p. 104, $\operatorname{ch}\theta-\operatorname{sh}\theta=e^{-\theta}=\exp(-\theta)$; also
$\cos\phi-i\sin\phi=\exp(-i\phi)$.

16. By eqn. (10), $X + iY = e^x(\cos y + i \sin y)$;

$\therefore$ (i) $X = e^c . \cos y$, $Y = e^c \sin y$; square and add;

(ii) $X = e^x \cos m$, $Y = e^x \sin m$; divide.

17. $\dfrac{x-a+iy}{x+a+iy} = \dfrac{(x-a+iy)(x+a-iy)}{(x+a+iy)(x+a-iy)} = \dfrac{x^2-a^2+y^2+2iay}{(x+a)^2+y^2}$;

$\therefore$ l.h.s. $= \exp\left(\dfrac{x^2-a^2+y^2}{(x+a)^2+y^2}\right) . \operatorname{cis} \dfrac{2ay}{(x+a)^2+y^2}$.

18. l.h.s. $= \dfrac{1}{(1-a^2 \operatorname{cis} 2\theta) . \operatorname{cis}(-\theta)} = \dfrac{1}{\operatorname{cis}(-\theta) - a^2 \operatorname{cis}(\theta)}$

$= \dfrac{\operatorname{cis}(\theta) - a^2 \operatorname{cis}(-\theta)}{\{\operatorname{cis}(-\theta) - a^2 \operatorname{cis}(\theta)\}\{\operatorname{cis}(\theta) - a^2 \operatorname{cis}(-\theta)\}}$

$= \dfrac{\cos\theta(1-a^2) + i \sin\theta(1+a^2)}{1 + a^4 - a^2\{\operatorname{cis}(2\theta) + \operatorname{cis}(-2\theta)\}}$.

19. $z' = \exp(z) = \exp(x+iy) = e^x . (\cos y + i \sin y)$.

(i) $|z'| = 1$; $\therefore e^x = 1$; $\therefore x = 0$; also $2n\pi + y = \operatorname{am}(z')$. For one revolution of P′ round the circle, am (z') decreases from π to $-\pi$; $\therefore$ P starts from any of the points $y = (2k+1)\pi$ on the y-axis and moves downwards a distance 2π;

(ii) If $z' \equiv x' + iy'$, $x' = 0$ and y' is negative; $\therefore e^x \cos y = 0$ and $e^x \sin y$ is negative; $\therefore \cos y = 0$ and $\sin y$, being negative, $= -1$; $\therefore y = 2n\pi - \dfrac{\pi}{2}$. Also y' decreases steadily from 0 to $-\infty$ and $y' = e^x \sin y = -e^x$; $\therefore e^x$ increases from 0 to $+\infty$; $\therefore x$ increases from $-\infty$ to $+\infty$; $\therefore$ P moves from left to right along any one of the lines, $y = 2n\pi - \dfrac{\pi}{2}$.

20. By eqn. (10), $e^a \operatorname{cis} b = e^{2b} \operatorname{cis} a$, $\therefore e^a = \pm e^{2b}$, but e^a and e^{2b} are each positive; $\therefore e^a = e^{2b}$; $\therefore a = 2b$; also cis b = cis a; $\therefore b = a + 2n\pi$.

21. (i) $iv + a = \exp(iv) = \cos v + i \sin v$; $\therefore a = \cos v$, $v = \sin v$, but $\dfrac{v}{\sin v} > 1$ for $v \neq 0$; $\therefore$ the only solution is $v = 0$. [This may also be seen graphically.]

(ii) $a + u + iv = \exp(u + iv) = e^u(\cos v + i \sin v)$; $\therefore v = e^u \sin v$;

$\therefore e^u = \dfrac{v}{\sin v} > 1$, for $v \neq 0$; $\therefore u > 0$.

22. (i) $\exp(z) = \exp(\cos a + i \sin a)$

$= e^{\cos a} . \{\cos(\sin a) + i \sin(\sin a)\}$;

the series $= 1 + \sum \dfrac{\cos na + i \sin na}{n!}$.

(ii) $\exp(z) = \exp(1 + i \tan\beta) = e . \{\cos(\tan\beta) + i \sin(\tan\beta)\}$;

also $z^n = \left(\dfrac{\cos\beta + i \sin\beta}{\cos\beta}\right)^n = \sec^n\beta . (\cos n\beta + i \sin n\beta)$.

23. (i) $\exp\{(a+ib)x\} = \exp(ax) . \exp(ibx)$

$= e^{ax} . (\cos bx + i \sin bx)$;

$\therefore \exp\{(a-ib)x\} = e^{ax}(\cos bx - i \sin bx)$;

(ii) Put $a = \cos\theta$, $b = \sin\theta$; then $2ie^{x\cos\theta} . \sin(x \sin\theta)$

$= \exp\{x . \operatorname{cis}\theta\} - \exp\{x . \operatorname{cis}(-\theta)\}$

$= \sum \dfrac{x^n}{n!}\{\operatorname{cis}(n\theta) - \operatorname{cis}(-n\theta)\}$

$= \sum \dfrac{x^n}{n!}\{2i \sin n\theta\}$.

24. By eqn. (11), $2e^\theta . \cos\theta = \exp(\theta) . \{\exp(i\theta) + \exp(-i\theta)\}$

$= \exp(\theta + i\theta) + \exp(\theta - i\theta) = \exp\{\theta(1+i)\} + \exp\{\theta(1-i)\}$

$= \exp\left\{\theta\sqrt{2} . \operatorname{cis}\left(\dfrac{\pi}{4}\right)\right\} + \exp\left\{\theta\sqrt{2} . \operatorname{cis}\left(-\dfrac{\pi}{4}\right)\right\}$

$= \sum \dfrac{\theta^n . (\sqrt{2})^n}{n!}\left\{\operatorname{cis}\left(\dfrac{n\pi}{4}\right) + \operatorname{cis}\left(-\dfrac{n\pi}{4}\right)\right\}$

$= \sum \dfrac{\theta^n . 2^{\frac{1}{2}n}}{n!} . 2 \cos\dfrac{n\pi}{4}$.

25. As in No. 4,

l.h.s. $= e^{\cos\theta} . \operatorname{cis}(\sin\theta) - e^{-\cos\theta} . \operatorname{cis}(-\sin\theta)$

$= \cos(\sin\theta) . \{e^{\cos\theta} - e^{-\cos\theta}\} + i \sin(\sin\theta) . \{e^{\cos\theta} + e^{-\cos\theta}\}$.

26. $e^{x \sin a} . \{\cos(x \cos a) + i \sin(x \cos a)\}$

$= \exp(x \sin a) . \exp(ix \cos a) = \exp\{x(\sin a + i \cos a)\}$

$= \exp\left\{x . \operatorname{cis}\left(\dfrac{\pi}{2} - a\right)\right\} = 1 + \sum \dfrac{x^n}{n!} \operatorname{cis}\left(\dfrac{n\pi}{2} - na\right)$

$= 1 + \sum \dfrac{x^n}{n!} \cos\left(\dfrac{n\pi}{2} - na\right) + i \sum \dfrac{x^n}{n!} \sin\left(\dfrac{n\pi}{2} - na\right)$.

27. As on p. 191, since $|x| < 1$,

$$x \operatorname{cis} \theta + x^2 \operatorname{cis} 2\theta + \dots + x^n \operatorname{cis} n\theta + \dots$$

$$= \frac{x \operatorname{cis} \theta}{1 - x \operatorname{cis} \theta} = \frac{x \operatorname{cis} \theta\,[1 - x \operatorname{cis}(-\theta)]}{[1 - x \operatorname{cis} \theta]\,[1 - x \operatorname{cis}(-\theta)]}$$

$$= \frac{x \operatorname{cis} \theta - x^2}{1 - x[\operatorname{cis} \theta + \operatorname{cis}(-\theta)] + x^2} = \frac{x \cos \theta - x^2 + ix \sin \theta}{1 - 2x \cos \theta + x^2};$$

but $\Sigma x^n \operatorname{cis} n\theta = \Sigma x^n \cos n\theta + i \,.\, \Sigma x^n \sin n\theta$. **Or** use method of No. 28.

28. By eqn. (12),

$$2i \,.\, \text{sum} = \{x \exp(\alpha i) + x^2 \exp(\alpha i + \beta i) + \dots\} - \{x \exp(-\alpha i) + x^2 \exp(-\alpha i - \beta i) + \dots\}$$

$$= \frac{x \exp(\alpha i)}{1 - x \exp(\beta i)} - \frac{x \exp(-\alpha i)}{1 - x \exp(-\beta i)} \text{ since } |x| < 1,$$

$$= \frac{x\{\exp(\alpha i) - \exp(-\alpha i)\} - x^2\{\exp(\alpha - \beta)i - \exp(\beta - \alpha)i\}}{1 - x\{\exp(\beta i) + \exp(-\beta i)\} + x^2}$$

$$= \frac{x \,.\, 2i \sin \alpha - x^2 \,.\, 2i \sin(\alpha - \beta)}{1 - x \,.\, 2 \cos \beta + x^2}.$$

Or use method of No. 27.

29. $1 + \operatorname{cis} \theta + \dfrac{\operatorname{cis} 2\theta}{2!} + \dots = \exp(\operatorname{cis} \theta) =$, by No. 4,

$$e^{\cos \theta} \,.\, \{\cos(\sin \theta) + i \sin(\sin \theta)\};$$

equate "first parts" of these complex numbers.

30. $\operatorname{cis} \theta - \dfrac{\operatorname{cis} 2\theta}{2!} + \dfrac{\operatorname{cis} 3\theta}{3!} - \dots$

$$= 1 - \exp(-\operatorname{cis} \theta) = 1 - \exp(-\cos \theta - i \sin \theta)$$

$$= 1 - e^{-\cos \theta} \,.\, \{\cos(\sin \theta) - i \sin(\sin \theta)\};$$

equate "second parts."

31. $\operatorname{cis} \alpha - x \operatorname{cis}(\alpha + \beta) + \dfrac{x^2}{2!} \operatorname{cis}(\alpha + 2\beta) - \dots$

$$= \operatorname{cis} \alpha \,.\, \left\{1 - x \operatorname{cis} \beta + \frac{x^2}{2!} \operatorname{cis} 2\beta - \dots\right\} = \operatorname{cis} \alpha \,.\, \exp(-x \operatorname{cis} \beta)$$

$$= \exp(i\alpha) \,.\, \exp(-x \cos \beta - ix \sin \beta)$$

$$= e^{-x \cos \beta} \,.\, \exp\{i\alpha - ix \sin \beta\} = e^{-x \cos \beta} \,.\, \operatorname{cis}(\alpha - x \sin \beta);$$

equate "first parts."

32. $1 + \operatorname{cis} \theta \,.\, \tan \theta + \dfrac{1}{2!} \operatorname{cis} 2\theta \,.\, \tan^2 \theta + \dots = \exp(\operatorname{cis} \theta \,.\, \tan \theta) =$, as in No. 12, $e^{\sin \theta} \,.\, \operatorname{cis}(\sin \theta \tan \theta)$; equate "first parts."

33. $\operatorname{cis} \alpha + \cos \beta \,.\, \operatorname{cis}(\alpha + \beta) + \dfrac{\cos^2 \beta}{2!} \,.\, \operatorname{cis}(\alpha + 2\beta) + \dots$

$$= \operatorname{cis} \alpha \,.\, \left\{1 + \cos \beta \,.\, \operatorname{cis} \beta + \frac{\cos^2 \beta}{2!} \,.\, \operatorname{cis} 2\beta + \dots\right\}$$

$$= \operatorname{cis} \alpha \,.\, \exp(\cos \beta \,.\, \operatorname{cis} \beta)$$

$$= \exp(i\alpha) \,.\, \exp(\cos^2 \beta + i \cos \beta \sin \beta)$$

$$= e^{\cos^2 \beta} \,.\, \exp\{i\alpha + i \cos \sin \beta\}$$

$$= e^{\cos^2 \beta} \,.\, \operatorname{cis}(\alpha + \cos \beta \sin \beta);$$ equate "first parts."

34. $\operatorname{cis} \alpha + \dfrac{\operatorname{cis} 3\alpha}{3!} + \dfrac{\operatorname{cis} 5\alpha}{5!} + \dots = \frac{1}{2}\{\exp(\operatorname{cis} \alpha) - \exp(-\operatorname{cis} \alpha)\} =$, by No. 25, $\cos(\sin \alpha) \,.\, \operatorname{sh}(\cos \alpha) + i \,.\, \sin(\sin \alpha) \,.\, \operatorname{ch}(\cos \alpha)$; equate "first parts."

35. $C + iS = \dfrac{\operatorname{cis} 2\theta}{2!} + \dfrac{\operatorname{cis} 4\theta}{4!} + \dots = \frac{1}{2}\{\exp(\operatorname{cis} \theta) + \exp(-\operatorname{cis} \theta) - 2\}$

$$= \tfrac{1}{2}\{\exp(\tfrac{1}{2} \operatorname{cis} \theta) - \exp(-\tfrac{1}{2} \operatorname{cis} \theta)\}^2;$$

$$\therefore\; C - iS = \tfrac{1}{2}\{\exp[\tfrac{1}{2} \operatorname{cis}(-\theta)] - \exp[-\tfrac{1}{2} \operatorname{cis}(-\theta)]\}^2.$$

For brevity, put $\operatorname{cis} \theta = u$; $\therefore \operatorname{cis}(-\theta) = \dfrac{1}{u}$; also

$$u + \frac{1}{u} = 2 \cos \theta,\quad u - \frac{1}{u} = 2i \sin \theta.$$

Then $C^2 + S^2 = (C + iS)(C - iS) = \frac{1}{4}K^2$ where

$$K \equiv \left\{\exp\left(\frac{u}{2}\right) - \exp\left(-\frac{u}{2}\right)\right\}\left\{\exp\left(\frac{1}{2u}\right) - \exp\left(-\frac{1}{2u}\right)\right\}$$

$$= \exp\left(\frac{u}{2} + \frac{1}{2u}\right) + \exp\left(-\frac{u}{2} - \frac{1}{2u}\right) - \exp\left(\frac{u}{2} - \frac{1}{2u}\right) - \exp\left(-\frac{u}{2} + \frac{1}{2u}\right)$$

$$= \{\exp(\cos \theta) + \exp(-\cos \theta)\} - \{\exp(i \sin \theta) + \exp(-i \sin \theta)\}$$

=, using eqn. (11), $2 \operatorname{ch}(\cos \theta) - 2 \cos(\sin \theta)$.

EXERCISE X. b. (p. 200.)

1. $2 \cos \dfrac{\pi i}{2} = \exp\left(\dfrac{\pi i}{2} \,.\, i\right) + \exp\left(-\dfrac{\pi i}{2} \,.\, i\right) = e^{-\frac{\pi}{2}} + e^{\frac{\pi}{2}}$;

$$2i \sin \frac{\pi i}{2} = \exp\left(-\frac{\pi}{2}\right) - \exp\left(\frac{\pi}{2}\right).$$

2. $\sin 2z = \frac{1}{2i}\{\exp(2zi) - \exp(-2zi)\}$

$= \frac{1}{2i}\{\exp(zi) - \exp(-zi)\}\{\exp(zi) + \exp(-zi)\}$

$= \frac{1}{2i}\{2i \sin z\}\{2\cos z\}$; from eqn. (15), $\cos^2 z + \sin^2 z$

$= (\cos z + i \sin z)(\cos z - i \sin z)$

$= \exp(iz) . \exp(-iz) = \exp(0) = 1$.

3. $\text{ch}(zi) = \frac{1}{2}\{\exp(zi) + \exp(-zi)\} = \frac{1}{2}(2\cos z)$ by eqn. (13);
$\text{sh}(zi) = \frac{1}{2}\{\exp(zi) - \exp(-zi)\} = \frac{1}{2}(2i \sin z)$ by eqn. (14).

4. $\text{ch}(A + B) = \cos(Ai + Bi) = \cos(Ai) . \cos(Bi) - \sin(Ai) . \sin(Bi)$
$= \text{ch}\, A . \text{ch}\, B - i^2 \text{sh}\, A . \text{sh}\, B$;

$$\text{ch}\, C - \text{ch}\, D = \cos(Ci) - \cos Di = -2\sin\frac{Ci + Di}{2} . \sin\frac{Ci - Di}{2}$$

$$= -2i^2 \text{sh}\frac{C + D}{2} . \text{sh}\frac{C - D}{2}.$$

5. $\sin x \cos(iy) - \cos x \sin(iy)$; use (24), (25).

6. $\frac{1}{2}\{1 + \cos(2x + 2iy)\} = \frac{1}{2}\{1 + \cos 2x \cos(2iy) - \sin 2x \sin(2iy)\}$; use (24), (25).

7. $\dfrac{\cos(x + iy)\sin(x - iy)}{\sin(x + iy)\sin(x - iy)} = \dfrac{\sin 2x - \sin(2iy)}{\cos(2iy) - \cos 2x}$; use (24), (25).

8. $\text{ch}\, x\, \text{ch}(iy) + \text{sh}\, x\, \text{sh}(iy)$; use No. 3.

9. $\dfrac{\text{sh}(x - iy)\,\text{ch}(x + iy)}{\text{ch}(x - iy)\,\text{ch}(x + iy)} = \dfrac{\text{sh}\, 2x - \text{sh}(2yi)}{\text{ch}\, 2x + \text{ch}(2yi)}$; use No. 3.

10. $\dfrac{\sin(x - iy)}{\sin(x + iy) . \sin(x - iy)} = \dfrac{\sin x\, \text{ch}\, y - i \cos x\, \text{sh}\, y}{\frac{1}{2}[\cos(2iy) - \cos 2x]}$, by No. 5; use eqn. (24).

11. By eqn. (28), $\exp(\sin x\, \text{ch}\, y + i \cos x\, \text{sh}\, y)$; use eqn. (10).

12. $\text{sh}(x - iy) = \text{sh}\, x\, \text{ch}(iy) - \text{ch}\, x\, \text{sh}(iy) = \text{sh}\, x \cos y - i\, \text{ch}\, x \sin y$, by No. 3; use eqn. (10).

13. $\text{sh}(x - iy)\cos(y + ix) = \text{sh}(x - iy)\,\text{ch}(iy + i^2 x)$
$= \text{sh}(x - iy)\,\text{ch}(iy - x) = \text{sh}(x - iy)\,\text{ch}(x - iy)$
$= \frac{1}{2}\text{sh}(2x - 2iy)$
$=$, as in No. 12, $\frac{1}{2}\text{sh}\, 2x \cos 2y - \frac{1}{2}i\, \text{ch}\, 2x \sin 2y$.

14. (i) $\text{ch}(x + \frac{1}{2}\pi i) = \cos(ix + \frac{1}{2}\pi i^2) = \cos\left(ix - \frac{\pi}{2}\right) = \sin(ix) = i\, \text{sh}\, x$;

(ii) $\text{sh}(x + \frac{1}{2}\pi i) = \frac{1}{i}\sin\left(ix - \frac{\pi}{2}\right) = -\frac{1}{i}\cos(ix) = i\, \text{ch}\, x$;

(iii) From (i) and (ii), $i\, \text{ch}\, x \div i\, \text{sh}\, x$;

(iv) As in (i), $\cos(ix - \pi) = -\cos(ix) = -\text{ch}\, x$;

(v) As in (ii), $\frac{1}{i}\sin(ix - \pi) = -\frac{1}{i}\sin(ix) = -\text{sh}\, x$;

(vi) From (iv) and (v), $-\text{sh}\, x \div -\text{ch}\, x$.

15. $\dfrac{\cos^2(x + yi) + \cos^2(x - yi)}{\cos(x - yi) . \cos(x + yi)} = \dfrac{1 + \cos(2x + 2yi) + 1 + \cos(2x - 2yi)}{\cos 2x + \cos(2yi)}$

$$= \frac{2 + 2\cos 2x \cos(2yi)}{\cos 2x + \text{ch}\, 2y}.$$

16. By eqn. (28), $u = \sin x . \text{ch}\, y$, $v = \cos x . \text{sh}\, y$;

(i) $\left(\dfrac{u}{\sin x}\right)^2 - \left(\dfrac{v}{\cos x}\right)^2 = \text{ch}^2 y - \text{sh}^2 y = 1$;

(ii) $\left(\dfrac{u}{\text{ch}\, y}\right)^2 + \left(\dfrac{v}{\text{sh}\, y}\right)^2 = \sin^2 x + \cos^2 x = 1$.

17. $\cos(x - iy) = \cos\theta - i\sin\theta$;

$\therefore \cos 2x + \text{ch}\, 2y = \cos 2x + \cos(2yi)$
$= 2\cos(x + yi) . \cos(x - yi)$
$= 2(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 2$.

18. $\tan(x - iy) = u - iv$;

$$\therefore \tan 2x = \tan\{(x + iy) + (x - iy)\} = \frac{(u + iv) + (u - iv)}{1 - (u + iv)(u - iv)};$$

$$\text{th}\, 2y = \frac{1}{i}\tan(2yi) = -i\tan\{(x + iy) - (x - iy)\}$$

$$= -i . \frac{(u + iv) - (u - iv)}{1 + (u + iv)(u - iv)}.$$

19. By No. 8 and eqn. (27),

$(\text{ch}\, x \cos y + i\, \text{sh}\, x \sin y)(\cos u\, \text{ch}\, v - i \sin u\, \text{sh}\, v) = 1$;

equate "second parts,"

$\text{sh}\, x \sin y \cos u\, \text{ch}\, v - \text{ch}\, x \cos y \sin u\, \text{sh}\, v = 0$;

also $\text{ch}\, x \neq 0$, since $\text{ch}\, x \geqslant 1$, and $\text{ch}\, v \neq 0$; divide by $\cos y \cos u\, \text{ch}\, x\, \text{ch}\, v$.

20. $\sin(x - iy) = \tan(u - iv)$;

$$\therefore \frac{\sin(x + iy) + \sin(x - iy)}{\sin(x + iy) - \sin(x - iy)} = \frac{\tan(u + iv) + \tan(u - iv)}{\tan(u + iv) - \tan(u - iv)};$$

$$\therefore \frac{2\sin x \cos(iy)}{2\cos x \sin(iy)} = \frac{\sin\{(u + iv) + (u - iv)\}}{\sin\{(u + iv) - (u - iv)\}};$$

$$\therefore \frac{\tan x}{i\, \text{th}\, y} = \frac{\sin 2u}{\sin(2iv)} = \frac{\sin 2u}{i\, \text{sh}(2v)}.$$

21. By No. 8, $x+yi=c\,\{\text{ch}\,\alpha\cos\beta+i\,\text{sh}\,\alpha\sin\beta\}$;

$\therefore\ x=c\,\text{ch}\,\alpha\cos\beta,\ y=c\,\text{sh}\,\alpha\sin\beta$;

(i) For α constant, these are parametric eqns. to an ellipse, semi-axes $c\,\text{ch}\,\alpha$, $c\,\text{sh}\,\alpha$, eccentric angle β;

(ii) For β constant, $\left(\dfrac{x}{c\cos\beta}\right)^2-\left(\dfrac{y}{c\sin\beta}\right)^2=\text{ch}^2\alpha-\text{sh}^2\alpha=1$, they give a hyperbola, semi-axes $c\cos\beta$, $c\sin\beta$; the conics are confocal because

$$c^2\,\text{ch}^2\alpha-c^2\,\text{sh}^2\alpha=c^2=c^2\cos^2\beta+c^2\sin^2\beta;$$

$\therefore$ they cut orthogonally. **Or**, see No. 23.

22. $x+yi=\alpha^2-\beta^2+2i\alpha\beta$; $\therefore\ x=\alpha^2-\beta^2,\ y=2\alpha\beta$; for α constant, $\beta=\dfrac{y}{2\alpha}$; $\therefore\ x=\alpha^2-\dfrac{y^2}{4\alpha^2}$; $\therefore\ y^2=-4\alpha^2(x-\alpha^2)$; similarly for β constant, eliminate α, $y^2=4\beta^2(x+\beta^2)$; these are parabolas with origin as focus.

23. Put $w=\alpha+i\beta$; $\dfrac{\partial x}{\partial\alpha}+i\dfrac{\partial y}{\partial\alpha}=f'(w)\cdot\dfrac{\partial w}{\partial\alpha}=f'(w)$;

$$\frac{\partial x}{\partial\beta}+i\frac{\partial y}{\partial\beta}=f'(w)\cdot\frac{\partial w}{\partial\beta}$$
$$=f'(w)\cdot i=i\left(\frac{\partial x}{\partial\alpha}+i\frac{\partial y}{\partial\alpha}\right)=-\frac{\partial y}{\partial\alpha}+i\frac{\partial x}{\partial\alpha};$$

$$\therefore\ \frac{\partial x}{\partial\beta}=-\frac{\partial y}{\partial\alpha}\ \text{and}\ \frac{\partial y}{\partial\beta}=\frac{\partial x}{\partial\alpha};\quad\therefore\ \frac{\dfrac{\partial y}{\partial\beta}}{\dfrac{\partial x}{\partial\beta}}\cdot\frac{\dfrac{\partial y}{\partial\alpha}}{\dfrac{\partial x}{\partial\alpha}}=-1;$$

$$\therefore\ \left\{\frac{dy}{dx}\ (\text{for }\alpha\text{ constant})\right\}\cdot\left\{\frac{dy}{dx}\ (\text{for }\beta\text{ constant})\right\}=-1.$$

24. By eqn. (12), $2i\,(\text{sum})=$

$$\left\{\frac{\exp(\theta i)}{1!}-\frac{\exp(3\theta i)}{3!}+\dots\right\}-\left\{\frac{\exp(-\theta i)}{1!}-\frac{\exp(-3\theta i)}{3!}+\dots\right\}$$

$=$, by eqn. (17), $\sin\{\exp(\theta i)\}-\sin\{\exp(-\theta i)\}$

$=2\cos(\text{semi-sum})\,.\,\sin(\text{semi-diff.})=$, by eqns. (11), (12), $2\cos(\cos\theta)\,.\,\sin(i\sin\theta)$

$=$, by eqn. (25), $2i\cos(\cos\theta)\,.\,\text{sh}\,(\sin\theta)$.

25. As in No. 24, $2\,(\text{sum})=$

$$\left\{\frac{\exp(\theta i)}{2!}-\frac{\exp(2\theta i)}{4!}+\dots\right\}+\left\{\frac{\exp(-\theta i)}{2!}-\frac{\exp(-2\theta i)}{4!}+\dots\right\}$$

$=\{1-\cos[\exp(\tfrac12\theta i)]\}+\{1-\cos[\exp(-\tfrac12\theta i)]\}$

$=2-2\cos(\text{semi-sum})\,.\,\cos(\text{semi-diff.})$

$=2-2\cos(\cos\tfrac12\theta)\,.\,\cos(i\sin\tfrac12\theta)=2-2\cos(\cos\tfrac12\theta)\,.\,\text{ch}\,(\sin\tfrac12\theta)$.

26. As in No. 24,

$$2i\,(\text{sum})=\left\{\frac{\exp(2\theta i)}{2!}+\frac{\exp(4\theta i)}{4!}+\dots\right\}-\left\{\frac{\exp(-2\theta i)}{2!}+\dots\right\}$$

$=\{-1+\text{ch}\,[\exp(\theta i)]\}-\{-1+\text{ch}\,[\exp(-\theta i)]\}$

$=2\,\text{sh}\,(\text{semi-sum})\,.\,\text{sh}\,(\text{semi-diff.})$

$=2\,\text{sh}\,(\cos\theta)\,.\,\text{sh}\,(i\sin\theta)$; use No. 3.

27. $2i\dfrac{\sin^n\theta}{n!}\sin n\theta=\dfrac{\sin^n\theta}{n!}\{\exp(n\theta i)-\exp(-n\theta i)\}$

$$=\frac{\{\sin\theta\,.\,\exp(\theta i)\}^n}{n!}-\frac{\{\sin\theta\,.\,\exp(-\theta i)\}^n}{n!};$$

$\therefore$ by eqn. (23),

$2i\,.\,(\text{sum})=\text{sh}\{\sin\theta\,.\,\exp(\theta i)\}-\text{sh}\{\sin\theta\,.\,\exp(-\theta i)\}$

$=2\,\text{ch}\,(\text{semi-sum})\,.\,\text{sh}\,(\text{semi-diff.})$

$=2\,\text{ch}\,(\sin\theta\,.\,\cos\theta)\,.\,\text{sh}\,(\sin\theta\,.\,i\sin\theta)$;

use No. 3.

28. $1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\dots=\text{ch}\,x$, $1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dots=\cos x$; add;

$$\therefore\ 1+\frac{x^4}{4!}+\dots=\tfrac12(\text{ch}\,x+\cos x);$$

first put $x=\exp(\theta i)$, then put $x=\exp(-\theta i)$ and add; then

$4(\text{sum})=\text{ch}\{\exp(\theta i)\}+\cos\{\exp(\theta i)\}$
$\qquad+\text{ch}\{\exp(-\theta i)\}+\cos\{\exp(-\theta i)\}$

$=2\,\text{ch}\,(\text{semi-sum})\,\text{ch}\,(\text{semi-diff.})$
$\qquad+2\cos(\text{semi-sum})\,.\,\cos(\text{semi-diff.})$

$=2\,\text{ch}\,(\cos\theta)\,.\,\text{ch}\,(i\sin\theta)+2\cos(\cos\theta)\,.\,\cos(i\sin\theta)$;

use No. 3 and eqn. (24).

29. $2i\,.\,\dfrac{\sin nz}{n!}=\dfrac{\exp(nzi)-\exp(-nzi)}{n!}=\dfrac{\{\exp(zi)\}^n}{n!}-\dfrac{\{\exp(-zi)\}^n}{n!}$;

$\therefore\ 2i\,.\,(\text{sum})=\exp\{\exp(zi)\}-\exp\{\exp(-zi)\}=$, by eqn. (15), $\exp\{\cos z+i\sin z\}-\exp\{\cos z-i\sin z\}=$, by eqn. (7), $\exp(\cos z)\,.\,\{\exp(i\sin z)-\exp(-i\sin z)\}=$, by eqn. (14), $\exp(\cos z)\,.\,2i\sin(\sin z)$.

30. (i) $\cos z\,.\,\text{ch}\,z=\text{ch}(iz)\,.\,\text{ch}\,z=\tfrac12\{\text{ch}(1+i)z+\text{ch}\,(1-i)z\}=$, by eqn. (22), $1+\tfrac12\sum\dfrac{z^{2n}}{(2n)!}\{(1+i)^{2n}+(1-i)^{2n}\}$; also

$$(1+i)^{2n}+(1-i)^{2n}=\left(\sqrt2\,.\,\text{cis}\frac{\pi}{4}\right)^{2n}+\left(\sqrt2\,\text{cis}\frac{-\pi}{4}\right)^{2n}$$
$$=2^n\left\{\text{cis}\frac{n\pi}{2}+\text{cis}\frac{-n\pi}{2}\right\}=2^n\,.\,2\cos\frac{n\pi}{2};$$

this $=0$ if n is odd and $=2^{2p} \cdot 2(-1)^p$ if $n=2p$;

$$\therefore \text{ series} = 1+\sum \frac{z^{4p}}{(4p)!}\cdot(-1)^p \cdot 2^{2p};$$

(ii) $\sin z \cdot \text{ch}\, z = -i\,\text{sh}\, zi \cdot \text{ch}\, z = -\tfrac{1}{2}i\{\text{sh}\,(1+i)z - \text{sh}\,(1-i)z\}$

$$=, \text{ by eqn. (23)}, -\tfrac{1}{2}i\sum \frac{z^{2n-1}}{(2n-1)!}\{(1+i)^{2n-1}-(1-i)^{2n-1}\};$$

$\therefore$ coefficient of $z^{2n-1}=$, as in (i),

$$-\tfrac{1}{2}i \cdot \frac{1}{(2n-1)!}\cdot(\sqrt{2})^{2n-1}\cdot 2i\sin\frac{(2n-1)\pi}{4}$$

$$=\frac{2^n}{(2n-1)!}\cdot\frac{1}{\sqrt{2}}\sin\left(\frac{n\pi}{2}-\frac{\pi}{4}\right)$$

$$=\frac{2^n}{(2n-1)!}\cdot\left\{\tfrac{1}{2}\sin\frac{n\pi}{2}-\tfrac{1}{2}\cos\frac{n\pi}{2}\right\}.$$

EXERCISE X. c. (p. 202.)

1. $\text{cis}\,(2n+1)\dfrac{\pi}{2}=i\sin\left(n\pi+\dfrac{\pi}{2}\right)$.

2. $\exp\{(x^2-y^2)+i \cdot 2xy\}$; use eqn. (10).

3. $\dfrac{2\sin\frac{1}{2}(x+iy)\cos\frac{1}{2}(x-iy)}{2\cos\frac{1}{2}(x+iy)\cos\frac{1}{2}(x-iy)}=\dfrac{\sin x+\sin iy}{\cos x+\cos iy}$; use eqns. (25), (24).

4. $\dfrac{2\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$

$$=, \text{ by eqn. (27)}, \frac{2\{\cos x\,\text{ch}\, y+i\sin x\,\text{sh}\, y\}}{\cos 2x+\cos(2iy)}.$$

5. Write $-y$ for y in X. b, No. 10, or put $\dfrac{\pi}{2}-x$ for x in No. 4.

6. $\dfrac{2\,\text{sh}\,(x+iy)}{2\,\text{sh}\,(x-iy)\,\text{sh}\,(x+iy)}=\dfrac{2\{\text{sh}\, x\,\text{ch}\,(iy)+\text{ch}\, x\,\text{sh}\,(iy)\}}{\text{ch}\, 2x-\text{ch}(2iy)}$; and use X. b, No. 3.

7. $1=(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)$

$$=\sin(\alpha+i\beta)\cdot\sin(\alpha-i\beta)=\tfrac{1}{2}(\cos 2i\beta-\cos 2\alpha);$$

$\therefore\ 1+\cos 2\alpha=\text{ch}\, 2\beta-1$; $\therefore\ 2\cos^2\alpha=2\,\text{sh}^2\beta$; by eqn. (28).

$\cos\theta+i\sin\theta=\sin\alpha\cdot\text{ch}\,\beta+i\cos\alpha\,\text{sh}\,\beta$;

$\therefore\ \sin\theta=\cos\alpha\cdot\text{sh}\,\beta=\cos\alpha\cdot(\pm\cos\alpha)$.

8. By No. 3, $u+iv=\dfrac{\sin x+i\,\text{sh}\, y}{\cos x+\text{ch}\, y}$;

$$\therefore\ u=\frac{\sin x}{\cos x+\text{ch}\, y},\quad v=\frac{\text{sh}\, y}{\cos x+\text{ch}\, y};$$

divide for $\dfrac{u}{v}$. Also

$$u^2+v^2=(u+iv)(u-iv)=\tan\tfrac{1}{2}(x+iy)\cdot\tan\tfrac{1}{2}(x-iy)$$

$$=\frac{2\sin\frac{1}{2}(x+iy)\cdot\sin\frac{1}{2}(x-iy)}{2\cos\frac{1}{2}(x+iy)\cdot\cos\frac{1}{2}(x-iy)}=\frac{\cos iy-\cos x}{\cos iy+\cos x}$$

$$=\frac{\text{ch}\, y-\cos x}{\text{ch}\, y+\cos x};\quad\therefore\quad\frac{1-(u^2+v^2)}{1+(u^2+v^2)}$$

$$=\frac{(\text{ch}\, y+\cos x)-(\text{ch}\, y-\cos x)}{(\text{ch}\, y+\cos x)+(\text{ch}\, y-\cos x)}.$$

9. $\text{th}\, x=\dfrac{\sin a}{\cos(ib)}$, $\text{th}\, yi=i\tan y=\dfrac{i\,\text{sh}\, b}{\cos a}=\dfrac{\sin(ib)}{\cos a}$;

$$\therefore\ \text{th}\,(x+yi)=\frac{\text{th}\, x+\text{th}\,(yi)}{1+\text{th}\, x\,\text{th}\,(yi)}=\frac{\sin a\cos a+\sin(ib)\cos(ib)}{\cos a\cos(ib)+\sin a\sin(ib)}$$

$$=\frac{\frac{1}{2}[\sin 2a+\sin 2ib]}{\cos(a-ib)}=\sin(a+ib);$$

$$\therefore\ \text{sech}^2(x+yi)=1-\text{th}^2(x+yi)=1-\sin^2(a+ib)$$

$$=\cos^2(a+ib).$$

10. $a\cos\theta+b\cos 3\theta=c$, $a\sin\theta-b\sin 3\theta=0$;

$\therefore\ \sin\theta=0$ or $a=b(3-4\sin^2\theta)$;

If $\sin\theta=0$, $\cos\theta=\cos 3\theta=\pm 1$; $\therefore\ a+b=\pm c$.

If $a=b(3-4\sin^2\theta)$,

$$c=a\cos\theta+b\cos 3\theta=\cos\theta\{a+b(4\cos^2\theta-3)\}$$

$$=\cos\theta\{2a-b(3-4\sin^2\theta)+b(4\cos^2\theta-3)\}=\cos\theta\{2a-2b\};$$

$$\therefore\ a+b=b(3-4\sin^2\theta)+b=4b\cos^2\theta$$

$$=b(2\cos\theta)^2=b\cdot\left(\frac{c}{a-b}\right)^2.$$

11. $xe^\alpha+x^2e^{2\alpha}+\ldots=\dfrac{xe^\alpha}{1-xe^\alpha}$ if $|xe^\alpha|<1$; write $-\alpha$ for α and subtract;

$$2\,(\text{sum})=\frac{xe^\alpha}{1-xe^\alpha}-\frac{xe^{-\alpha}}{1-xe^{-\alpha}}$$

$$=\frac{(xe^\alpha-x^2)-(xe^{-\alpha}-x^2)}{1-x(e^\alpha+e^{-\alpha})+x^2}=\frac{x(e^\alpha-e^{-\alpha})}{1-2x\,\text{ch}\,\alpha+x^2}.$$

12. $2\,(\text{sum})=2-\{x\exp(\alpha i)-x^2\exp(\alpha i+\beta i)+\ldots\}$

$$-\{x\exp(-\alpha i)-x^2\exp(-\alpha i-\beta i)+\ldots\}$$

$$=2-\frac{x\exp(\alpha i)}{1+x\exp(\beta i)}-\frac{x\exp(-\alpha i)}{1+x\exp(-\beta i)},\ \text{if } |x|<1,$$

$$=2-\frac{x\{\exp(\alpha i)+\exp(-\alpha i)\}+x^2\{\exp[(\alpha-\beta)i]+\exp[-(\alpha-\beta)i]\}}{1+x\{\exp(\beta i)+\exp(-\beta i)\}+x^2};$$

13. In X. a, No. 31, write $\alpha - \frac{\pi}{2}$ for α and $\beta + \pi$ for β.

14. In No. 13, put $\alpha = 0$, $\beta = \theta$.

15. $C + iS \equiv 1 + \frac{\text{cis } 2\theta}{2!} + \frac{\text{cis } 4\theta}{4!} + \ldots =$, by eqn. (22), $\text{ch}(\text{cis } \theta)$

$= \text{ch}(\cos\theta + i\sin\theta)$

$= \text{ch}(\cos\theta) \,.\, \text{ch}(i\sin\theta) + \text{sh}(\cos\theta) \,.\, \text{sh}(i\sin\theta)$

$=$, by X. b, No. 3, $\text{ch}(\cos\theta) \,.\, \cos(\sin\theta) + i\,\text{sh}(\cos\theta) \,.\, \sin(\sin\theta)$.

Or $2C = (C + iS) + (C - iS)$

$= \text{ch}(\cos\theta + i\sin\theta) + \text{ch}(\cos\theta - i\sin\theta)$

$= 2\,\text{ch}(\text{semi-sum}) \,.\, \text{ch}(\text{semi-diff.})$.

16. As in No. 15, $C + iS = \frac{\text{cis } \theta}{1!} + \frac{\text{cis } 3\theta}{3!} + \ldots$

$= \text{sh}(\text{cis } \theta) = \text{sh}(\cos\theta + i\sin\theta)$

$= \text{sh}(\cos\theta) \,.\, \text{ch}(i\sin\theta) + \text{ch}(\cos\theta) \,.\, \text{sh}(i\sin\theta)$;

use X. b, No. 3.

17. If $w^3 = 1$, $w \neq 1$, then $1 + w + w^2 = 0$;

$\therefore \exp(x) + \exp(wx) + \exp(w^2x)$

$$= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + \left(1 + \frac{wx}{1!} + \frac{w^2x^2}{2!} + \frac{w^3x^3}{3!} + \ldots\right)$$

$$+ \left(1 + \frac{w^2x}{1!} + \frac{w^4x^2}{2!} + \frac{w^6x^3}{3!} + \ldots\right) = 3\left(1 + \frac{x^3}{3!} + \ldots\right);$$

also $\exp(wx) = \exp\left(x\cos\frac{2\pi}{3} + ix\sin\frac{2\pi}{3}\right)$

$$= \exp\left(x\cos\frac{2\pi}{3}\right) \,.\, \text{cis}\left(x\sin\frac{2\pi}{3}\right)$$

$$= e^{-\frac{1}{2}x} \,.\, \text{cis}\left(\frac{x\sqrt{3}}{2}\right);$$

similarly $\exp(w^2x) = e^{-\frac{1}{2}x} \,.\, \text{cis}\left(-\frac{x\sqrt{3}}{2}\right)$.

18. By eqns. (23), (17), $\frac{1}{2}(\text{sh } z - \sin z) = \frac{z^3}{3!} + \frac{z^7}{7!} + \frac{z^{11}}{11!} + \ldots$; put $z = x\,\text{cis}\frac{\pi}{4}$, then

$$\tfrac{1}{2}\left\{\text{sh}\frac{x(1+i)}{\sqrt{2}} - \sin\frac{x(1+i)}{\sqrt{2}}\right\} = \text{cis}\frac{3\pi}{4}\left\{\frac{x^3}{3!} - \frac{x^7}{7!} + \frac{x^{11}}{11!} - \ldots\right\};$$

write $-i$ for i and add;

$$\text{cis}\frac{3\pi}{4} + \text{cis}\left(-\frac{3\pi}{4}\right) = 2\cos\frac{3\pi}{4} = -\sqrt{2};$$

$$\therefore -\sqrt{2} \,.\, \left\{\frac{x^3}{3!} - \frac{x^7}{7!} + \ldots\right\}$$

$$= \tfrac{1}{2}\left\{\text{sh}\frac{x(1+i)}{\sqrt{2}} + \text{sh}\frac{x(1-i)}{\sqrt{2}}\right\} - \tfrac{1}{2}\left\{\sin\frac{x(1+i)}{\sqrt{2}} + \sin\frac{x(1-i)}{\sqrt{2}}\right\}$$

$$= \text{sh}\frac{x}{\sqrt{2}}\,\text{ch}\frac{xi}{\sqrt{2}} - \sin\frac{x}{\sqrt{2}}\cos\frac{xi}{\sqrt{2}}$$

$=$, by X. b, No. 3, $\text{sh}\frac{x}{\sqrt{2}}\cos\frac{x}{\sqrt{2}} - \sin\frac{x}{\sqrt{2}}\,\text{ch}\frac{x}{\sqrt{2}}$.

19. $e^{ax}\,\text{cis } bx = \exp(ax) \,.\, \exp(ibx)$

$$= \exp(ax + ibx) = 1 + \sum\frac{x^n(a+ib)^n}{n!};$$

put $a = r\cos\theta$, $b = r\sin\theta$, so that

$$\cos\theta : \sin\theta : 1 = a : b : \sqrt{(a^2 + b^2)};$$

then $(a+ib)^n = r^n(\cos\theta + i\sin\theta)^n = r^n\,\text{cis } n\theta$; $\therefore$ coefficient of x^n in $e^{ax} \,.\, \cos bx$ is $\frac{r^n\cos n\theta}{n!}$.

20. $|\exp z - 1| = |\exp x \,.\, \text{cis } y - 1| = |e^x\cos y - 1 + ie^x\sin y|$

$$= +\sqrt{\{(e^x\cos y - 1)^2 + (e^x\sin y)^2\}}$$

$$= +\sqrt{(1 - 2e^x\cos y + e^{2x})};$$

similarly $|\exp z + 1| = \sqrt{(1 + 2e^x\cos y + e^{2x})}$;

$\therefore$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$, $|\exp z - 1| < |\exp z + 1|$;

$\therefore$ $|Z| < 1$, i.e. $X^2 + Y^2 < 1$.

21. Solving, $\alpha = \frac{\pi}{2}(1+i)$, $\beta = \frac{\pi}{2}(1-i)$, $\alpha + \beta = \pi$;

$$\sin\alpha\sin\beta = \sin^2\alpha = \sin^2\left(\frac{\pi}{2} + \frac{\pi i}{2}\right) = \cos^2\left(\frac{\pi i}{2}\right) = \text{ch}^2\left(\frac{\pi}{2}\right).$$

22. $\exp\{\exp(\theta i)\} = \exp(\cos\theta + i\sin\theta) = e^{\cos\theta} \,.\, \text{cis}(\sin\theta)$;

$\exp\{-\exp(-\theta i)\} = \exp(-\cos\theta + i\sin\theta) = e^{-\cos\theta} \,.\, \text{cis}(\sin\theta)$;

expression $= \text{cis}(\sin\theta) \,.\, \{e^{\cos\theta} - e^{-\cos\theta}\}$.

23. $e^{x\cos\beta} \,.\, \text{cis}(\alpha + x\sin\beta) = \exp(x\cos\beta) \,.\, \exp(\alpha i + xi\sin\beta)$

$= \exp(x\cos\beta + \alpha i + xi\sin\beta) = \exp(\alpha i) \,.\, \exp(x\,\text{cis}\,\beta)$

$$= \text{cis}\,\alpha \,.\, \left\{1 + \sum\frac{x^n\,\text{cis}\,n\beta}{n!}\right\} = \text{cis}\,\alpha + \sum\frac{x^n}{n!} \,.\, \text{cis}(\alpha + n\beta).$$

24. $1+2z+3z^2+4z^3+\dots$ to n terms $=\dfrac{1}{(1-z)^2}-\dfrac{z^n}{(1-z)^2}-\dfrac{nz^n}{1-z}$; to prove this, multiply the series by $(1-z)^2$. Put $z=\sin\theta \,.\, \text{cis}\,\theta$, then $|z|<1$ since $\theta \neq k\pi+\dfrac{\pi}{2}$;

$$\therefore\ z^n \to 0 \text{ when } n\to\infty;$$

$$\therefore\ \text{sum to infinity} = (1-\sin\theta\,\text{cis}\,\theta)^{-2}$$

$$=(1-\sin\theta\cos\theta-i\sin^2\theta)^{-2}$$

$$=(1-\sin\theta\cos\theta+i\sin^2\theta)^2/\{(1-\sin\theta\cos\theta)^2+(\sin^2\theta)^2\}^2;$$

equate "second parts";

$$\therefore\ 2\sin\theta\,.\,\sin\theta+3\sin^2\theta\,.\,\sin 2\theta+\dots=\sin\theta\,.\,\{\text{given series}\}$$

$$=2\sin^2\theta(1-\sin\theta\cos\theta)/\{1-2\sin\theta\cos\theta+\sin^2\theta(\cos^2\theta+\sin^2\theta)\}^2.$$

25. $C+iS \equiv \text{cis}\,\alpha+\dfrac{\text{cis}\,(\alpha+2\beta)}{3!}+\dots$

$$=\text{cis}\,(\alpha-\beta)\,.\left\{\text{cis}\,\beta+\frac{\text{cis}\,3\beta}{3!}+\dots\right\}$$

$=$, by eqn. (23), $\text{cis}\,(\alpha-\beta)\,.\,\text{sh}\,(\text{cis}\,\beta)$

$$=\text{cis}\,(\alpha-\beta)\,.\,\text{sh}\,(\cos\beta+i\sin\beta)$$

$$=\text{cis}\,(\alpha-\beta)\times\{\text{sh}\,(\cos\beta)\,.\,\text{ch}\,(i\sin\beta)+\text{ch}\,(\cos\beta)\,.\,\text{sh}(i\sin\beta)\}$$

$=$, by X. b, No. 3, $\{\cos(\alpha-\beta)+i\sin(\alpha-\beta)\}\times\{\text{sh}(\cos\beta)\,.\,\cos(\sin\beta)+i\,\text{ch}(\cos\beta)\,.\,\sin(\sin\beta)\}$;

equate "first parts."

26. $C+iS=\exp\{e^{\cos\theta}\,.\,\text{cis}\,(\sin\theta)\}$

$$=\exp\{e^{\cos\theta}\,.\,[\cos(\sin\theta)+i\sin(\sin\theta)]\}$$

$$=\exp\{e^{\cos\theta}\,.\,\cos(\sin\theta)\}\,.\,\exp\{ie^{\cos\theta}\,.\,\sin(\sin\theta)\};$$

use eqn. (10) and equate "first parts."

27. By eqns. (23), (17), $\tfrac{1}{2}(\text{sh}\,z+\sin z)=\dfrac{z}{1!}+\dfrac{z^5}{5!}+\dots$; put $z=\text{cis}\,\theta$, then put $z=\text{cis}\,(-\theta)$ and subtract; then $4i$ (sum) $=$, as in No. 18,

$$\text{sh}\,(\cos\theta+i\sin\theta)-\text{sh}\,(\cos\theta-i\sin\theta)+\sin(\cos\theta+i\sin\theta)-\sin(\cos\theta-i\sin\theta)$$

$$=2\,\text{ch}\,(\cos\theta)\,.\,\text{sh}\,(i\sin\theta)+2\cos(\cos\theta)\sin(i\sin\theta);$$

use X. b, No. 3.

28. By eqns. (23), (17), $\tfrac{1}{2}(\text{sh}\,z-\sin z)=\dfrac{z^3}{3!}+\dfrac{z^7}{7!}+\dots$; put $z=\text{cis}\,\theta$, then put $z=\text{cis}\,(-\theta)$ and add; then 4(sum) $=$, as in No. 27,

$$\text{sh}\,(\cos\theta+i\sin\theta)+\text{sh}\,(\cos\theta-i\sin\theta)-\sin(\cos\theta+i\sin\theta)-\sin(\cos\theta-i\sin\theta)$$

$$=2\,\text{sh}\,(\cos\theta)\,.\,\text{ch}\,(i\sin\theta)-2\sin(\cos\theta)\,.\,\cos(i\sin\theta);$$

use X. b, No. 3.

29. See No. 17, and compare p. 161, Ex. VIII. h, No. 31.

$$3(\text{sum})=\exp x+w^2\exp(wx)+w\exp(w^2x)$$

$$=\exp x+\text{cis}\left(-\frac{2\pi}{3}\right).\,e^{-\frac{1}{2}x}\,.\,\text{cis}\,\frac{x\sqrt{3}}{2}+\text{cis}\,\frac{2\pi}{3}\,.\,e^{-\frac{1}{2}x}\,.\,\text{cis}\left(-\frac{x\sqrt{3}}{2}\right)$$

$$=e^x+e^{-\frac{1}{2}x}\,.\left\{\text{cis}\left(\frac{x\sqrt{3}}{2}-\frac{2\pi}{3}\right)+\text{cis}\left[-\left(\frac{x\sqrt{3}}{2}-\frac{2\pi}{3}\right)\right]\right\}.$$

30. See No. 17, and compare p. 161, Ex. VIII. h, No. 31.

$$3(\text{sum})=\exp x+w\exp(wx)+w^2\exp(w^2x)$$

$$=\exp x+\text{cis}\,\frac{2\pi}{3}\,.\,e^{-\frac{1}{2}x}\,.\,\text{cis}\,\frac{x\sqrt{3}}{2}+\text{cis}\left(-\frac{2\pi}{3}\right).\,e^{-\frac{1}{2}x}\,.\,\text{cis}\left(-\frac{x\sqrt{3}}{2}\right)$$

$$=e^x+e^{-\frac{1}{2}x}\,.\left\{\text{cis}\left(\frac{x\sqrt{3}}{2}+\frac{2\pi}{3}\right)+\text{cis}\left[-\left(\frac{x\sqrt{3}}{2}+\frac{2\pi}{3}\right)\right]\right\}.$$

CHAPTER XI

EXERCISE XI. a. (p. 207.)

1. As in Ex. 1 the roots of $4c^3-3c=2c^2-1$ are $\cos 0$, $\cos 72°$, $\cos 144°$; this eqn. may be written $(c-1)(4c^2+2c-1)=0$. $\cos 72°$ is the positive root of $4c^2+2c-1=0$, namely $\dfrac{-1+\sqrt{5}}{4}$; $-\cos 36°=\cos 144°=$ the negative root.

2. $\cos\dfrac{\pi}{7}=-\cos\dfrac{6\pi}{7}$, $\cos\dfrac{3\pi}{7}=-\cos\dfrac{4\pi}{7}$, $\cos\dfrac{5\pi}{7}=-\cos\dfrac{2\pi}{7}$; $\therefore$ write $-c$ for c in result of Ex. 1. **Or** use method of Ex. 1, starting with $\cos 4\theta=-\cos 3\theta=\cos(\pi-3\theta)$, which is satisfied by $\theta=\dfrac{(2n+1)\pi}{7}$; remove the root, $c=-1$.

3. Use the result of No. 2; l.h.s. = sum of products, two together, of the reciprocals of the roots = +(coeff. of c^2) ÷ (constant term).

4. $\theta = 0,\ (2n+1)\frac{\pi}{7}$ satisfy

$\sin 4\theta = \sin 3\theta$ or $4\sin\theta\cos\theta\cos 2\theta = 3\sin\theta - 4\sin^3\theta$;

$\therefore\ \theta = (2n+1)\frac{\pi}{7}$ satisfies $16\cos^2\theta\cos^2 2\theta = (3-4\sin^2\theta)^2$; this reduces to $64x^3 - 112x^2 + 56x - 7 = 0$, where $x = \sin^2\theta$.

Or, Use the first method of Example 2;

$$\sin^2\frac{r\pi}{7} = \tfrac{1}{2}\left(1 - \cos\frac{2r\pi}{7}\right);$$

eliminate c between $x = \frac{1}{2}(1-c)$ and the result of Example 1;
$\therefore\ 8(1-2x)^3 + 4(1-2x)^2 - 4(1-2x) - 1 = 0.$

5. The eqn. $\cot 7\theta = 0$ is satisfied by $\theta = \frac{(2n+1)\pi}{14}$. Put $\tan\theta = t$ and use eqn. (6) p. 172; thus $1 - 21t^2 + 35t^4 - 7t^6 = 0$ is satisfied by $\pm\tan\frac{\pi}{14},\ \pm\tan\frac{3\pi}{14},\ \pm\tan\frac{5\pi}{14}$. Put x for t^2.

6. By Ex. 2 the roots of $t^6 - 21t^4 + 35t^2 - 7 = 0$ are $\tan\frac{r\pi}{7}$, $r = 1, 2, \ldots 6$; the eqn. may be written $t^6 - 14t^4 + 49t^2 = 7(t^2+1)^2$ or $t(t^2-7) = \pm(t^2+1)\sqrt{7}$. $\tan\frac{\pi}{7}$, $\tan\frac{2\pi}{7}$ are positive and have squares less than $\tan^2\frac{\pi}{3} = 3 < 7$, thus they make $t(t^2-7) = -(t^2+1)\sqrt{7}$. For the same reason $-\tan\frac{\pi}{7}$, $-\tan\frac{2\pi}{7}$ are not roots of this equation; but the product of the roots is negative (being $-\sqrt{7}$); $\therefore$ the third root is negative and is $\therefore\ \tan\frac{4\pi}{7}$. (Compare Ex. 3.)

7. By Ex. 2, $\cot^2\frac{\pi}{7}$, $\cot^2\frac{2\pi}{7}$, $\cot^2\frac{3\pi}{7}$ are the roots of

$$\frac{1}{x^3} - \frac{21}{x^2} + \frac{35}{x} - 7 = 0 \text{ or } 7x^3 - 35x^2 + 21x - 1 = 0.$$

Expression (i) $= 3 +$ sum of roots $= 3 + 5$; (ii) By Ex. 2, $\tan^2\frac{\pi}{7}$, $\tan^2\frac{2\pi}{7}$, $\tan^2\frac{3\pi}{7}$ are the roots of

$$x^3 - 21x^2 + 35x - 7 = 0;$$

$\therefore\ \Sigma t^2 = 21,\ \Sigma t^4 = 21^2 - 2\,.\,35 = 371$, but $\sec^4\theta = (1+\tan^2\theta)^2 = 1 + 2\tan^2\theta + \tan^4\theta$; $\therefore$ expression $= 3 + 2\,.\,21 + 371$.

8. Expression $= \sin\frac{4\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{8\pi}{7} =$ half sum of roots of eqn. in Ex. 3. See also No. 10.

9. $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$; $\therefore$ expression $= \cos^2\frac{3\pi}{7} + \cos^2\frac{2\pi}{7} + \cos^2\frac{\pi}{7}$ = sum of squares of roots of eqn. in Ex. $1 = (\frac{1}{2})^2 - 2\,.\,(-\frac{1}{2})$.

10. Sum of roots $= (1 + k + k^2 + k^3 + k^4 + k^5 + k^6) - 1 = \frac{1-k^7}{1-k} - 1 = -1$;

product $= \Sigma k^r$ for $r = 4, 6, 7, 5, 7, 8, 7, 9, 10$
$= \Sigma k^r$ for $r = 1, 2, 3, 4, 5, 6, 7, 7, 7 = 2k^7 = 2$.

One root is $\text{cis}\frac{2\pi}{7} + \text{cis}\frac{4\pi}{7} + \text{cis}\frac{8\pi}{7}$

$$= \left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}\right) + i\left(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}\right),$$

but the roots are $\frac{-1 \pm i\sqrt{7}}{2}$; $\therefore$ expression $= \pm\frac{1}{2}\sqrt{7}$, and is positive because $\sin\frac{2\pi}{7} + \sin\frac{8\pi}{7} = \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} > 0$.

11. $\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$ satisfy $\cos 3\theta = -\frac{1}{2}$; $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$;
$\therefore\ 4x^3 - 3x = -\frac{1}{2}$ is satisfied by $\cos\frac{2\pi}{9}$, etc.

12. Expression $= -\sec\frac{8\pi}{9} - \sec\frac{4\pi}{9} - \sec\frac{2\pi}{9} = -$sum of reciprocals of roots of eqn. in No. $11 = -\frac{6}{1}$.

13. $-\frac{x}{2} = -\cos\frac{\pi}{9} = \cos\frac{8\pi}{9}$ satisfies the eqn. in No. 11;

$$\therefore\ 8\left(-\frac{x}{2}\right)^3 - 6\left(-\frac{x}{2}\right) + 1 = 0:\ \therefore\ x^3 - 3x - 1 = 0;$$

$\therefore\ x^2(x^2-3)^2 = 1.$

14. $\tan^2\theta = \sec^2\theta - 1$; $\therefore\ 3 +$ reqd. sum = sum of squares of reciprocals of roots of eqn. in No. $11 = 6^2 - 2\,.\,0 = 36$. **Or,** take sum of squares of roots of eqn. in No. 15, $(3\sqrt{3})^2 - 2(-3)$.

15. $\theta = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}$ satisfy $\tan 3\theta = +\sqrt{3}$,

i.e. $\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = +\sqrt{3}$, put $\tan\theta = x$.

16. Ex. IX. e, No. 12, applied to $\cos 6\theta = \cos 5\theta$ gives
$$(2c)^6 - 6(2c)^4 + 9(2c)^2 - 2 = (2c)^5 - 5(2c)^3 + 5(2c),$$
or putting
$$x = 2c = 2\cos\theta,\ x^6 - x^5 - 6x^4 + 5x^3 + 9x^2 - 5x - 2 = 0;$$
by the argument of Ex. 1, this has roots $2\cos\frac{2r\pi}{11}$ for $r=0$ to 5. Divide by the factor $x-2$ corresponding to $r=0$.

17. $\frac{1}{4}$ of sum of squares of roots of eqn. in No. $16 = \frac{1}{4}\{1^2 - 2(-4)\}$.

18. $\theta = 55°, 65°, 175°$ satisfy
$$\cos 3\theta = -\cos 15° = -\frac{\sqrt{3}+1}{2\sqrt{2}} \text{ or } 4\cos^3\theta - 3\cos\theta + \frac{\sqrt{3}+1}{2\sqrt{2}} = 0;$$
put $x = \cos\theta$.

19. As in the second method of Example 2, from the eqn. $\tan 13\theta = 0$, it follows that $\tan^2\frac{r\pi}{13}$ for $r=1$ to 6 are the roots of $x^6 - \binom{13}{2}x^5 + \ldots - \binom{13}{3}x + 13 = 0$;
$$\therefore \sum_1^{12}\operatorname{cosec}^2\frac{r\pi}{13} \equiv 12 + 2\sum_1^6\cot^2\frac{r\pi}{13}$$
$$= 12 + 2\{\tbinom{13}{3} \div 13\} = 12 + 44.$$
Also
$$\sum_1^{12}\sec^2\frac{r\pi}{13} \equiv 12 + 2\sum_1^6\tan^2\frac{r\pi}{13}$$
$$= 12 + 2\tbinom{13}{2} = 12 + 156.$$

Or, use Ex. IX. e, Nos. 25, 17 to express $\sin 13\theta = 0$ as an eqn. with roots $\sin^2\frac{r\pi}{13}$, $\cos^2\frac{r\pi}{13}$ respectively for parts (i), (ii).

20. Expression is sum of reciprocals of roots of the eqn. in No. 5.

21. $\theta = \frac{2\pi}{15}, \frac{4\pi}{15}, \frac{8\pi}{15}, \frac{14\pi}{15}$ satisfy $\cos 5\theta = -\frac{1}{2}$, i.e. by Ex. IX. e, No. 12, $(2c)^5 - 5(2c)^3 + 5(2c) = -1$, or putting $x = 2c \equiv 2\cos\theta$, $x^5 - 5x^3 + 5x + 1 = 0$, divide by the factor $x+1$ corresponding to $x = 2\cos\frac{10\pi}{15}$.

22. Put $x = \tan\theta$, equation becomes $1 = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = \tan 4\theta$,
$$4\theta = n\pi + \frac{\pi}{4},\ \theta = \frac{(4n+1)\pi}{16}.$$

23. Ex. IX. e, No. 12, applied to $\cos 9\theta = \cos 8\theta$ gives
$$(2c)^9 - 9(2c)^7 + 27(2c)^5 - 30(2c)^3 + 9(2c)$$
$$= (2c)^8 - 8(2c)^6 + 20(2c)^4 - 16(2c)^2 + 2 \text{ or } 1 - 9c - 32c^2 + \ldots = 0;$$
by the argument of Ex. 1 this has roots $\cos\frac{2r\pi}{17}$ for $r=0$ to 8; the sum of the squares of their reciprocals is $9^2 - 2(-32) = 145$, and $\sec^2\frac{2r\pi}{17}$ for $r=0$ is unity.

Or, as in the second method of Example 2, from $\tan 17\theta = 0$, obtain an eqn. with roots $\tan^2\frac{r\pi}{17}$. Compare No. 19.

24. If $\theta = 10°$, $70°$ or $130°$, $+\frac{1}{\sqrt{3}} = \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$; put $\tan\theta = x$.

EXERCISE XI. b. (p. 210.)

1. Eqn. is satisfied by $x = \alpha$, $\alpha + \frac{2\pi}{3}$, or $\alpha + \frac{4\pi}{3}$, and may be written $\cos 3\alpha = 4\cos^3 x - 3\cos x = 4y^3 - 3y$; this eqn. for y is therefore satisfied by $\cos\alpha$, $\cos\left(\alpha + \frac{2\pi}{3}\right)$, $\cos\left(\alpha + \frac{4\pi}{3}\right)$, and, since these are all different, they are the roots of the cubic. Hence their product $= \frac{-\text{constant term}}{\text{coeff. } y^3} = \frac{\cos 3\alpha}{4}$.

2. Method of No. 1. $\sin 3x = \sin 3\alpha$ is satisfied by $x = \alpha$, $\alpha + \frac{2\pi}{3}$, $\alpha + \frac{4\pi}{3}$ and $\sin 3\alpha = 3\sin x - 4\sin^3 x = 3z - 4z^3$ is satisfied by $z = \sin\alpha$, $\sin\left(\alpha + \frac{2\pi}{3}\right)$, $\sin\left(\alpha + \frac{4\pi}{3}\right)$. Hence the product of these $= -\frac{1}{4}\sin 3\alpha$; also
$$\sin\left(\alpha + \frac{\pi}{3}\right) = -\sin\left(\alpha + \frac{4\pi}{3}\right).$$

3. Use the cubic for z in No. 2; the sum of the reciprocals of the roots $= -\frac{\text{coeff. } z}{\text{constant term}} = \frac{3}{\sin 3\alpha}$.

4. $\tan 3\theta = \tan 3x$ is satisfied by $x = \theta$, $\theta + \frac{\pi}{3}$, $\theta + \frac{2\pi}{3}$, and can be written $\tan 3\theta = \frac{3t - t^3}{1 - 3t^2}$ where $t = \tan x$; hence the roots of $(1 - 3t^2)\tan 3\theta = 3t - t^3$ are
$$t = \tan\theta,\ \tan\left(\theta + \frac{\pi}{3}\right),\ \tan\left(\theta + \frac{2\pi}{3}\right).$$
Sum of roots $= 3\tan 3\theta$.

5. Differentiate the identity of No. 4 w.r.t. θ.

6. The eqn. in Ex. 6 may be written

$$x^n \tan n\alpha - nx^{n-1} - \frac{n(n-1)}{2!}x^{n-2}\tan n\alpha + \dots = 0;$$

the sum of the squares of roots

$$=(n\cot n\alpha)^2 - 2\left(\frac{-n(n-1)}{2!}\right) = n^2\cot^2 n\alpha + n^2 - n.$$

7. The eqn. $\tan n\alpha = \dfrac{\binom{n}{1}t - \binom{n}{3}t^3+\dots}{1-\binom{n}{2}t^2+\dots}$ found in Ex. 6 has roots $t=\tan\left(\alpha+\frac{r\pi}{n}\right)$. For n odd, it can be written

$$\tan n\alpha\left\{1-\binom{n}{2}t^2+\dots+(-1)^{\frac{n-1}{2}}\binom{n}{n-1}t^{n-1}\right\} = \binom{n}{1}t - \binom{n}{3}t^3+\dots+(-1)^{\frac{n-1}{2}}\binom{n}{n}t^n$$

or $t^n - n\tan n\alpha \,.\, t^{n-1}+\dots=0.$

$$\Sigma\tan\left(\alpha+\frac{r\pi}{n}\right) = \text{sum of roots} = n\tan n\alpha,$$

and $\Sigma\sec^2\left(\alpha+\frac{r\pi}{n}\right)=\frac{d}{d\alpha}(n\tan n\alpha)=n^2\sec^2 n\alpha.$

8. Compare No. 7. With $2n$ for n, the eqn. in Ex. 6 is

$$\tan 2n\alpha = \frac{\binom{2n}{1}t - \binom{2n}{3}t^3+\dots+(-1)^{n-1}\binom{2n}{2n-1}t^{2n-1}}{1-\binom{2n}{2}t^2+\dots+(-1)^n\binom{2n}{2n}t^{2n}}$$

or $t^{2n} \,.\, \tan 2n\alpha + 2n \,.\, t^{2n-1}+\dots=0$;

sum of roots $= -2n\cot 2n\alpha$.

9. In No. 6, write $\frac{\pi}{2}+\alpha$ for α.

10. For n even, from equation $\tan n\theta=0$, $\tan\frac{r\pi}{n}$ for $r=0$ to $n-1$ gives roots of $nt^{n-1}-\binom{n}{3}t^{n-3}+\dots+kt=0$; remove factor t and put $x=t^2$, then $\tan^2\frac{r\pi}{n}$ for $r=1$ to $\frac12 n-1$ gives roots of

$$nx^{\frac12 n-1}-\binom{n}{3}x^{\frac12 n-2}+\dots=0;$$

$$\therefore \text{ expression } =\tfrac12 n - 1 + \sum_1^{\frac12 n-1}\tan^2\frac{r\pi}{n}$$

$$= \tfrac12 n-1+\binom{n}{3}\div n = \tfrac12 n - 1 + \frac{(n-1)(n-2)}{6}.$$

Or, take the equation $\sin n\theta=0$ and use Ex. IX. e, No. 16.

11. For n odd, put $\alpha=0$ in No. 9, then $2\sum_1^{\frac12(n-1)}\tan^2\frac{r\pi}{n}=n^2-n$;

$$\therefore \sum_1^{\frac12(n-1)}\sec^2\frac{r\pi}{n} = \tfrac12(n-1)+\sum_1^{\frac12(n-1)}\tan^2\frac{r\pi}{n}$$

$$= \tfrac12(n-1)+\tfrac12(n^2-n)=\tfrac12(n^2-1).$$

Or, use either method of No. 10 (Ex. IX. e, No. 17).

12. $\tan n\theta$, n odd, $=\dfrac{\binom n1 t - \dots + (-1)^{\frac{n-1}{2}}\binom nn t^n}{1-\binom n2 t^2+\dots}$, and is zero for $\theta=\frac{r\pi}{n}$; hence the values of $\tan\frac{r\pi}{n}$ for $r=0$ to $(n-1)$ are the roots of

$$\binom n1 t - \dots - (-1)^{\frac{n-1}{2}}\binom{n}{n-2}t^{n-2}+(-1)^{\frac{n-1}{2}}\binom nn t^n=0$$

or $t^n - \binom n2 t^{n-2}+\dots=0$; the sum of the products two together $=$ coeff. of $t^{n-2} = -\frac12 n(n-1)$.

13. $\cos n\theta = \cos n\alpha$ is satisfied by $\theta=\alpha+\frac{2r\pi}{n}$. By Ex. IX. e, Nos. 12, 15 $\cos n\theta = 2^{n-1}c^n - n \,.\, 2^{n-3}c^{n-2}+\dots+(-1)^{\frac{n-1}{2}}nc$, thus the values of $\cos\left(\alpha+\frac{2r\pi}{n}\right)$ for $r=0$ to $n-1$ are the roots of $2^{n-1}c^n - \dots + (-1)^{\frac{n-1}{2}}nc - \cos n\alpha = 0$, and the sum of the reciprocals of the roots $=(-1)^{\frac{n-1}{2}}n\sec n\alpha$.

14. $\sin nx = \sin n\theta$ is satisfied by

$$x=\theta,\ -\left(\theta+\frac{\pi}{n}\right),\ \left(\theta+\frac{2\pi}{n}\right),\ -\left(\theta+\frac{3\pi}{n}\right),\ \dots .$$

By Ex. IX. e, No. 25, for n odd,

$$\sin nx = n\sin x - \dots + (-1)^{\frac{n-1}{2}} \,.\, 2^{n-1}\sin^n x;$$

$\therefore$ $ns - \dots + (-1)^{\frac{n-1}{2}} \,.\, 2^{n-1} \,.\, s^n - \sin n\theta = 0$ has roots

$$\sin\theta,\ -\sin\left(\theta+\frac{\pi}{n}\right),\ \sin\left(\theta+\frac{2\pi}{n}\right),\ -\sin\left(\theta+\frac{3\pi}{n}\right),\ \dots;$$

the sum of the reciprocals of the roots $=\dfrac{n}{\sin n\theta}$.

15. Put $n=2m$ and $\theta=\phi-\frac{\pi}{n}$;

$$\text{l.h.s.}=\cot\theta-\cot\phi+\cot\left(\theta+\frac{\pi}{m}\right)-\cot\left(\phi+\frac{\pi}{m}\right)+\dots+\cot\left(\theta+\frac{\overline{m-1}\pi}{m}\right)-\cot\left(\phi+\frac{\overline{m-1}\pi}{m}\right)$$

=, by Example 6,

$$m\cot m\theta-m\cot m\phi=\frac{n}{2}\left\{\cot\frac{n\theta}{2}-\cot\left(\frac{n\theta}{2}+\frac{\pi}{2}\right)\right\}$$

$$=\frac{n}{2}\left(\cot\frac{n\theta}{2}+\tan\frac{n\theta}{2}\right)=n\operatorname{cosec} n\theta.$$

16. (i) $=\frac{1}{2^2}\cdot\sigma=\frac{1}{4}\cdot\frac{\pi^2}{6}$;

(ii) $=\sigma-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\dots\right)$ on account of absolute convergence, $=\frac{\pi^2}{6}-\frac{\pi^2}{24}$.

17. Absolute convergence. Use results of No. 16. Sum $=\frac{\pi^2}{8}-\frac{\pi^2}{24}$.

18. Absolute convergence. Sum $=\sigma-\frac{1}{3^2}-\frac{1}{6^2}-\frac{1}{9^2}-\dots=\sigma-\frac{1}{9}\sigma$.

19. $$\frac{1}{n^2(n+1)^2}=\left(\frac{1}{n}-\frac{1}{n+1}\right)^2$$

$$=\frac{1}{n^2}+\frac{1}{(n+1)^2}-\frac{2}{n(n+1)}=\frac{1}{n^2}+\frac{1}{(n+1)^2}-\left(\frac{2}{n}-\frac{2}{n+1}\right);$$

sum to n terms $=\left(\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{n^2}\right)+\left(\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{(n+1)^2}\right)$

$$-\left(\frac{2}{1}-\frac{2}{n+1}\right)\to\sigma+(\sigma-1)-2.$$

20. $$\frac{1}{n^3(n+1)^3}=\left(\frac{1}{n}-\frac{1}{n+1}\right)^3$$

$$=\frac{1}{n^3}-\frac{3}{n^2(n+1)}+\frac{3}{n(n+1)^2}-\frac{1}{(n+1)^3}$$

$$=\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right)-\frac{3}{n(n+1)}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right)-\frac{3}{n^2(n+1)^2};$$

sum to n terms $=\left(\frac{1}{1^3}-\frac{1}{(n+1)^3}\right)$

$$-3\left(\frac{1}{1^2\cdot2^2}+\frac{1}{2^2\cdot3^2}+\dots+\frac{1}{n^2(n+1)^2}\right)$$

$\rightarrow\ 1-3\times$(answer to No. 19).

21. The eqn. in Ex. 7 with roots $\operatorname{cosec}^2\frac{r\pi}{n}$ is

$$x^{\frac{n-1}{2}}-\frac{n^2-1}{6}x^{\frac{n-3}{2}}+\frac{(n^2-1)(n^2-9)}{5!}x^{\frac{n-5}{2}}-\dots=0;$$

$$\therefore\ \Sigma\operatorname{cosec}^4\left(\frac{r\pi}{n}\right)=\left(\frac{n^2-1}{6}\right)^2-\frac{2}{5!}(n^2-1)(n^2-9)$$

$$=\frac{n^2-1}{180}\{5(n^2-1)-3(n^2-9)\}.$$

As in Ex. 7, $\frac{1}{\phi^4}<\operatorname{cosec}^4\phi<\left(1+\frac{1}{\phi^2}\right)^2$, hence

$$\sum\frac{n^4}{r^4\pi^4}<\Sigma\operatorname{cosec}^4\left(\frac{r\pi}{n}\right)<\frac{n-1}{2}+2\sum\frac{n^2}{r^2\pi^2}+\sum\frac{n^4}{r^4\pi^4};$$

$$\therefore\ \sum\frac{1}{r^4}<\frac{\pi^4}{n^4}\frac{(n^2-1)(n^2+11)}{90}$$

$$<\frac{\pi^4}{n^4}\left\{\frac{n-1}{2}+2\sum\frac{n^2}{r^2\pi^2}+\sum\frac{n^4}{r^4\pi^4}\right\}$$

$=\frac{\pi^4(n-1)}{2n^4}+\frac{2\pi^2}{n^2}\sum\frac{1}{r^2}+\sum\frac{1}{r^4}$. But $\frac{1}{1^4}+\frac{1}{2^4}+\dots$ is convergent; $\therefore$ if T is its sum, taking limits of the inequalities, $T\leqslant\frac{\pi^4}{90}\leqslant T$.

22. On account of absolute convergence,

$$2\Sigma\Sigma=\left(\sum\frac{1}{r^2}\right)^2-\left(\sum\frac{1}{r^4}\right)=\left(\frac{\pi^2}{6}\right)^2-\frac{\pi^4}{90}=\frac{\pi^4}{60}.$$

23. Follows from No. 7, since $\sec^2\left(\alpha+\frac{r\pi}{n}\right)$, for $=1, 3, 5, \dots (n-2)$,

$$=\sec^2\left(\alpha+\frac{\overline{n+r}\pi}{n}\right)=\sec^2\left(\alpha+\frac{2s\pi}{n}\right)$$

for $s=\frac{n+1}{2},\ \frac{n+3}{2},\ \dots (n-1)$.

24. Put $n=2m$; l.h.s. $=\sum_{0}^{2m-1}\sec^2\left(\theta+\frac{r\pi}{m}\right)=2\sum_{0}^{m-1}\sec^2\left(\theta+\frac{r\pi}{m}\right)$

since $\sec^2\left(\theta+\frac{(m+p)\pi}{m}\right)=\sec^2\left(\theta+\frac{p\pi}{m}\right)$,

$=2m+2\Sigma\tan^2\left(\theta+\frac{r\pi}{m}\right)=$, by No. 9,

$$2m+\left\{2m^2\operatorname{cosec}^2\left(m\theta+\frac{m\pi}{2}\right)-2m\right\}$$

$$=\frac{4m^2}{2\sin^2\left(m\theta+\frac{m\pi}{2}\right)}=\frac{n^2}{1-\cos\left(n\theta+\frac{n\pi}{2}\right)}.$$

EXERCISE XI. c. (p. 214.)

1. $\tan(\alpha+\beta+\gamma)=\frac{\Sigma\tan\alpha-\tan\alpha\tan\beta\tan\gamma}{1-\Sigma\tan\beta\tan\gamma}=\frac{-\frac{1}{a}-\left(-\frac{1}{a}\right)}{1-\frac{b}{a}}=0,$

since $a\neq b$. If $a=b$, the cubic becomes $(x^2+1)(ax+1)=0$; but there is no angle θ such that $\tan^2\theta=-1$; $\therefore$ the data imply $a\neq b$. [Even in Ch. XIII, $\tan^{-1}(i)$ remains undefined.]

2. $0=1-\Sigma\tan\beta\tan\gamma=1-q$.

3. $(c-b\sin\alpha)^2=a^2\cos^2\alpha=a^2-a^2\sin^2\alpha$; $\therefore\sin\alpha$ satisfies
$$(c-bx)^2=a^2-a^2x^2;$$
similarly for $\sin\beta$. If $\sin\alpha\neq\sin\beta$, they are the two roots, and their sum $=-\frac{\text{coeff. }x}{\text{coeff. }x^2}=\frac{2bc}{b^2+a^2}$.

4. $(\cos\theta-\cos\alpha\cos\beta)^2=\sin^2\alpha\sin^2\beta(1-k^2\sin^2\theta)$
$$=\sin^2\alpha\sin^2\beta(1-k^2+k^2\cos^2\theta);$$
$\therefore\cos x,\cos y$ are the roots of the equation in z,
$$z^2(1-k^2\sin^2\alpha\sin^2\beta)-2z\cos\alpha\cos\beta+\ldots=0;$$
$\cos x+\cos y=$sum of roots.

5. $\theta=\alpha,\beta,\gamma$ satisfy $\tan3\theta=k$ or $3\tan\theta-\tan^3\theta=k(1-3\tan^2\theta)$;
$\therefore\tan\alpha,\ \tan\beta,\ \tan\gamma$ are roots of $x^3-3kx^2-3x+k=0$;
$\therefore\Sigma(\cot\alpha)=\frac{3}{k}$ and $\Sigma(\tan\alpha)=3k$.

6. If k is the common value and $t=\tan\theta$, $\frac{3t-t^3}{1-3t^2}-t=k$, i.e.
$2t^3+3kt^2+2t-k=0$ has roots $t=\tan\alpha,\tan\beta,\tan\gamma$.
$$\Sigma(\tan\alpha\tan\beta)=1;\quad\therefore\ \alpha+\beta+\gamma=(2n+1)\frac{\pi}{2};$$
also $\frac{k}{2}=$product of roots$=\tan\alpha\tan\beta\tan\gamma$.

7. Put $\tan x=t$, $\tan\theta=t_1$, etc., then $\tan\alpha,\tan\beta,\tan\gamma$ are the roots of the equation in t,
$$\sum\left(\frac{t-t_1}{1+tt_1}\right)=0\ \text{ or }\ \Sigma\{(t-t_1)(1+tt_2)(1+tt_3)\}=0$$
or $t^3\Sigma(t_2t_3)-t^2(3t_1t_2t_3-2\Sigma t_1)+t(3-2\Sigma t_2t_3)-\Sigma t_1=0$;
$$\therefore\ \tan(\alpha+\beta+\gamma)=\frac{(3t_1t_2t_3-2\Sigma t_1)-\Sigma t_1}{\Sigma t_2t_3-(3-2\Sigma t_2t_3)}$$
$$=\frac{\Sigma t_1-t_1t_2t_3}{1-\Sigma t_1t_2}=\tan(\theta+\phi+\psi).$$

8. $a-\sin\theta=\cos\theta(1-2b\sin\theta)$;
$\therefore\ (a-\sin\theta)^2=(1-\sin^2\theta)(1-2b\sin\theta)^2$,
or $4b^2s^4-4bs^3+\ldots=0$ has roots $\sin\alpha_1$, etc.;
$$\therefore\ \Sigma(\sin\alpha_1)=+\frac{1}{b}.$$

9. Put $\tan\frac{x}{2}=t$, so that
$\sin x=\frac{2t}{1+t^2}$, $\cos x=\frac{1-t^2}{1+t^2}$, and $\sin2x=\frac{4t(1-t^2)}{(1+t^2)^2}$, then
$$(1+t^2)\sin2\theta\{2at+b(1-t^2)\}$$
$$=4t(1-t^2)(a\sin\theta+b\cos\theta)$$
is a quartic for t with roots $\tan\frac{\alpha}{2}$, $\tan\frac{\beta}{2}$, $\tan\frac{\gamma}{2}$, $\tan\frac{\delta}{2}$.
$$\text{Product of roots}=\frac{\text{constant term}}{\text{coeff. }t^4}=\frac{b\sin2\theta}{-b\sin2\theta}.$$

10. If $t=\tan\theta$, the method of Ex. 9 gives
$$4at(1-t^2)+b(1-6t^2+t^4)=c(1+t^2)^2,$$
or $(b-c)t^4-4at^3+\ldots+4at+(b-c)=0$;
since constant term$=$coeff. t^4, product of roots$=1$. For (ii),
$$\Sigma\operatorname{cosec}2\theta=\sum\frac{1+t^2}{2t}=\tfrac{1}{2}\left(\sum\frac{1}{t}+\Sigma t\right)=0$$
Or, by inspection roots are $\theta_1,\ \theta_2,\ \theta_1+\frac{1}{2}\pi,\ \theta_2+\frac{1}{2}\pi$;
$\therefore\ \tan\theta_1\tan\theta_3=-1=\tan\theta_2\tan\theta_4$
and $\operatorname{cosec}2\theta_1=-\operatorname{cosec}2\theta_3$, etc.

11. Put $\theta - a = \phi$, $a - \beta = \gamma$, $\tan\dfrac{\phi}{2} = t$. Then eqn. becomes

$$a\cos 2\phi + b\cos(\phi+\gamma) + c = 0, \text{ or, as in No. 10,}$$

$$a(1-6t^2+t^4) + b(1+t^2)[\cos\gamma(1-t^2) - 2t\sin\gamma] + c(1+t^2)^2 = 0.$$

Here, coeff. of t^3 equals coeff. of t;

$$\therefore\ \Sigma(t_1) - \Sigma(t_2t_3t_4) = 0;\quad \therefore\ \tan(\Sigma\tfrac{1}{2}\phi) = 0;$$

$$\therefore\ \tfrac{1}{2}\Sigma(\theta - a) = k\pi;\quad \therefore\ \Sigma(\theta) - 4a = 2k\pi.$$

Or, in Ex. 9, p. 212, write $a\cos 2a$, $a\sin 2a$ for a, b; then

$$\tan(\Sigma\tfrac{1}{2}\theta) = \tan 2a.$$

12. (i) Method of Ex. 9, p. 212; but $a = 0$, hence

$$\cot\sum\frac{a}{2} = 0;\quad \therefore\ \sum\frac{a}{2} = \frac{\pi}{2} + n\pi;$$

(ii) $\sin\theta(2\cos\theta - n) = m\cos\theta - r$;

$$\therefore\ (1-\cos^2\theta)(2\cos\theta - n)^2 = (m\cos\theta - r)^2,$$

$$\text{or } 4\cos^4\theta - 4n\cos^3\theta + \ldots = 0;$$

$$\Sigma(\cos a) = \text{sum of roots} = n;$$

(iii) $\cos\theta(2\sin\theta - m) = n\sin\theta - r$ gives a similar quartic for $\sin a$ with m, n interchanged.

13. $\dfrac{\tan\theta + \tan a}{1 - \tan\theta\tan a} = \tan(\theta + a)$

$$= k\tan n\theta = k\frac{\binom{n}{1}\tan\theta - \binom{n}{3}\tan^3\theta + \ldots}{1 - \binom{n}{2}\tan^2\theta + \ldots}$$

is an eqn. of degree $(n+1)$ for $\tan\theta$. It may be written:

$$\tan a + \tan\theta\{1 - k\tbinom{n}{1}\} + \ldots = 0;$$

the sum of the cotangents = sum of reciprocals of roots $= -\{1 - k\,.\,\binom{n}{1}\}/\tan a$.

14. $\tan\theta_1$, etc., are the roots of the eqn. in t,

$$a\frac{3t - t^3}{1 - 3t^2} + b\frac{2t}{1 - t^2} + ct + d = 0,$$

or $a(t^3 - 3t)(t^2 - 1) - 2bt(3t^2 - 1) + (ct + d)(3t^2 - 1)(t^2 - 1) = 0$,

or $(3c + a)t^5 + 3dt^4 - (4a + 6b + 4c)t^3 - 4dt^2 + (c + 2b + 3a)t + d = 0$;

$$\tan(\Sigma\theta) = \frac{\Sigma_1 - \Sigma_3 + \Sigma_5}{1 - \Sigma_2 + \Sigma_4}$$

$$= \frac{(-3d) - 4d + (-d)}{(3c+a) + (4a+6b+4c) + (c+2b+3a)} = \frac{-d}{a+b+c}.$$

15. The eqn. may be written

$$a\sin\theta + b\cos\theta = c\sin\theta\cos\theta \text{ or } \sin\theta(a - c\cos\theta) = -b\cos\theta,$$

$$\text{or } (1-\cos^2\theta)(a - c\cos\theta)^2 = b^2\cos^2\theta;$$

$$\Sigma(\cos\theta) = -\frac{\text{coeff. }\cos^3\theta}{\text{coeff. }\cos^4\theta} = \frac{2ac}{c^2}.$$

(ii) is proved similarly by writing the eqn. as

$$(1-\sin^2\theta)(b - c\sin\theta)^2 = a^2\sin^2\theta.$$

For (iii) put $\tan\dfrac{\theta}{2} = t$; $\therefore$ $\operatorname{cosec}\theta = \dfrac{1+t^2}{2t}$, $\sec\theta = \dfrac{1+t^2}{1-t^2}$;

eqn. becomes $a(1+t^2)\,.\,2t + b(1+t^2)(1-t^2) = c\,.\,2t\,.\,(1-t^2)$
or $bt^4 - 2(a+c)t^3 + 2(c-a)t - b = 0$, and

$$1 - \Sigma\left(\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\right) + \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2}$$

$$= 1 - 0 + \frac{-b}{b} = 0;\quad \therefore\ \Sigma\left(\frac{\theta}{2}\right) = (2n+1)\frac{\pi}{2}.$$

One condition for the normals at θ_1, θ_2, θ_3, θ_4 to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ to be concurrent is $\Sigma\theta = (2n+1)\pi$; the eqn. to a normal is $ax\sec\theta - by\operatorname{cosec}\theta = c^2$, thus the four normals pass through (x_1y_1) if $\theta = \theta_1, \theta_2, \theta_3, \theta_4$ satisfy

$$ax_1\sec\theta - by_1\operatorname{cosec}\theta = c^2;$$

this is of the form $a'\sec\theta + b'\operatorname{cosec}\theta = c'$, hence the condition follows.

16. (i) Eqn. is $\{(a+\cos\theta)\cos\theta\cos\gamma - b\}^2 = (a+\cos\theta)^2\sin^2\theta\sin^2\gamma$

$$= (a+\cos\theta)^2(1-\cos^2\theta)\sin^2\gamma,$$

$$\text{or } \cos^4\theta(\cos^2\gamma + \sin^2\gamma) + 2a\cos^3\theta(\cos^2\gamma + \sin^2\gamma) + \ldots = 0;$$

$\Sigma\cos\theta$ = sum of roots;

(ii) Eqn. may also be written

$$\cos\theta(a\cos\gamma + \sin\theta\sin\gamma) = b - a\sin\theta\sin\gamma - \cos^2\theta\cos\gamma$$
$$= b - a\sin\theta\sin\gamma + (\sin^2\theta - 1)\cos\gamma,$$

$$\text{or } (1-\sin^2\theta)(a\cos\gamma + \sin\theta\sin\gamma)^2 = (b - \cos\gamma - a\sin\theta\sin\gamma + \sin^2\theta\cos\gamma)^2,$$

in which coeff. of $\sin^3\theta$ is zero; $\therefore$ $\Sigma\sin\theta$ = sum of roots = 0;

(iii) Put $\theta - \dfrac{\gamma}{2} = \phi$ and $\gamma = 2\beta$. Eqn. is

$$[a + \cos(\phi+\beta)]\cos(\phi-\beta) = b$$

$$\text{or } 2a\cos(\phi-\beta) + \cos 2\phi + \cos 2\beta = 2b.$$

Put $\tan\frac{\phi}{2}=t$, then $\cos 2\phi=\frac{1-6t^2+t^4}{(1+t^2)^2}$,

and eqn. becomes

$$2a(1+t^2)[\cos\beta(1-t^2)+\sin\beta\,.\,2t]$$
$$+(1-6t^2+t^4)+(1+t^2)^2(\cos 2\beta-2b)=0;$$

coeff. of t^3 = coeff. of t;

$$\therefore\ \Sigma_1-\Sigma_3=0;\qquad \therefore\ \tan(\Sigma\tfrac{1}{2}\phi)=0;$$
$$\therefore\ \tfrac{1}{2}\Sigma(\theta-\tfrac{1}{2}\gamma)=k\pi;\quad \therefore\ \Sigma(\theta)-2\gamma=2k\pi.$$

17. $\theta=x$, $\theta=y$ are essentially distinct roots of

$$a\cos z\cos\theta+b\sin z\sin\theta=c;$$

put $\tan\frac{\theta}{2}=t$, eqn. becomes

$$a\cos z(1-t^2)+2bt\sin z-c(1+t^2)=0$$

or $\quad t^2(c+a\cos z)-2bt\sin z+c-a\cos z=0;$

$$\therefore\ \tan\frac{x+y}{2}=\frac{t_1+t_2}{1-t_1t_2}=\frac{2b\sin z}{(c+a\cos z)-(c-a\cos z)}$$
$$=\frac{b}{a}\tan z;\ \text{similarly}\ \tan\frac{x+z}{2}=\frac{b}{a}\tan y;$$
$$\therefore\ \tan\frac{y-z}{2}=\tan\left(\frac{x+y}{2}-\frac{x+z}{2}\right)$$
$$=\frac{\frac{b}{a}(\tan z-\tan y)}{1+\frac{b^2}{a^2}\tan z\tan y}=\frac{ab\sin(z-y)}{a^2\cos y\cos z+b^2\sin y\sin z};$$

divide by $\sin\frac{y-z}{2}$, which is not zero;

$$\therefore\ a^2\cos y\cos z+b^2\sin y\sin z$$
$$=-2ab\cos^2\frac{y-z}{2}=-ab[1+\cos(y-z)];$$
$$\therefore\ (a^2+ab)\cos y\cos z+(b^2+ab)\sin y\sin z+ab=0;$$
$$\therefore\ (a+b)(a\cos y\cos z+b\sin y\sin z)+ab=0;$$
$$\therefore\ (a+b)c+ab=0.$$

18. (i) $x=\alpha$, $x=\beta$ are distinct roots of

$$a\cos\gamma\cos x+b(\sin\gamma+\sin x)+c=0;$$

put $\tan\frac{x}{2}=t$, eqn. becomes

$$a\cos\gamma(1-t^2)+b\sin\gamma(1+t^2)+2bt+c(1+t^2)=0;$$

then as in No. 17,

$$\tan\frac{\alpha+\beta}{2}=\frac{b}{a\cos\gamma};\ \text{similarly}\ \tan\frac{\alpha+\gamma}{2}=\frac{b}{a\cos\beta};$$

$$\therefore\ \tan\frac{\beta-\gamma}{2}=\tan\left(\frac{\alpha+\beta}{2}-\frac{\alpha+\gamma}{2}\right)=\frac{ab(\cos\beta-\cos\gamma)}{a^2\cos\beta\cos\gamma+b^2}$$
$$=\frac{-2ab\sin\frac{\beta+\gamma}{2}\sin\frac{\beta-\gamma}{2}}{a^2\cos\beta\cos\gamma+b^2};$$

divide by $\sin\frac{\beta-\gamma}{2}$, which $\neq 0$;

$$\therefore\ a^2\cos\beta\cos\gamma+b^2=-2ab\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}$$
$$=-ab(\sin\beta+\sin\gamma);$$
$$\therefore\ a[a\cos\beta\cos\gamma+b(\sin\beta+\sin\gamma)]+b^2=0;$$
$$\therefore\ a(-c)+b^2=0.$$

(ii) $\cos\gamma\tan\frac{\alpha+\beta}{2}=\frac{b}{a}=\cos\beta\tan\frac{\alpha+\gamma}{2}$;

$$\therefore\ \cos\gamma\,.\,\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\gamma}{2}=\cos\beta\,.\,\sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\beta}{2};$$
$$\therefore\ \cos\gamma\cdot\left[\sin\left(\alpha+\frac{\beta+\gamma}{2}\right)+\sin\frac{\beta-\gamma}{2}\right]$$
$$=\cos\beta\cdot\left[\sin\left(\alpha+\frac{\beta+\gamma}{2}\right)-\sin\frac{\beta-\gamma}{2}\right];$$
$$\therefore\ \sin\left(\alpha+\frac{\beta+\gamma}{2}\right)(\cos\beta-\cos\gamma)=\sin\frac{\beta-\gamma}{2}(\cos\beta+\cos\gamma);$$
$$\therefore\ 2\sin\left(\alpha+\frac{\beta+\gamma}{2}\right)\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma-\beta}{2}$$
$$=\sin\frac{\beta-\gamma}{2}(\cos\beta+\cos\gamma);$$
$$\therefore\ \cos\alpha-\cos(\alpha+\beta+\gamma)=-(\cos\beta+\cos\gamma).$$

(iii) From (i), $\sin\frac{\alpha+\beta}{2}=\frac{b}{a\cos\gamma}\cdot\cos\frac{\alpha+\beta}{2}$;

$$\therefore\ \sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$$
$$=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2\cos\left(\gamma+\frac{\alpha+\beta}{2}\right)\sin\frac{\alpha+\beta}{2}$$
$$=\frac{2b}{a\cos\gamma}\left\{\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-\cos\left(\gamma+\frac{\alpha+\beta}{2}\right)\cos\frac{\alpha+\beta}{2}\right\}$$
$$=\frac{b}{a\cos\gamma}\left\{\cos\alpha+\cos\beta-\cos(\alpha+\beta+\gamma)-\cos\gamma\right\}$$
$$=\frac{b}{a\cos\gamma}\{-2\cos\gamma\},\ \text{by (ii)},\ =-\frac{2b}{a}.$$

19. Use the method of No. 17. If in No. 17,

$$a = -c\sec^2\alpha,\ b = -c\operatorname{cosec}^2\alpha,$$

the condition $bc + ca + ab = 0$ is satisfied; $\therefore$ the third eqn. can be deduced from the other two by using the processes of No. 17.

20. $a(a - 2\sin\phi) = (\sin\theta + \sin\phi)(\sin\theta - \sin\phi)$

$$= \sin^2\theta - \sin^2\phi = \cos^2\phi - \cos^2\theta = b(2\cos\phi - b),$$

and similarly with θ for ϕ; $\therefore$ if $t = \tan\frac{1}{2}\theta$ or $\tan\frac{1}{2}\phi$,

$$a\left(a - \frac{4t}{1+t^2}\right) = b\left\{\frac{2(1-t^2)}{1+t^2} - b\right\};$$

$$\therefore\ (a^2+b^2)(1+t^2) - 4at - 2b(1-t^2) = 0.$$

Thus $$\cot\tfrac{1}{2}\theta + \cot\tfrac{1}{2}\phi = \frac{4a}{a^2+b^2-2b}$$

and $$\cot\tfrac{1}{2}\theta\cot\tfrac{1}{2}\phi = \frac{a^2+b^2+2b}{a^2+b^2-2b};$$

$\therefore\ \cot^3\frac{1}{2}\theta + \cot^3\frac{1}{2}\phi$

$$= \frac{4a}{a^2+b^2-2b}\left\{\frac{16a^2}{(a^2+b^2-2b)^2} - \frac{3(a^2+b^2+2b)}{a^2+b^2-2b}\right\}$$

$$= \frac{4a}{(a^2+b^2-2b)^3}\{16a^2 - 3(a^2+b^2-2b)(a^2+b^2+2b)\}.$$

$$\tan\frac{\theta+\phi}{2} = \frac{\sin\theta+\sin\phi}{\cos\theta+\cos\phi} = \frac{a}{b};$$

$$\therefore\ \sin(\theta+\phi) = \frac{2\tan\frac{\theta+\phi}{2}}{1+\tan^2\frac{\theta+\phi}{2}} = \frac{2\frac{a}{b}}{1+\frac{a^2}{b^2}}.$$

EXERCISE XI. d. (p. 216.)

1. By Ex. IX. c, No. 1, $5x - 20x^3 + 16x^5 = \sin 5\theta = \sin\frac{\pi}{4}$. Roots are found from $5\theta = 2n\pi + \frac{\pi}{4}$ or $2n\pi + \frac{3\pi}{4}$ which give

$$\theta = \frac{(8n+1)\pi}{20}\ \text{ or }\ \frac{(8n+3)\pi}{20};$$

$$\therefore\ x = \sin\theta = \sin\frac{\pi}{20},\ \sin\frac{3\pi}{20},\ \sin\frac{9\pi}{20},\ \sin\frac{-7\pi}{20},\ \sin\frac{-5\pi}{20}.$$

2. As in No. 1, from $\sin 5\theta = 0$, roots of $16x^5 - 20x^3 + 5x = 0$ are 0, $\pm\sin\frac{2\pi}{5}$, $\pm\sin\frac{4\pi}{5}$; divide by x and put $x^2 = y$; $\therefore\ \sin^2\frac{2\pi}{5}$, $\sin^2\frac{4\pi}{5}$ are roots of $16y^2 - 20y + 5 = 0$; $\therefore$ their sum is $\frac{20}{16}$.

Since $\sin^2\frac{6\pi}{5} = \sin^2\frac{4\pi}{5}$ and $\sin^2\frac{8\pi}{5} = \sin^2\frac{2\pi}{5}$,

$$\sum_1^4 \sin^2\frac{2n\pi}{5} = 2\times\tfrac{20}{16}.$$

3. $\cos\frac{8\pi}{7} = \cos\frac{6\pi}{7}$; put $x = 2c$ in Example 1, p. 204.

4. By No. 3, $2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{8\pi}{7} =$ sum of roots $= -1$,

but $\cos\frac{4\pi}{7} = -\cos\frac{3\pi}{7}$, and $\cos\frac{8\pi}{7} = -\cos\frac{\pi}{7}$.

5. Use Example 2, p. 205, second method; expression = sum of reciprocals of roots of $x^3 - 21x^2 + 35x - 7 = 0$; this is $\frac{35}{7}$.

6. From $\tan 3\theta = \tan\frac{\pi}{3}$ it follows that $\tan\frac{\pi}{9}, \tan\frac{4\pi}{9}, \tan\frac{7\pi}{9}$ are roots of $\frac{3t - t^3}{1-3t^2} = \sqrt{3}$; $\therefore$ putting $t = \frac{1}{x}$, $\cot\frac{\pi}{9}, \cot\frac{4\pi}{9}$, $-\cot\frac{2\pi}{9}$ are roots of $\sqrt{3}\,.\,x^3 - 3x^2 - 3\sqrt{3}\,.\,x + 1 = 0$;

$$\therefore\ \text{expression} = \left(\frac{3}{\sqrt{3}}\right)^2 - 2\left(-\frac{3\sqrt{3}}{\sqrt{3}}\right) = 3 + 6.$$

7. From $\frac{\sin 9\theta}{\sin\theta} = 0$, using IX. e, No. 13, $\cos\frac{r\pi}{9}$ is a root of $(2c)^8 - 7(2c)^6 + 15(2c)^4 - 10(2c)^2 + 1 = 0$; put $x = (2c)^2$, then $\left(2\cos\frac{r\pi}{9}\right)^2$ is a root of $x^4 - 7x^3 + 15x^2 - 10x + 1 = 0$.

8. Use second method of Example 2. From $\tan 11\theta = 0$, $\tan\frac{r\pi}{11}$ is root of $11t - 165t^3 + 462t^5 - 330t^7 + 55t^9 - t^{11} = 0$, and so $\tan^2\frac{r\pi}{11}$, for $r = 1$ to 4, is root of

$$x^5 - 55x^4 + 330x^3 - 462x^2 + 165x - 11 = 0;$$

(i) sum of roots $= 55$;

(ii) sum of squares $= 55^2 - 2\times 330 = 2365$;

(iii) $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$; reqd. sum $= 5 +$ sum of reciprocals of roots $= 5 + \frac{165}{11}$;

(iv) from the eqn. in XI. *a*, No. 16, $\Sigma\left(2\cos\frac{2r\pi}{11}\right) = -1$, and $\cos^2\frac{r\pi}{11} = \frac{1}{2}\left(1 + \cos\frac{2r\pi}{11}\right)$; $\therefore$ reqd. sum $= \frac{1}{2}[5 + (-\frac{1}{2})] = 2\frac{1}{4}$.

9. The eqn. $\cos 7\theta = 0$ is satisfied by $\theta = \dfrac{(2r+1)\pi}{14}$;

$$\cos 7\theta \equiv 2^6 \cos^7 \theta \ldots - 7\cos\theta;$$

$\therefore \cos^2 \dfrac{n\pi}{14}$, for $n = 1, 3, 5$, are roots of $2^6 x^3 \ldots - 7 = 0$;

$\therefore \cos^2 \dfrac{\pi}{14} \cdot \cos^2 \dfrac{3\pi}{14} \cdot \cos^2 \dfrac{5\pi}{14} = \frac{7}{64}$; take square root and note that each factor of left side is positive.

10. As in the second method of Example 2, from $\tan 13\theta = 0$, $\tan^2 \dfrac{n\pi}{13}$, for $n = 1$ to 6, are roots of

$$x^6 - \binom{13}{11} x^5 + \ldots - \binom{13}{3} x + 13 = 0;$$

$\therefore$ square of reqd. expression = product of roots = 13. Take square root and note that each factor of left side is positive.

11. Sum of roots $= \cos \dfrac{2\pi}{13} + \cos \dfrac{4\pi}{13} + \ldots + \cos \dfrac{12\pi}{13} =$, by eqn. (11) of

Ch. VII, $\dfrac{\cos \frac{7\pi}{13} \sin \frac{6\pi}{13}}{\sin \frac{\pi}{13}} = \dfrac{-\cos \frac{6\pi}{13} \sin \frac{6\pi}{13}}{\sin \frac{12\pi}{13}} = -\frac{1}{2}$. Product of roots is the sum of terms like $\cos x \cdot \cos y$; expressing each as $\frac{1}{2}\{\cos(x-y) + \cos(x+y)\}$, the product is $\frac{1}{2}\Sigma \cos \dfrac{r\pi}{13}$ for $r = 2, 6, 2, 10, 4, 12, 8, 12, 4, 16, 2, 18, 10, 14, 6, 18, 4, 20$, $= \frac{3}{2}\Sigma \cos \dfrac{r\pi}{13}$ for $r = 2, 4, 6, 8, 10, 12$, $= \frac{3}{2} \cdot \dfrac{-1}{2}$.

12. l.h.s. $= 64 \cos 2\alpha \cos 3\alpha \cos \dfrac{\alpha}{2} \cos \dfrac{5\alpha}{2} \cos \alpha \cos 5\alpha = -64 \prod_{1}^{6} \cos \dfrac{r\pi}{13}$;

by Ex. IX. e, No. 17,

$$\cos^2 \frac{r\pi}{13} \text{ satisfies } 1 - \frac{13^2 - 1}{2!} x + \ldots + 2^{12} \cdot x^6 = 0,$$

and the product of the roots is $\dfrac{1}{2^{12}}$;

$$\therefore \text{ given expression} = -64 \cdot \frac{\pm 1}{2^6} = \mp 1;$$

the sign is $-$, because l.h.s. $= -64 \cdot$ (product of cosines of acute angles).

13. Sum of the two expressions $= \cos \alpha + \cos 3\alpha + \ldots + \cos 15\alpha =$, by eqn. (11) of Ch. VII,

$$\frac{\cos 8\alpha \cdot \sin 8\alpha}{\sin \alpha} = \frac{\frac{1}{2}\sin 16\alpha}{-\sin 16\alpha} = -\tfrac{1}{2}.$$

Product of the two expressions, as in No. 11, is $\frac{1}{2}\Sigma \cos r\alpha$ for $r = 2, 4, 4, 6, 6, 8, 10, 12, 6, 12, 4, 14, 2, 16, 2, 20, 10, 16, 8, 18, 6, 20, 2, 24, 12, 18, 10, 20, 8, 22, 4, 26$, $= 2\Sigma \cos r\alpha$ for

$$r = 1 \text{ to } 8, = 2\frac{\cos \frac{9\pi}{17} \sin \frac{8\pi}{17}}{\sin \frac{\pi}{17}} = -\frac{\sin \frac{16\pi}{17}}{\sin \frac{\pi}{17}} = -1.$$

Hence the expressions are the roots of $x^2 + \frac{1}{2}x - 1 = 0$, and are $\therefore \dfrac{-1 \pm \sqrt{17}}{4}$;

$$\text{but (ii)} = \cos \frac{10\pi}{17} + \cos \frac{14\pi}{17} + 2\cos \frac{14\pi}{17} \cos \frac{8\pi}{17}$$

has each term negative, and is $\therefore -\dfrac{1 + \sqrt{17}}{4}$.

14. Sum of roots $= k + k^2 + \ldots + k^{10} = \dfrac{k - k^{11}}{1 - k} = \dfrac{k - 1}{1 - k} = -1$; using $k^{11} = 1$, the product is found to be $5 + 2\Sigma k^r$ for $r = 1$ to 10, $= 5 - 2$; the quadratic is $x^2 + x + 3 = 0$.

$$\sum_{1}^{10} \operatorname{cis} \frac{2r\pi}{11} = \text{sum of roots} = -1,$$

$$\therefore \sum_{1}^{10} \cos \frac{2r\pi}{11} = -1,$$

$$\text{but} \quad \sum_{1}^{10} \cos \frac{2r\pi}{11} = 2\sum_{1}^{5} \cos \frac{2r\pi}{11}.$$

15. $\tan 3\alpha = \tan 3\theta = \dfrac{3t - t^3}{1 - 3t^2}$ is a cubic for t, with roots

$$\tan \alpha, \tan\left(\alpha + \frac{\pi}{3}\right), \tan\left(\alpha + \frac{2\pi}{3}\right);$$

it may be written $t^3 - 3t^2 \tan 3\alpha - 3t + \tan 3\alpha = 0$;

$\therefore$ sum of squares $= (3 \tan 3\alpha)^2 - 2 \cdot (-3)$.

16. $\tan 2n\theta = \tan 2nx = \dfrac{2nt - \ldots - (-1)^n \cdot 2nt^{2n-1}}{1 \ldots + (-1)^n \cdot t^{2n}}$ is an eqn. of degree $2n$, satisfied by $t = t_r \equiv \tan\left[\theta + \dfrac{(r-1)\pi}{2n}\right]$ for $r = 1$

to $2n$; it may be written $t^{2n} . \tan 2n\theta + 2nt^{2n-1} - \ldots = 0$;

$\therefore\ \Sigma t_r = \Sigma \tan\left[\theta + \frac{(r-1)\pi}{2n}\right] = -2n \cot 2n\theta$; differentiate w.r.t. θ; $\therefore\ \Sigma \sec^2\left[\theta + \frac{(r-1)\pi}{2n}\right] = 4n^2 \operatorname{cosec}^2 2n\theta$.

17. From No. 16, since $\frac{d}{d\theta}(\sec^2\theta) = \frac{2\sin\theta}{\cos^3\theta} = 2\tan\theta(1+\tan^2\theta)$, by differentiation

$$\Sigma(2t_r + 2t_r^3) = \frac{d}{d\theta}(4n^2 \operatorname{cosec}^2 2n\theta)$$

$$= -16n^3 \cot 2n\theta \operatorname{cosec}^2 2n\theta;$$

but $\Sigma t_r = -2n \cot 2n\theta$;

$$\therefore\ \Sigma t_r^3 = -8n^3 \cot 2n\theta \operatorname{cosec}^2 2n\theta + 2n \cot 2n\theta;$$

differentiate again,

$$\frac{d}{d\theta}\tan^3\theta = 3\tan^2\theta \sec^2\theta = 3\tan^2\theta(1+\tan^2\theta);$$

$\therefore\ 3\Sigma(t_r^2 + t_r^4)$

$$= -8n^3(-2n \operatorname{cosec}^4 2n\theta - 4n \cot^2 2n\theta \operatorname{cosec}^2 2n\theta) - 4n^2 \operatorname{cosec}^2 2n\theta$$

$$= 48n^4 \operatorname{cosec}^4 2n\theta - 32n^4 \operatorname{cosec}^2 2n\theta - 4n^2 \operatorname{cosec}^2 2n\theta;$$

but $\Sigma t_r^2 = \Sigma \sec^2\left[\theta + \frac{(r-1)\pi}{2n}\right] - 2n = 4n^2 \operatorname{cosec}^2 2n\theta - 2n$;

$$\therefore\ \Sigma t_r^4 = 16n^4 \operatorname{cosec}^4 2n\theta - \frac{4n^2}{3}\operatorname{cosec}^2 2n\theta(8n^2+1) - (4n^2 \operatorname{cosec}^2 2n\theta - 2n)$$

$$= 16n^4 \operatorname{cosec}^4 2n\theta - \frac{16n^2}{3}(2n^2+1)\operatorname{cosec}^2 2n\theta + 2n.$$

18. Let $y = \sin^{-1}x$, $z = \cos(\frac{1}{3}\sin^{-1}x) = \cos\frac{1}{3}y$;

$\therefore\ 4z^3 - 3z = \cos y = \pm\sqrt{(1-x^2)}$ $\therefore\ (4z^3-3z)^2 + x^2 - 1 = 0$;

product of roots is $\frac{x^2-1}{4^2}$.

19. Use $\operatorname{cosec}^2\alpha = 1 + \cot^2\alpha$ and the method of No. 22.

Or, By Ex. IX. e, No. 25,

$\sin\frac{r\pi}{n}$ satisfies $ns - \frac{n(n^2-1)}{3!}s^3 + \ldots + (-1)^{\frac{n-1}{2}} 2^{n-1}s^n = 0$

for $r = 0, \pm 1, \pm 2, \ldots \pm\frac{n-1}{2}$; dividing by s and putting $x = \frac{1}{s^2}$ we get $nx^{\frac{n-1}{2}} - \frac{n(n^2-1)}{3!}x^{\frac{n-3}{2}} + \ldots = 0$, which has

roots $\operatorname{cosec}^2\frac{r\pi}{n}$ for $r = 1$ to $\frac{n-1}{2}$; thus $\Sigma \operatorname{cosec}^2\frac{r\pi}{n}$ for these values $= \frac{n^2-1}{6}$. Since $\operatorname{cosec}^2\frac{r\pi}{n} = \operatorname{cosec}^2\frac{(n-r)\pi}{n}$, required sum is twice $\frac{n^2-1}{6}$.

20. $2\sin^2\theta = 1 - \cos 2\theta$; $\therefore\ 4\sin^4\theta = 1 - 2\cos 2\theta + \cos^2 2\theta$;

$$\therefore\ 8\sin^4\theta = 2 - 4\cos 2\theta + (1 + \cos 4\theta);$$

$$\therefore\ 8\Sigma\sin^4\theta = 3n - 4\Sigma\cos 2\theta + \Sigma\cos 4\theta;$$

by eqn. (11) of Ch. VII,

$$\sum_1^n \cos\frac{r\pi}{n} = \cos\left(\frac{\pi}{2} + \frac{\pi}{2n}\right)\sin\frac{\pi}{2}\operatorname{cosec}\frac{\pi}{2n} = -1, \text{ and}$$

$$\sum_1^n \cos\frac{2r\pi}{n} = \cos\left(\pi + \frac{\pi}{n}\right)\sin\pi \operatorname{cosec}\frac{\pi}{n} = 0;$$

hence required sum $= \frac{3n}{8} - \frac{4}{8} . (-1)$.

21. By Ex. IX. e, No. 22,

$$\cos 2n\theta = 1 - \frac{(2n)^2}{2!}s^2 + \ldots + (-1)^n 2^{2n-1} . s^{2n},$$

where $s = \sin\theta$; now

$\cos 2n\theta = 0$ for $\theta = \frac{(2r-1)\pi}{4n}$, so $\sin^2\frac{(2r-1)\pi}{4n}$

for $r = 1$ to n are the roots of $1 - \frac{(2n)^2}{2!}x + \ldots = 0$. Required sum is sum of reciprocals of roots $= \frac{(2n)^2}{2!}$.

Or, use

$\operatorname{cosec}^2\alpha = 1 + \cot^2\alpha$, and in $\cot 2n\theta = 0$, put $\tan^2\theta = x$.

22. $\theta = \frac{r\pi}{n}$ satisfies $\tan n\theta = 0$, and

$$\therefore\ \tan\frac{r\pi}{n} \text{ satisfies } nt - \binom{n}{3}t^3 + \ldots + (-1)^{\frac{1}{2}(n-1)}t^n = 0,$$

for $r = 0, \pm 1, \pm 2, \ldots \pm\frac{1}{2}(n-1)$. Divide by t and put x for t^2; then $\tan^2\frac{r\pi}{n}$ for $r = 1, 2, \ldots \frac{1}{2}(n-1)$ satisfies

$$n - \binom{n}{3}x + \ldots x^{\frac{1}{2}(n-1)} = 0; \ \therefore\ \Sigma\cot^2\frac{r\pi}{n} = \binom{n}{3} \div n.$$

23. From the eqn. in No. 22, $\tan^2\frac{r\pi}{n}$ for $r=1$ to $\frac{n-1}{2}$ satisfies

$\binom{n}{1}-\binom{n}{3}x+\dots+(-1)^{\frac{n-1}{2}}x^{\frac{n-1}{2}}=0$; the sum of the squares of the reciprocals of the roots is

$$\left\{\frac{\binom{n}{3}}{\binom{n}{1}}\right\}^2-2\frac{\binom{n}{5}}{\binom{n}{1}}=\frac{(n-1)^2(n-2)^2}{36}-\frac{(n-1)(n-2)(n-3)(n-4)}{60}$$

$$=\frac{(n-1)(n-2)}{180}\{5(n-1)(n-2)-3(n-3)(n-4)\}$$

$$=\frac{(n-1)(n-2)}{90}(n^2+3n-13).$$

24. By eqn. (11), of Ch. VII,

$$\sum_{r=1}^{2n}\cos\frac{rs\pi}{n}=\frac{\cos\frac{(2n+1)s\pi}{2n}\cdot\sin s\pi}{\sin\frac{s\pi}{2n}}=0.$$

Ex. IX. b, No. 15 gives

$$\left(2\cos\frac{r\pi}{n}\right)^{2n}=2\cos\left(2n\cdot\frac{r\pi}{n}\right)+\sum+\frac{(2n)!}{(n!)^2},$$

where Σ stands for a series of terms of the form

$$\binom{2n}{s}2\cos\left\{(2n-2s)\frac{r\pi}{n}\right\}$$

and s goes from 1 to $n-1$. Taking $r=1, 2, \dots 2n$ and adding the results, the sums of terms $\cos\left\{(2n-2s)\frac{r\pi}{n}\right\}$ are zero by the first part, hence

$$\sum_{r=1}^{2n}\left(2\cos\frac{r\pi}{n}\right)^{2n}=2\sum_{r=1}^{2n}\cos 2r\pi+\sum_{r=1}^{2n}\frac{(2n)!}{(n!)^2}$$

$$=2\cdot 2n+2n\frac{(2n)!}{(n!)^2}=4n\left\{1+\frac{(2n)!}{2(n!)^2}\right\}.$$

25. (i) $$\frac{1}{n^4(n+1)^4}=\left(\frac{1}{n}-\frac{1}{n+1}\right)^4$$

$$=\frac{1}{n^4}-\frac{4}{n^3(n+1)}+\frac{6}{n^2(n+1)^2}-\frac{4}{n(n+1)^3}+\frac{1}{(n+1)^4}$$

$$=\frac{1}{n^4}+\frac{1}{(n+1)^4}-\frac{4}{n(n+1)}\left\{\frac{1}{n^2}+\frac{1}{(n+1)^2}\right\}+\frac{6}{n^2(n+1)^2}$$

$$=\frac{1}{n^4}+\frac{1}{(n+1)^4}+\frac{6}{n^2(n+1)^2}-\frac{4}{n(n+1)}\left\{\left(\frac{1}{n}-\frac{1}{n+1}\right)^2+\frac{2}{n(n+1)}\right\}$$

$$=\frac{1}{n^4}+\frac{1}{(n+1)^4}-\frac{2}{n^2(n+1)^2}-\frac{4}{n^3(n+1)^3};$$

but $\sum\frac{1}{n^4}=\frac{\pi^4}{90}$ by Ex. XI. b, No. 21,

$\sum\frac{1}{n^2(n+1)^2}=\frac{\pi^2}{3}-3$ by XI. b, No. 19, and

$\sum\frac{1}{n^3(n+1)^3}=10-\pi^2$ by XI. b, No. 20,

thus the sum $=\frac{2\pi^4}{90}-1-2\left(\frac{\pi^2}{3}-3\right)-4(10-\pi^2)$;

(ii) As in part (i) it can be shown that

$$\frac{1}{n^5(n+1)^5}=\frac{1}{n^5}-\frac{1}{(n+1)^5}-\frac{5}{n^3(n+1)^3}-\frac{5}{n^4(n+1)^4},$$

also $\sum\left(\frac{1}{n^5}-\frac{1}{(n+1)^5}\right)=1$, hence, using previous results, sum $=1-5(10-\pi^2)-5\left(\frac{\pi^4}{45}+\frac{10\pi^2}{3}-35\right)$.

26. (i) Compare Example 7. From the equation

$$x^{\frac{n-1}{2}}-\frac{n^2-1}{3!}x^{\frac{n-3}{2}}+\frac{(n^2-1)(n^2-9)}{5!}x^{\frac{n-5}{2}}-\frac{(n^2-1)(n^2-9)(n^2-25)}{7!}x^{\frac{n-7}{2}}+\dots=0$$

whose roots are $\operatorname{cosec}^2\frac{r\pi}{n}$, the sum of the cubes of the roots is found, by Newton's equations, to be $\frac{n^6+\dots}{945}$. Hence, using the method of Ex. XI. b, No. 21, with the inequalities $\frac{1}{\phi^6}<\operatorname{cosec}^6\phi<\left(1+\frac{1}{\phi^2}\right)^3$, the limit of $\sum_1^{\frac{1}{2}(n-1)}\frac{1}{r^6}$ is found to be $\frac{\pi^6}{945}$;

(ii) If $x-y=xy$, $x^2+y^2=x^2y^2+2xy$;

$\therefore\ x^4+y^4=(x^2y^2+2xy)^2-2x^2y^2=x^4y^4+4x^3y^3+2x^2y^2$;

$\therefore\ x^6+y^6\equiv(x^2+y^2)(x^4-x^2y^2+y^4)$

$=(x^2y^2+2xy)(x^4y^4+4x^3y^3+x^2y^2)$

$=x^6y^6+6x^5y^5+9x^4y^4+2x^3y^3.$

Put $x=\frac{1}{n}$, $y=\frac{1}{n+1}$, thus

$$\frac{1}{n^6(n+1)^6}=\frac{1}{n^6}+\frac{1}{(n+1)^6}-\frac{2}{n^3(n+1)^3}-\frac{9}{n^4(n+1)^4}-\frac{6}{n^5(n+1)^5}$$

and use results of No. 25 and of XI. b, No. 20;

(iii) $$\frac{1}{n^7(n+1)^7}=\frac{1}{n^7}-\frac{1}{(n+1)^7}-\frac{7}{n^6(n+1)^6}-\frac{14}{n^5(n+1)^5}-\frac{7}{n^4(n+1)^4}$$

Proceed as in (ii) and No. 25.

27. $2c^2=2a^2\cos^2\theta+2b^2\sin^2\theta+4ab\sin\theta\cos\theta$

$=a^2(1+\cos 2\theta)+b^2(1-\cos 2\theta)+2ab\sin 2\theta$;

∴ $(2c^2-a^2-b^2)\sec 2\theta=a^2-b^2+2ab\tan 2\theta$;

squaring $(2c^2-a^2-b^2)^2(1+\tan^2 2\theta)=(a^2-b^2+2ab\tan 2\theta)^2$, a quadratic for $\tan 2\theta$, the sum of whose roots is

$$4ab(a^2-b^2)/\{(2c^2-a^2-b^2)^2-(2ab)^2\}.$$

28. $c=a(4\cos^3\theta-3\cos\theta)+b(3\sin\theta-4\sin^3\theta)$

$=a\cos\theta(1-4\sin^2\theta)+b(3\sin\theta-4\sin^3\theta)$;

∴ $(c-3b\sin\theta+4b\sin^3\theta)^2=a^2(1-\sin^2\theta)(1-4\sin^2\theta)^2$;

∴ cosec θ_r, $r=1$ to 6, are the roots of

$$\left(c-\frac{3b}{x}+\frac{4b}{x^3}\right)^2+a^2\left(1-\frac{1}{x^2}\right)\left(1-\frac{4}{x^2}\right)^2=0,$$

or $(c^2-a^2)x^6-6bc\,x^5+\dots=0$; Σ cosec θ = sum of roots.

29. $\sec(\theta-\beta)=\cot(\alpha+\beta)-\tan(\theta-\alpha)$

$$=\frac{\cos\{(\alpha+\beta)+(\theta-\alpha)\}}{\sin(\alpha+\beta)\cos(\theta-\alpha)}=\frac{\cos(\theta+\beta)}{\sin(\alpha+\beta)\cos(\theta-\alpha)};$$

∴ $\cos(\theta-\beta)\cos(\theta+\beta)=\cos(\theta-\alpha)\sin(\alpha+\beta)$;

∴ $\cos 2\theta+\cos 2\beta=2\sin(\alpha+\beta)(\cos\theta\cos\alpha+\sin\theta\sin\alpha)$;

put $\tan\frac{1}{2}\theta=t$, then, as in Example 9,

$$\sin\theta=\frac{2t}{1+t^2},\quad \cos\theta=\frac{1-t^2}{1+t^2},\quad\text{and}\quad \cos 2\theta=\frac{1-6t^2+t^4}{(1+t^2)^2};$$

∴ $\tan\frac{1}{2}\theta_1$, etc., are the roots of

$1-6t^2+t^4+\cos 2\beta(1+t^2)^2$

$=2(1+t^2)\sin(\alpha+\beta)\{(1-t^2)\cos\alpha+2t\sin\alpha\}$;

the coeffs. of t^3, t in this quartic are equal;

∴ $\Sigma(\tan\frac{1}{2}\theta_1)=\Sigma(\tan\frac{1}{2}\theta_1\tan\frac{1}{2}\theta_2\tan\frac{1}{2}\theta_3)$;

∴ $\tan\Sigma(\frac{1}{2}\theta)=0$.

30. (i) Use the same method as for XI. c, Nos. 17, 18.

$\theta=\beta$, $\theta=\gamma$ satisfy $a\cos(\theta-\alpha)+b(\cos\theta+\cos\alpha)+c=0$;

put $\tan\frac{\theta}{2}=t$; then eqn. becomes

$$t^2[(b-a)\cos\alpha+c-b]+2a\sin\alpha\,.\,t+[(b+a)\cos\alpha+c+b]=0;$$

$$\therefore\ \tan\frac{\beta+\gamma}{2}=\frac{t_1+t_2}{1-t_1t_2}=\frac{a\sin\alpha}{b+a\cos\alpha};$$

similarly, $\tan\frac{\alpha+\gamma}{2}=\frac{a\sin\beta}{b+a\cos\beta}$;

$$\therefore\ \tan\frac{\alpha-\beta}{2}=\tan\left[\frac{\alpha+\gamma}{2}-\frac{\beta+\gamma}{2}\right]$$

$$=\frac{a\sin\beta(b+a\cos\alpha)-a\sin\alpha(b+a\cos\beta)}{(b+a\cos\beta)(b+a\cos\alpha)+a^2\sin\alpha\sin\beta}$$

$$=\frac{ab(\sin\beta-\sin\alpha)+a^2\sin(\beta-\alpha)}{b^2+ab(\cos\alpha+\cos\beta)+a^2\cos(\alpha-\beta)};$$

divide by $\sin\frac{\alpha-\beta}{2}$, which $\neq 0$;

$b^2+ab(\cos\alpha+\cos\beta)+a^2\cos(\alpha-\beta)$

$$=-\cos\frac{\alpha-\beta}{2}\left\{2ab\cos\frac{\alpha+\beta}{2}+2a^2\cos\frac{\beta-\alpha}{2}\right\}$$

$=-ab(\cos\alpha+\cos\beta)-a^2[1+\cos(\alpha-\beta)]$;

∴ $2a^2\cos(\alpha-\beta)+2ab(\cos\alpha+\cos\beta)+a^2+b^2=0$;

∴ $2a\{a\cos(\alpha-\beta)+b(\cos\alpha+\cos\beta)\}+a^2+b^2=0$;

∴ $2a(-c)+a^2+b^2=0$;

(ii) $(b+a\cos\beta)\sin\frac{\alpha+\gamma}{2}=a\sin\beta\cos\frac{\alpha+\gamma}{2}$;

$$\therefore\ b\sin\frac{\alpha+\gamma}{2}=a\sin\left\{\beta-\frac{\alpha+\gamma}{2}\right\};$$

similarly, $b\sin\frac{\alpha+\beta}{2}=a\sin\left\{\gamma-\frac{\alpha+\beta}{2}\right\}$;

$$\therefore\ \sin\frac{\alpha+\gamma}{2}\sin\left(\gamma-\frac{\alpha+\beta}{2}\right)=\sin\frac{\alpha+\beta}{2}\sin\left(\beta-\frac{\alpha+\gamma}{2}\right);$$

$$\therefore\ \cos\left(\alpha-\frac{\gamma}{2}+\frac{\beta}{2}\right)-\cos\left(\frac{3\gamma}{2}-\frac{\beta}{2}\right)$$

$$=\cos\left(\alpha-\frac{\beta}{2}+\frac{\gamma}{2}\right)-\cos\left(\frac{3\beta}{2}-\frac{\gamma}{2}\right);$$

$$\therefore\ 2\sin\alpha\sin\frac{\gamma-\beta}{2}=\cos\left(\frac{3\gamma}{2}-\frac{\beta}{2}\right)-\cos\left(\frac{3\beta}{2}-\frac{\gamma}{2}\right)$$

$$=2\sin\frac{\gamma+\beta}{2}\sin(\beta-\gamma)$$

$$=-4\sin\frac{\gamma+\beta}{2}\sin\frac{\gamma-\beta}{2}\cos\frac{\gamma-\beta}{2};$$

$$\therefore\ \sin\alpha=-2\sin\frac{\gamma+\beta}{2}\cos\frac{\gamma-\beta}{2}=-\sin\beta-\sin\gamma;$$

(iii) $\Sigma(\cos\alpha)=\cos\alpha+2\cos\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}$

$$=\cos\alpha+2\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}\cot\frac{\beta+\gamma}{2}$$

$$=\cos\alpha+(\sin\beta+\sin\gamma)\cot\frac{\beta+\gamma}{2}$$

$$=\cos\alpha+(-\sin\alpha)\cdot\frac{b+a\cos\alpha}{a\sin\alpha}, \text{ from (i)}, \; =-\frac{b}{a}.$$

CHAPTER XII

EXERCISE XII. a. (p. 220.)

1. As on p. 219, $x^3-1=0$ if $x=\text{cis}\frac{2r\pi}{3}$ for $r=0, \pm 1$.
2. In eqn. (2), put $n=3$.
3. As on pp. 219-220,

$$x^4+1=\left(x-\text{cis}\frac{\pi}{4}\right)\left(x-\text{cis}\frac{-\pi}{4}\right)\left(x-\text{cis}\frac{3\pi}{4}\right)\left(x-\text{cis}\frac{-3\pi}{4}\right)$$

$$=\left(x^2-2x\cos\frac{\pi}{4}+1\right)\left(x^2-2x\cos\frac{3\pi}{4}+1\right).$$

$$x^4+1\equiv(x^2+1)^2-(x\sqrt{2})^2.$$

4. $x^4+x^3+x^2+x+1=\frac{x^5-1}{x-1}=\prod\left(x-\text{cis}\frac{2r\pi}{5}\right)$ for $r=\pm1, \pm2$;

and $\left(x-\text{cis}\frac{2\pi}{5}\right)\left(x-\text{cis}\frac{-2\pi}{5}\right)=x^2-2x\cos\frac{2\pi}{5}+1$

$$=x^2-\tfrac{1}{2}x(\sqrt{5}-1)+1;$$

and similarly for $\left(x-\text{cis}\frac{4\pi}{5}\right)\left(x-\text{cis}\frac{-4\pi}{5}\right)$

$$x^2\left(x^2+x+1+\frac{1}{x}+\frac{1}{x^2}\right)=x^2\{1+y+(y^2-2)\}$$

$$=x^2\{(y+\tfrac{1}{2})^2-\tfrac{5}{4}\}=x^2\left(y+\tfrac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(y+\tfrac{1}{2}-\frac{\sqrt{5}}{2}\right)$$

$$=\left(x^2+1+\tfrac{1}{2}x+\frac{\sqrt{5}}{2}x\right)\left(x^2+1+\tfrac{1}{2}x-\frac{\sqrt{5}}{2}x\right).$$

5. (i) $(x^2+1)(x^4-x^2+1)=(x^2+1)\{(x^2+1)^2-(x\sqrt{3})^2\}$

Or the method used for eqn. (3);

(ii) $x^8=-y^8=y^8\,\text{cis}\,(2r-1)\pi$; $\therefore$ when $x=y\,\text{cis}\,\frac{(2r-1)\pi}{8}$;

factors are $\left(x-y\,\text{cis}\frac{k\pi}{8}\right)\left(x-y\,\text{cis}\frac{-k\pi}{8}\right)$ for $k=1, 3, 5, 7$ or $x^2-2xy\cos\frac{k\pi}{8}+y^2$;

(iii) $(x^3-a^3)(x^3+a^3)$. **Or,** use eqn. (1), writing $\frac{x}{a}$ for x;

(iv) $(x^4-16)(x^4+16)=(x^2-4)(x^2+4)\{(x^2+4)^2-(2x\sqrt{2})^2\}$.

Or, use eqn. (1), writing $\frac{x}{2}$ for x.

6. Expression $=\frac{x^{15}+1}{x^5+1}=$ by eqn. (4),

$$\prod_1^7\left(x^2-2x\cos\frac{(2r-1)\pi}{15}+1\right)\div\prod_1^2\left(x^2-2x\cos\frac{(2s-1)\pi}{5}+1\right)$$

$$=\prod\left(x^2-2x\cos\frac{(2r-1)\pi}{15}+1\right)$$

for $r=1, 3, 4, 6, 7$, since $\cos\frac{(2s-1)\pi}{5}$ for $s=1, 2$ is the same as $\cos\frac{(2r-1)\pi}{15}$ for $r=2, 5$.

Compare also the method of eqn. (10);

$$x^{10}-x^5+1\equiv x^{10}-2x^5\cos\frac{\pi}{3}+1$$

$$\equiv\prod_0^4\left(x^2-2x\cos\frac{(6r+1)\pi}{15}+1\right).$$

7. (i) By Ch. IX, eqn. (12), there are not more than n values. $n\theta=(2r+1)\frac{\pi}{2}$; $\therefore$ values are those of $\cos(2r+1)\frac{\pi}{2n}$, and $r=0, 1, 2, \dots n-1$ all give different values because $(2n-1)\frac{\pi}{2n}<\pi$.

(ii) As in (i), but $\cos\theta=\cos\frac{2r\pi}{n}$. For n even, $r=0, 1, 2, \dots, \frac{n}{2}$; of these, $r=1$ to $\frac{n}{2}-1$ give repeated roots of the equation in Ch. IX, because $1-\cos n\theta$ equals $2\sin^2\frac{1}{2}n\theta$.

For n odd, $r=0, 1, 2 \dots \frac{n-1}{2}$; of these $r=1$ to $\frac{n-1}{2}$ give repeated roots because

$$\frac{\cos n\theta - 1}{\cos\theta - 1} = \frac{\sin^2\frac{1}{2}n\theta}{\sin^2\frac{1}{2}\theta} = \left\{\frac{\sin\frac{1}{2}(n+1)\theta + \sin\frac{1}{2}(n-1)\theta}{\sin\theta}\right\}^2.$$

(iii) As in (i) and (ii), but $\cos\theta = \cos\frac{(2r-1)\pi}{n}$.

For n even, $r=1, 2, \dots, \frac{n}{2}$; all of these given repeated roots. For n odd, $r=1, 2, \dots, \frac{n+1}{2}$; of these, $r=1$ to $\frac{n-1}{2}$ give repeated roots.

(iv) As in (ii), but $n\theta = 2r\pi \pm n\alpha$; $\therefore \cos\theta = \cos\left(\frac{2r\pi}{n} \pm \alpha\right)$. But $\cos\theta = \cos\left(\frac{2r\pi}{n} + \alpha\right)$ for $r=0, 1, 2, \dots (n-1)$ give n different values, unless $\left(\frac{2r_1\pi}{n} + \alpha\right) + \left(\frac{2r_2\pi}{n} + \alpha\right) = 2k\pi$, which requires $\alpha = \frac{s\pi}{n}$, i.e. $\cos n\alpha = \pm 1$; also there cannot be more than n values; $\therefore$ there are n distinct roots, $\cos\left(\frac{2r\pi}{n} + \alpha\right)$ where $r=0, 1, \dots (n-1)$.

(v) By Ch. IX., eqn. (13), there are not more than $(n-1)$ values. Numerator is zero for $n\theta = r\pi$; $\therefore \cos\theta = \cos\frac{r\pi}{n}$ for $r=0$ to $n-1$; $r=0$ makes the denominator zero.

8. (i) By IX. e, No. 18, there are not more than n values. $n\theta = (2r+1)\frac{\pi}{2}$; $\therefore$ values are those of $\sin(2r+1)\frac{\pi}{2n}$, and $2r+1 = \pm 1, \pm 3, \dots, \pm(n-1)$ gives n different values because $(n-1)\frac{\pi}{2n} < \frac{\pi}{2}$;

(ii) By IX. e, No. 20, there are not more than $(n-2)$ values. Numerator is zero for $n\theta = r\pi$; $\therefore \sin\theta = \sin\frac{r\pi}{n}$ for $r=0, \pm 1, \pm 2, \dots, \pm\frac{1}{2}n$, but $r=0$ or $\pm\frac{1}{2}n$ makes the denominator zero.

9. (i) As in No. 8 (ii), $\sin\theta = \sin\frac{r\pi}{n}$; $r=0, \pm 1, \pm 2, \dots, \pm\frac{1}{2}(n-1)$ gives n distinct values;

(ii) As in No. 8 (i), $\sin\theta = \sin(2r+1)\frac{\pi}{2n}$;

$2r+1 = \pm 1, \pm 3, \dots, \pm n$ gives $n+1$ values, but $r = \pm n$ makes the denominator zero.

10. $(x^n - \cos n\alpha)^2 = -1 + \cos^2 n\alpha = i^2 \sin^2 n\alpha$;

$\therefore x^n = \cos n\alpha \pm i\sin n\alpha = \cos(n\alpha + 2r\pi) \pm i\sin(n\alpha + 2r\pi)$.

11. Put $2n$ for n in eqn. (3). Change x into $\frac{1+x}{1-x}$, then

$$\left(\frac{1+x}{1-x}\right)^{2n} + 1 = \prod_1^n\left\{\left(\frac{1+x}{1-x}\right)^2 - 2\left(\frac{1+x}{1-x}\right)\cos\frac{(2r-1)\pi}{2n} + 1\right\};$$

multiply by $(1-x)^{2n}$, thus factors are

$$\prod_1^n\left\{(1+x)^2 - 2(1-x^2)\cos\frac{(2r-1)\pi}{2n} + (1-x)^2\right\}$$
$$= 2^n\prod_1^n\left\{1 + x^2 - (1-x^2)\cos\frac{(2r-1)\pi}{2n}\right\}$$
$$= 2^n\prod_1^n\left\{2\sin^2\frac{(2r-1)\pi}{4n} + x^2 \cdot 2\cos^2\frac{(2r-1)\pi}{4n}\right\}$$
$$= \mathsf{A} \cdot \prod_1^n\left\{x^2 + \tan^2\frac{(2r-1)\pi}{4n}\right\}$$

where A is independent of x. By comparing coefficients of x^{2n} it follows that $\mathsf{A} = 2$. (Cf. No. 12.)

12. Put $x = i\tan\theta$, then $(\cos\theta + i\sin\theta)^{2n} + (\cos\theta - i\sin\theta)^{2n} = 0$;

$\therefore 2\cos 2n\theta = 0$; $\therefore \theta = \frac{(2r-1)\pi}{4n}$; the different values of $i\tan\theta$ are given by $2r-1 = \pm 1, \pm 3, \dots, \pm(2n-1)$ and are the values of $\pm i\tan\theta$ for $r=1, 2, \dots, n$. Hence by the argument on p. 219,

exprn. $= p_0\prod(x - i\tan\theta)(x + i\tan\theta) = p_0\prod(x^2 + \tan^2\theta)$; equating coeffs. of x^{2n} gives $2 = p_0$; equating coeffs. of x^{2n-2} gives $2 \cdot \binom{2n}{2} = p_0\Sigma(\tan^2\theta)$;

$$\therefore \Sigma(\tan^2\theta) = \frac{2}{p_0}\,\frac{2n(2n-1)}{2!} = n(2n-1),$$

and $\sec^2\theta = 1 + \tan^2\theta$.

13. $u_{n+1} + a^2x^2u_{n-1} = x^{2n+2} + a^{2n+2} + a^2x^2(x^{2n-2} + a^{2n-2})$
$\qquad - 2x^{n+1}a^{n+1}\{\cos(n+1)\theta + \cos(n-1)\theta\}$
$= (x^{2n} + a^{2n})(x^2 + a^2) - 4x^{n+1}a^{n+1}\cos n\theta\cos\theta$
$= (x^{2n} + a^{2n})(u_1 + 2ax\cos\theta) - 4x^{n+1}a^{n+1}\cos n\theta\cos\theta$
$= (x^{2n} + a^{2n})u_1 + 2ax\cos\theta \cdot u_n$.

$u_2 \equiv x^4 - 2x^2a^2 \cos 2\theta + a^4$
$= (x^2+a^2)^2 - 2x^2a^2(1+\cos 2\theta) = (x^2+a^2)^2 - (2xa\cos\theta)^2$;
$\therefore$ u_2 has u_1 for a factor, hence the first part proves that u_1 is a factor of u_3, then of u_4, etc.
$u_n \equiv x^{2n} - 2x^n a^n \cos n\phi + a^{2n}$, where $\phi = \theta + \frac{2r\pi}{n}$, and therefore has $x^2 - 2ax\cos\phi + a^2$ for a factor.

14. By No. 13, for $a=1$, $\theta=0$, $x^2 - 2x\cos\frac{2r\pi}{n} + 1$ is a factor of $x^{2n} - 2x^n + 1 \equiv (x^n-1)^2$, where r is any integer or zero. For $r=0$, this gives $(x-1)^2$ as a factor. For $r=\frac{n}{2}$, it gives $(x+1)^2$ as a factor. For other integral values of r *between* 0 and $\frac{n}{2}$, $x^2 - 2x\cos\frac{2r\pi}{n}+1$ has no factors in real algebra and so, being a factor of $(x^n-1)^2$, must be also a factor of $x^n - 1$; $\therefore$ $x^n - 1 \equiv \mathsf{A}(x+1)(x-1)\prod\left(x^2 - 2x\cos\frac{2r\pi}{n}+1\right)$ for $r=1$ to $\frac{1}{2}n-1$; equating coeffs. of x^n, we see that $1=\mathsf{A}$.

EXERCISE XII. b. (p. 223.)

1. (i) and (ii) See Nos. 8, 9 below;
(iii) A polynomial of degree 5, zero when $5\theta = n\pi + (-1)^n 5a$; $\therefore$ factors are

$$\mathsf{A}\cdot\prod\left\{\sin\theta - \sin\left(\frac{n\pi}{5} + (-1)^n a\right)\right\} \text{ for } n=0, \pm 1, \pm 2;$$

A = coeff. of $\sin^5\theta = 2^4$ by Ex. IX. e, No. 25.

2. See Nos. 4, 3 below.

3. By Ex. IX. e, No. 12, $\cos n\theta - \cos na$ is a polynomial in $\cos\theta$, with $2^{n-1}\cos^n\theta$ for term of highest degree; it is zero when $\cos\theta = \cos\left(\frac{2r\pi}{n} + a\right)$ for $r=0$ to $n-1$ by Ex. XII. a, No. 7 (iv), $r=0$, $r=n$ give the same factor.

4. Method of No. 3, using XII. a, No. 7 (i).

5. By IX. e, No. 12, $\frac{\cos(2n+1)\theta}{\cos\theta}$ = polynomial of degree n in $\cos^2\theta$, which, as in No. 4, vanishes for $\cos\theta = \cos\frac{(2r-1)\pi}{2(2n+1)}$, for $r=1$ to n, and $\therefore$ also vanishes for $\cos\theta = -\cos\frac{(2r-1)\pi}{2(2n+1)}$,

so that $\cos^2\theta - \cos^2\frac{2(r-1)\pi}{2(2n+1)}$ is a factor; also the coeff. of $\cos^{2n+1}\theta$ in $\cos(2n+1)\theta$ is 2^{2n}; $\therefore$ result follows as in No. 3. Further $\cos^2\theta - \cos^2 a \equiv \sin^2 a - \sin^2\theta$.

6. As in No. 5; $\cos 2n\theta$ is a polynomial of degree n in $\cos^2\theta$ which vanishes for $\cos\theta = \cos\frac{(2r-1)\pi}{4n}$ and $\therefore$ for

$$\cos\theta = -\cos\frac{(2r-1)\pi}{4n}; \text{ etc.}$$

7. By Ex. IX. e, No. 13, $\frac{\sin n\theta}{\sin\theta}$ is a polynomial in $\cos\theta$ with $2^{n-1}\cos^{n-1}\theta$ for term of highest degree; it is zero when $\cos\theta = \cos\frac{r\pi}{n}$ for $r=1$ to $n-1$ by Ex. XII. a, No. 7 (v).

8. Use No. 7; this gives factors $\left(\cos\theta - \cos\frac{r\pi}{2n+1}\right)$ for $r=1$ to $2n$, and the values k, $2n+1-k$ of r give

$$\left(\cos\theta - \cos\frac{k\pi}{2n+1}\right)\left(\cos\theta + \cos\frac{k\pi}{2n+1}\right),$$

since $$\frac{k\pi}{(2n+1)} + \frac{(2n+1-k)\pi}{(2n+1)} = \pi,$$

and this $= \cos^2\theta - \cos^2\frac{k\pi}{2n+1} = \sin^2\frac{k\pi}{2n+1} - \sin^2\theta$.

Or, $\frac{\sin(2n+1)\theta}{\sin\theta}$ is a polynomial of degree n in $\cos^2\theta$, and use the method of No. 5.

9. Use No. 7; this gives factors $\left(\cos\theta - \cos\frac{r\pi}{2n}\right)$ for $r=1$ to $2n-1$; the value $r=n$ gives a factor $\cos\theta$, and values k, $2n-k$ of r give factors of the form

$$(\cos\theta - \cos a)(\cos\theta + \cos a)$$

as in No. 8; hence

$$\sin 2n\theta = 2^{2n-1}\sin\theta\cos\theta\prod_1^{n-1}\left(\cos^2\theta - \cos^2\frac{r\pi}{2n}\right)$$

and $2\sin\theta\cos\theta = \sin 2\theta$.

10. By No. 5, expression $= \mathsf{A}\prod_1^{\frac{1}{2}(n-1)}\left\{1 - \sin^2\theta\,\text{cosec}^2\frac{(2r-1)\pi}{2n}\right\}$, where A is independent of θ; put $\theta=0$, then $1=\mathsf{A}$.

11. From No. 6 by the same method as No. 10.

12. From No. 8, $\dfrac{\sin n\theta}{\sin\theta} = A\prod_{1}^{\frac{1}{2}(n-1)}\left(1-\sin^2\theta\,\mathrm{cosec}^2\dfrac{r\pi}{n}\right)$, where A is independent of θ. Make $\theta\to 0$; $\therefore$ $A=\lim\limits_{\theta\to 0}\dfrac{\sin n\theta}{\sin\theta}=n$.

13. From No. 9, as in No. 12, $A=\lim\limits_{\theta\to 0}\dfrac{\sin n\theta}{\sin\theta\cos\theta}=n$.

14. $\dfrac{\cos 3\theta}{\cos\theta}$ and $\dfrac{\sin 4\theta}{\cos\theta}$ are polynomials in $\sin\theta$, of degrees 2, 3; $\therefore$ expression is a polynomial of degree 3, and is zero for $\cos 3\theta=\sin 4\theta=\cos\left(\dfrac{\pi}{2}-4\theta\right)$, whence

$$\theta=(4n+1)\frac{\pi}{14}\ \text{ or }\ (4n+1)\frac{\pi}{2};$$

rejecting $(4n+1)\dfrac{\pi}{2}$, which makes $\cos\theta=0$, factors are $\left\{\sin\theta-\sin(4n+1)\dfrac{\pi}{14}\right\}$, and $n=0$, 1, and -1 give different values;

$\therefore$ expn. $=A\left(\sin\theta-\sin\dfrac{\pi}{14}\right)\left(\sin\theta-\sin\dfrac{5\pi}{14}\right)\left(\sin\theta+\sin\dfrac{3\pi}{14}\right)$

but $\dfrac{\sin 4\theta}{\cos\theta}=2\cos 2\theta\,.\,2\sin\theta=4\sin\theta(1-2\sin^2\theta)$, in which the coeff. of $\sin^3\theta$ is -8; $\therefore$ $A=+8$.

15. By Ch. IX, eqn. (13), expression is only a polynomial in $\sin\theta$ if n is odd; in that case, as in No. 1, factors are $A\,.\prod\left\{\sin\theta-\sin\left[\dfrac{r\pi}{n}+(-1)^r a\right]\right\}$ for

$$r=0,\ \pm1,\ \pm2,\ \ldots\pm\frac{n-1}{2},$$

and $A=$ coeff. of $\sin^n\theta=(-1)^{\frac{n-1}{2}}\,.\,2^{n-1}$ by Ex. IX. e, No. 25.

16. Put $a=\dfrac{\pi}{2n}$; No. 3 becomes

$$\cos n\theta=2^{n-1}\prod_{1}^{n}\left\{\cos\theta-\cos(4r+1)\frac{\pi}{2n}\right\};$$

in this product, such of the values of $4r+1$ as are between $2n$ and $4n$ may be replaced by the corresponding values of $4n-4r-1$, since $(4r+1)\dfrac{\pi}{2n}+(4n-4r-1)\dfrac{\pi}{2n}=2\pi$, and the value $4n+1$ may be replaced by 1; the values are then 1, 3, 5, ... $(2n-1)$ as in No. 4.

17. For n odd, $[\frac{1}{2}n]=\frac{1}{2}(n-1)=[\frac{1}{2}(n-1)]$; also

$$\cos n\left(\frac{\pi}{2}-\theta\right)=\sin\frac{n\pi}{2}\,.\,\sin n\theta=(-1)^{\frac{1}{2}(n-1)}.\sin n\theta,$$

and $\sin n\left(\dfrac{\pi}{2}-\theta\right)=\sin\dfrac{n\pi}{2}\cos n\theta=(-1)^{\frac{1}{2}(n-1)}.\cos n\theta.$

For n even, $[\frac{1}{2}n]=\frac{1}{2}n$ and $[\frac{1}{2}(n-1)]=\frac{1}{2}n-1$; also

$$\cos n\left(\frac{\pi}{2}-\theta\right)=\cos\frac{n\pi}{2}\cos n\theta=(-1)^{\frac{n}{2}}\cos n\theta,$$

and $\sin n\left(\dfrac{\pi}{2}-\theta\right)=-\cos\dfrac{n\pi}{2}\sin n\theta=-(-1)^{\frac{1}{2}n}.\sin n\theta.$

Hence first expression $=$ r.h.s. of No. 10 if n is odd and of No. 11 if n is even, and $\therefore$ $=(-1)^{\frac{1}{2}(n-1)}\dfrac{\sin n\theta}{\sin\theta}$ and $(-1)^{\frac{n}{2}}\cos n\theta$ in the two cases; also second expn. $=$ r.h.s. of No. 12 if n is odd and of No. 13 if n is even, and $\therefore$ $=(-1)^{\frac{1}{2}(n-1)}\cos n\theta/n\cos\theta$

and $-(-1)^{\frac{1}{2}n}\sin n\theta/(n\cos\theta\sin\theta)$.

18. For n odd, from No. 12, $\prod\limits_{1}^{\frac{1}{2}(n-1)}\left(1-\dfrac{\sin^2\theta}{\sin^2\frac{r\pi}{n}}\right)=\dfrac{1}{n}\dfrac{\sin n\theta}{\sin\theta}=$, from No. 8, $\dfrac{1}{n}\cdot 2^{n-1}\prod\limits_{1}^{\frac{1}{2}(n-1)}\left(\sin^2\dfrac{r\pi}{n}-\sin^2\theta\right)$. Make $\theta\to 0$ and take the square root; ambiguous sign is + because $\dfrac{r\pi}{n}<\pi$; also $\frac{1}{2}(n-1)=[\frac{1}{2}(n-1)]$. For n even, from No. 13,

$$\prod_{1}^{\frac{1}{2}n-1}\left(1-\frac{\sin^2\theta}{\sin^2\frac{r\pi}{n}}\right)=\frac{1}{n}\,\frac{\sin n\theta}{\sin\theta\cos\theta}$$

$=$, from No. 9, $\dfrac{1}{n}\cdot 2^{n-1}\prod\limits_{1}^{\frac{1}{2}n-1}\left(\sin^2\dfrac{r\pi}{n}-\sin^2\theta\right)$. Make $\theta\to 0$, etc. Here $\frac{1}{2}n-1=[\frac{1}{2}(n-1)]$. The result can also be deduced from eqn. (12); divide each side by $\sin\beta$, make $\beta\to 0$, and take the square root.

19. Method of No. 18. For n odd, compare the form in No. 10 with that in No. 5. For n even, compare No. 11 and No. 6. The result can be deduced from XII. c. No. 1, by putting $\beta=0$.

20. In No. 4 put $\theta=0$;

$$\text{r.h.s.}=2^{n-1}\prod_1^n\left\{1-\cos\frac{(2r-1)\pi}{2n}\right\}=2^{2n-1}\prod_1^n\sin^2\frac{(2r-1)\pi}{4n};$$

take the square root, all the angles are $<\pi$ and $\therefore$ have positive sines.

21. In No. 7 divide by $\sin\theta$ and make $\theta\to 0$;

$$\therefore\ n=2^{n-1}\prod_1^{n-1}\left(1-\cos\frac{r\pi}{n}\right)=2^{2n-2}\prod_1^{n-1}\sin^2\frac{r\pi}{2n},$$

take the square root; all the angles are $<\pi$ and $\therefore$ have positive sines.

22. $\cos\frac{r\pi}{n}=\sin\left(\frac{\pi}{2}-\frac{r\pi}{n}\right)=\sin\frac{n-2r}{2n}\pi=\sin\frac{s\pi}{n}$, where $s=\frac{n}{2}-r$, and $\therefore$ takes the same values, 1 to $\frac{1}{2}n-1$, as r, in the reverse order,

$$\cos\frac{(2r-1)\pi}{2n}=\sin\left(\frac{\pi}{2}-\frac{(2r-1)\pi}{2n}\right)$$
$$=\sin(n-2r+1)\frac{\pi}{2n}=\sin(2s-1)\frac{\pi}{2n},$$

where $s=\frac{n}{2}-r+1$, and $\therefore$ takes the same values, 1 to $\frac{1}{2}n$, as r in the reverse order.

23. As in No. 22, $\cos\frac{(2r-1)\pi}{2n}=\sin(n-2r+1)\frac{\pi}{2n}=\sin\frac{s\pi}{n}$, where $s=\frac{1}{2}(n+1)-r$ and assumes the same values, 1 to $\frac{1}{2}(n-1)$, as r. Also

$$\cos\frac{r\pi}{n}=\sin\frac{n-2r}{2n}\pi=\sin\frac{(2s-1)\pi}{2n},$$

where $s=\frac{1}{2}(n+1)-r$, etc.

24. (i) n odd, by No. 23, expression $=\prod_1^{\frac{1}{2}(n-1)}\sin\frac{(2r-1)\pi}{2n}=$, by No. 19, $2^{\frac{1}{2}(1-n)}$; n even, by No. 22,

$$\text{expression}=\prod_1^{\frac{1}{2}n-1}\sin\frac{r\pi}{n}=\text{, by No. 18, }\sqrt{\{n\,.\,2^{1-n}\}};$$

(ii) As in (i), use Nos. 23, 18 and Nos. 22, 19.

25. $\cos\frac{(2r-1)\pi}{4n}=\sin\left\{\frac{\pi}{2}-\frac{(2r-1)\pi}{4n}\right\}$

$$=\sin(2n-2r+1)\frac{\pi}{4n}=\sin\frac{(2s-1)\pi}{4n}$$

if $s=n-r+1$, and for $r=1$ to n, $s=n$ to 1;

$$\text{also }\cos\frac{r\pi}{2n}=\sin\frac{s\pi}{2n}$$

if $r+s=n$, and for $r=1$ to $n-1$, $s=n-1$ to 1.

26. Put $n=7$ in No. 19. **Or,** roots of $\frac{\cos 7\theta}{\cos\theta}=0$, expressed in terms of $\sin\theta$, are $\pm\sin\frac{\pi}{14}$, $\pm\sin\frac{3\pi}{14}$, $\pm\sin\frac{5\pi}{14}$; use Ch. IX., Ex. IX. e, No. 21, p. 183; then eqn. is $-(2s)^6\ldots+1=0$; take product of roots.

27. Put $n=11$ in No. 24 (i). **Or,** as in No. 26, take the product of the roots of $\frac{\sin 11\theta}{\sin\theta}=0$, expressed in terms of $\cos\theta$.

28. Put $n=7$ in No. 18. **Or,** as in No. 26, take the product of the roots of $\frac{\sin 7\theta}{\sin\theta}=0$, expressed in terms of $\sin\theta$.

29. For n odd, each is zero. For n even, put $\frac{\pi}{2}$ for θ in No. 4. **Or,** find the product of the roots of the eqn. in $\cos\theta$ given by $\cos n\theta=0$.

30. Use Nos. 20 and 25.

31. Put $\theta=\pi$, $a=-\frac{\pi}{2n}$ in No. 3, thus

$$\cos n\pi=2^{n-1}\prod_1^n\left\{-1-\cos(4r-1)\frac{\pi}{2n}\right\}$$
$$=(-1)^n2^{2n-1}\prod_1^n\cos^2(4r-1)\frac{\pi}{4n};$$

$\therefore\ \prod_1^n\cos(4r-1)\frac{\pi}{4n}=\pm\sqrt{(2^{1-2n})}$; now $0<(4r-1)\frac{\pi}{4n}<\pi$, and $(4r-1)\frac{\pi}{4n}>\frac{\pi}{2}$ if $4r>2n+1$, i.e. if $r>\frac{n}{2}$ (n even) and if $r>\frac{n-1}{2}$ (n odd); $\therefore$ for $[\frac{1}{2}(n+1)]$ values of r in both cases.

32. By No. 3,

$$\cos 5\theta+1=\cos 5\theta-\cos 5\pi=2^4\prod_1^5\left\{\cos\theta-\cos\frac{(2r+5)\pi}{5}\right\}$$
$$=16\left(\cos\theta+\cos\frac{2\pi}{5}\right)\left(\cos\theta+\cos\frac{4\pi}{5}\right)$$
$$\times\left(\cos\theta+\cos\frac{6\pi}{5}\right)\left(\cos\theta+\cos\frac{8\pi}{5}\right)(\cos\theta+1).$$

33. By No. 3,

$$\cos 5\theta+\tfrac{1}{2}=\cos 5\theta-\cos\frac{10\pi}{3}=2^4\cdot\prod_1^5\left\{\cos\theta-\cos(3r+5)\frac{2\pi}{15}\right\}$$

$$=16\left(\cos\theta+\cos\frac{\pi}{15}\right)\left(\cos\theta+\cos\frac{7\pi}{15}\right)$$

$$\times\left(\cos\theta-\cos\frac{2\pi}{15}\right)\left(\cos\theta-\cos\frac{4\pi}{15}\right)(\cos\theta+\tfrac{1}{2}).$$

If $\cos\theta=c$, $\cos 5\theta+\frac{1}{2}$ may also be written

$$16c^5-20c^3+5c+\tfrac{1}{2}=(c+\tfrac{1}{2})(16c^4-8c^3-16c^2+8c+1);$$

$\therefore$ expression = sum of squares of reciprocals of roots of

$$16c^4-8c^3-16c^2+8c+1=0;\quad\therefore\quad =8^2-2\,.\,(-16).$$

34. See p. 226, eqn. (10).

35. In the result of No. 34 put $x=e^y$, then

$$e^{2ny}-2e^{ny}\cos n\theta+1=\prod_0^{n-1}\left\{e^{2y}-2e^y\cos\left(\alpha+\frac{2r\pi}{n}\right)+1\right\};$$

divide by e^{ny}, using $e^{ny}+e^{-ny}=2\,\mathrm{ch}(ny)$, $e^y+e^{-y}=2\,\mathrm{ch}\,(y)$,

$$\text{thus}\quad 2\,\mathrm{ch}\,(ny)-2\cos(n\theta)=\prod_0^{n-1}\left\{2\,\mathrm{ch}\,(y)-2\cos\left(\alpha+\frac{2r\pi}{n}\right)\right\}$$

$$=2^n\cdot\prod_0^{n-1}\left\{\mathrm{ch}\,y-\cos\frac{n\alpha+2r\pi}{n}\right\}.$$

36. r.h.s. of No. $3=2^{n-1}\prod_1^n\left\{1-2\sin^2\frac{\theta}{2}-1+2\sin^2\left(\frac{\alpha}{2}+\frac{r\pi}{n}\right)\right\}$;

$$\therefore\quad \cos n\theta-\cos n\alpha=2^{2n-1}\prod\left\{\sin^2\left(\frac{\alpha}{2}+\frac{r\pi}{n}\right)-\sin^2\frac{\theta}{2}\right\},$$

put $\theta=0$, thus $1-\cos n\alpha=2^{2n-1}\prod\left\{\sin^2\left(\frac{\alpha}{2}+\frac{r\pi}{n}\right)\right\}$; divide.

EXERCISE XII. c. (p. 229.)

1. See (iii) on p. 227; put $x=1$, $\alpha=2\beta+\frac{\pi}{n}$.

2 and 3. See (iii) on p. 227; put $x=-1$, $\alpha=2\beta$.

4 and 5. See (iii) on p. 227; put $x=-1$, $\alpha=2\beta+\frac{\pi}{n}$.

6. Change β into $\beta+\frac{\pi}{2}$; $\cos n\beta$ becomes

$$\cos n\beta\cos\frac{n\pi}{2}-\sin n\beta\sin\frac{n\pi}{2},$$

=, for n odd, $+(-1)^{\frac{1}{2}(n+1)}\sin n\beta$, and, for n even, $(-1)^{\frac{1}{2}n}\cos n\beta$. Similarly from formula (12) $\sin n\beta$ becomes $(-1)^{\frac{1}{2}(n-1)}\cos n\beta$ for n odd, and $(-1)^{\frac{1}{2}n}\sin n\beta$ for n even.

7. $\log\cos n\beta=(n-1)\log 2+\Sigma\log\sin\left\{\beta+\frac{(2r+1)\pi}{2n}\right\}$;

$$\therefore\quad \frac{1}{\cos n\beta}(-\sin n\beta)\,.\,(n)=\Sigma\cot\left\{\beta+\frac{(2r+1)\pi}{n}\right\}.$$

8. Put $n=3$, $n=4$ in equation (12).

9. Put $n=3$ in Ex. XII. b, No. 3.

10. Equate the values of $\sin 2n\beta$ found by putting $2n$ for n in equation (12) and No. 3.

11. l.h.s. $=2^n\cdot\prod_0^{n-1}\sin\left(\phi+\frac{r\pi}{n}\right)\cdot\prod_0^{n-1}\cos\left(\phi+\frac{r\pi}{n}\right)$

$$=2\sin n\phi\prod_0^{n-1}\cos\left(\phi+\frac{r\pi}{n}\right)$$

by equation (12); also equation 12, with $\phi+\frac{\pi}{2}$ for β gives $\prod_0^{n-1}\cos\left(\phi+\frac{r\pi}{n}\right)=2^{1-n}\sin n\left(\phi+\frac{\pi}{2}\right)$.

12. See No. 7.

13. Use eqn. 13; differentiate w.r.t. β.

14. $\angle\mathrm{POA}_{r+1}=\theta+\frac{2r\pi}{n}$, $\mathrm{OP}=\mathrm{OA}_{r+1}=a$; $\therefore$ $\mathrm{PA}_{r+1}=2a\sin\left(\frac{\theta}{2}+\frac{r\pi}{n}\right)$;

$$\therefore\ \text{by eqn. 12, product}=(2a)^n\cdot\frac{\sin\frac{n\theta}{2}}{2^{n-1}}.$$

15. If $\mathrm{AOB}_1=2\theta$, $\mathrm{AB}_1\,.\,\mathrm{AB}_2,\ldots,\mathrm{AB}_5=2a^5\sin 5\theta$ by No. 14; for the other positions of A replace 2θ by $2\theta+\frac{\pi}{2}$, $2\theta+\frac{2\pi}{2}$, $2\theta+\frac{3\pi}{2}$; hence continued product

$$=(2a^5)^4\sin 5\theta\sin\left(5\theta+\frac{5\pi}{4}\right)\sin\left(5\theta+\frac{10\pi}{4}\right)\sin\left(5\theta+\frac{15\pi}{4}\right)$$

$$=16a^{20}\sin 5\theta\sin\left(5\theta+\frac{\pi}{4}\right)\sin\left(5\theta+\frac{\pi}{2}\right)\sin\left(5\theta+\frac{3\pi}{4}\right)$$

and use No. 8 (ii) or eqn. (12).

16. Perpendiculars are $a\sin\theta$, $a\sin\left(\theta+\frac{\pi}{n}\right)$, $a\sin\left(\theta+\frac{2\pi}{n}\right)$, ...; product $=a^n\prod_0^{n-1}\sin\left(\theta+\frac{r\pi}{n}\right)$ and use eqn. (12). **Or,** if PM_r

is perpendicular to OA_r, $M_1, M_2, \dots, M_n$ is a regular polygon inscribed in a circle, diameter OP; and arc PM_1 subtends 2θ at centre; ∴ by No. 14, product $=2\left(\frac{a}{2}\right)^n \cdot \sin\frac{n(2\theta)}{2}$.

17. In No. 14, write $2n+1$ for n and take $\theta=\frac{\pi}{2n+1}$; then

$$CA_0 \cdot CA_1, \dots CA_{2n}=2a^{2n+1} \cdot \sin\tfrac{1}{2}(2n+1)\cdot\frac{\pi}{2n+1}=2a^{2n+1};$$

but $CA_0=2a$, $CA_1=CA_{2n}$, $CA_2=CA_{2n+1}$, etc.;

$$\therefore (CA_1 \cdot CA_2, \dots CA_n)^2 \cdot 2a=2a^{2n+1}.$$

Or, from Cotes' second property, p. 228, when $x=a$, $CA_0 \cdot CA_1, \dots CA_{2n}=2a^{2n+1}$, etc.

18. In No. 14, divide by PA_1 and make $P\to A_1$; ∴ $\theta\to 0$, thus

$$A_1A_2 \cdot A_1A_3, \dots A_1A_n=\lim_{\theta\to 0}\frac{2a^n\sin\frac{n\theta}{2}}{2a\sin\frac{\theta}{2}}=na^{n-1};$$

similarly, make $P\to A_2, A_3, \dots A_n$; each product is na^{n-1}; multiply and take square root as each chord occurs twice; continued product $=\sqrt{\{n^n \cdot a^{n(n-1)}\}}$.

19. Let BC be diameter; if p_r=perpendicular from A_r to BC, $p_r \cdot BC=A_rB \cdot A_rC$. As in No. 17 or by Cotes' second property, $A_1B \cdot A_2B, \dots A_{2n}B=2a^{2n}=A_1C \cdot A_2C, \dots A_{2n}C$;

$$\therefore p_1p_2, \dots p_{2n} \cdot (2a)^{2n}=2a^{2n} \cdot 2a^{2n}.$$

20. From Ex. IX. e, No. 24, and Ex. XII. b, No. 13,

$$1-\frac{n^2-4}{6}s^2+\dots\equiv\prod\left(1-\frac{s^2}{\sin^2\frac{r\pi}{n}}\right);$$

equate coeffs. of s^2.

21. From Ex. IX. e, No. 22, and Ex. XII. b, No. 11,

$$1-\frac{n^2}{2}s^2+\dots=\prod\left\{1-\frac{s^2}{\sin^2\frac{(2r-1)\pi}{2n}}\right\};$$

equate coeffs. of s^2.

22. From Ex. IX. e, No. 23, and Ex. XII. b, No. 10,

$$1-\frac{n^2-1}{2}s^2+\dots=\prod\left\{1-\frac{s^2}{\sin^2\frac{(2r-1)\pi}{2n}}\right\};$$

equate coeffs. of s^2.

23. From Ex. IX. e, No. 25, and Ex. XII. b, No. 12,

$$ns-\frac{n(n^2-1)}{6}s^3+\frac{n(n^2-1)(n^2-9)}{120}s^5-\dots$$
$$\equiv ns\prod\left\{1-\frac{s^2}{\sin^2\frac{r\pi}{n}}\right\};$$

equate coeffs. of s^3, thus $\frac{n^2-1}{6}=\Sigma\operatorname{cosec}^2\frac{r\pi}{n}$, and coeffs. of s^5, thus $\frac{(n^2-1)(n^2-9)}{120}$=sum of products;

$$\Sigma\operatorname{cosec}^4\frac{r\pi}{n}=\left(\Sigma\operatorname{cosec}^2\frac{r\pi}{n}\right)^2-2\,(\text{sum of products})$$
$$=\left(\frac{n^2-1}{6}\right)^2-\tfrac{1}{60}(n^2-1)(n^2-9).$$

See also Ex. XI. b, No. 21.

24. See No. 20; equate coeffs. of s^4, thus sum of products two together $=\frac{(n^2-4)(n^2-16)}{120}$;

$$\text{sum of 4th powers}=\left(\frac{n^2-4}{6}\right)^2-\frac{(n^2-4)(n^2-16)}{60}.$$

25. For n even, see No. 21; equate coeffs. of s^4, thus sum of products two together $=\frac{n^2(n^2-4)}{24}$;

$$\text{sum of 4th powers}=\left(\frac{n^2}{2}\right)^2-\frac{n^2(n^2-4)}{12}.$$

Similarly, for n odd, see No. 22;

$$\text{sum of 4th powers}=\left(\frac{n^2-1}{2}\right)^2-\frac{(n^2-1)(n^2-9)}{12}.$$

EXERCISE XII. d. (p. 234.)

1. Use the method of Example 1.

$(x^5-1)\equiv(x-1)(x-\epsilon)(x-\epsilon^2)(x-\epsilon^3)(x-\epsilon^4)$, where

$$\epsilon=\operatorname{cis}\frac{2\pi}{5},\ \frac{1}{x^5-1}=\frac{A}{x-1}+\frac{B}{x-\epsilon}+\frac{C}{x-\epsilon^2}+\frac{D}{x-\epsilon^3}+\frac{E}{x-\epsilon^4},$$

where $A=\lim_{x\to 1}\frac{x-1}{x^5-1}=\frac{1}{5}$, $B=\lim_{x\to\epsilon}\frac{x-\epsilon}{x^5-1}=\frac{1}{5\epsilon^4}=\frac{\epsilon}{5}$,

$$E=\lim_{x\to\epsilon^4}\frac{x-\epsilon^4}{x^5-1}=\frac{1}{5\epsilon^{16}}=\frac{\epsilon^4}{5};$$

$$\therefore \frac{B}{x-\epsilon}+\frac{E}{x-\epsilon^4}=\frac{(B+E)x-(B\epsilon^4+E\epsilon)}{x^2-x(\epsilon+\epsilon^4)+1};$$

but $B+E=\frac{1}{5}(\epsilon+\epsilon^4)=\frac{1}{5}\left(\operatorname{cis}\frac{2\pi}{5}+\operatorname{cis}\frac{-2\pi}{5}\right)=\frac{2}{5}\cos\frac{2\pi}{5}$ and

$$B\epsilon^4+E\epsilon=\tfrac{1}{5}+\tfrac{1}{5};$$

$$\therefore \frac{B}{x-\epsilon}+\frac{E}{x-\epsilon^4}=\frac{2}{5}\,\frac{x\cos\frac{2\pi}{5}-1}{x^2-2x\cos\frac{2\pi}{5}+1}.$$

Similarly, $$\frac{C}{x-\epsilon^2}+\frac{D}{x-\epsilon^3}=\frac{2}{5}\cdot\frac{x\cos\frac{4\pi}{5}-1}{x^2-2x\cos\frac{4\pi}{5}+1}.$$

Or, $x^5-1=(x-1)\left(x^2-2x\cos\frac{2\pi}{5}+1\right)\left(x^2-2x\cos\frac{4\pi}{5}+1\right);$

$$\therefore \frac{1}{x^5-1}=\frac{A}{x-1}+\frac{Px+Q}{x^2-2x\cos\frac{2\pi}{5}+1}+\frac{Rx+S}{x^2-2x\cos\frac{4\pi}{5}+1},$$

etc., by the ordinary method of real algebra.

2. As in No. 1, $x^8+1=\prod_1^8\left\{x-\operatorname{cis}(2r-1)\frac{\pi}{8}\right\};$

$$\therefore \frac{1}{x^8+1}=\sum_1^8\frac{A_r}{x-\operatorname{cis}(2r-1)\frac{\pi}{8}},$$

where $$A_r=\lim\frac{x-\operatorname{cis}(2r-1)\frac{\pi}{8}}{x^8+1}$$

when $x\to\operatorname{cis}(2r-1)\frac{\pi}{8}$,

$$=\lim\frac{1}{8x^7}=-\tfrac{1}{8}\lim x=-\tfrac{1}{8}\operatorname{cis}(2r-1)\frac{\pi}{8};$$

taking together the partial fractions given by $r=1$, $r=8$, their sum is

$$\frac{\left\{-\frac{1}{8}\operatorname{cis}\frac{\pi}{8}\left(x-\operatorname{cis}\frac{15\pi}{8}\right)-\frac{1}{8}\operatorname{cis}\frac{15\pi}{8}\left(x-\operatorname{cis}\frac{\pi}{8}\right)\right\}}{\left\{\left(x-\operatorname{cis}\frac{\pi}{8}\right)\left(x-\operatorname{cis}\frac{15\pi}{8}\right)\right\}},$$

which, since $\operatorname{cis}\frac{15\pi}{8}=\operatorname{cis}\frac{-\pi}{8}$,

$$=\left(-\tfrac{1}{4}x\cos\frac{\pi}{8}+\tfrac{1}{4}\right)\Big/\left(x^2-2x\cos\frac{\pi}{8}+1\right);$$

similarly the fractions given by $r=2, 7$ and $3, 6$ and $4, 5$ can be combined.

3. The denr. is a quadratic in x^2, and is zero when $\frac{1+x}{1-x}=(-1)^{\frac{1}{5}}$,

which gives, as in Example 2, $x=i\tan\frac{(2r-1)\pi}{10}$;

$$\therefore x^2=-\tan^2\frac{\pi}{10}\ \text{ or }\ -\tan^2\frac{3\pi}{10};$$

$\therefore$ expn. $=\dfrac{A}{x^2+\tan^2\frac{\pi}{10}}+\dfrac{B}{x^2+\tan^2\frac{3\pi}{10}}$; but coeff. of x^4 in denr. of expn. is 10;

$$\therefore 8\sqrt{5}\equiv 10A\left(x^2+\tan^2\frac{3\pi}{10}\right)+10B\left(x^2+\tan^2\frac{\pi}{10}\right);$$

$$\therefore A+B=0 \text{ and } A\tan^2\frac{3\pi}{10}+B\tan^2\frac{\pi}{10}=\frac{4}{\sqrt{5}};$$

$$\therefore -B=A=\frac{4}{\sqrt{5}}\div\left(\tan^2\frac{3\pi}{10}-\tan^2\frac{\pi}{10}\right)$$

$$=\frac{4}{\sqrt{5}}\div\left\{\left(1+\frac{2}{\sqrt{5}}\right)-\left(1-\frac{2}{\sqrt{5}}\right)\right\}=1.$$

4. $\{(1+x)^7-(1-x)^7\}/x$ is a cubic in x^2, and as in Example 2, is zero when $x=i\tan\frac{r\pi}{7}$,

$$\therefore \text{ when } x^2=-\tan^2\frac{\pi}{7},\ -\tan^2\frac{2\pi}{7},\ -\tan^2\frac{3\pi}{7};$$

$$\therefore \text{expn.}=\sum_1^3\frac{A_r}{x^2+\tan^2\frac{r\pi}{7}} \text{ where } A_r=\lim\frac{x\left(x^2+\tan^2\frac{r\pi}{7}\right)}{(1+x)^7-(1-x)^7}$$

for $x\to i\tan\frac{r\pi}{7}$,

$$=\lim\frac{3x^2+\tan^2\frac{r\pi}{7}}{7\{(1+x)^6+(1-x)^6\}}$$

$$=\frac{-3\tan^2\frac{r\pi}{7}+\tan^2\frac{r\pi}{7}}{7\left\{\sec^6\frac{r\pi}{7}\left[\left(\text{cis}\frac{r\pi}{7}\right)^6+\left(\text{cis}\frac{-r\pi}{7}\right)^6\right]\right\}}$$

$$=\frac{-2\tan^2\frac{r\pi}{7}\cos^6\frac{r\pi}{7}}{7\left(2\cos\frac{6r\pi}{7}\right)}=(-1)^{r-1}\cdot\tfrac{1}{7}\sin^2\frac{r\pi}{7}\cos^3\frac{r\pi}{7}$$

since $\cos\frac{6r\pi}{7}=(-1)^r\cos\frac{r\pi}{7}$.

5. See Example 3, $\frac{\cos^2\theta}{\cos n\theta}=\sum\frac{A_r}{\cos\theta-\cos(2r-1)\frac{\pi}{2n}}$, for $r=1$ to n

but excluding $r=\frac{1}{2}(n+1)$ for n odd; in that case expression reduces to $(\cos\theta)/$(a polynomial of degree $n-1$), but has partial fractions of the same form.

$$A_r=\lim\frac{(\cos\theta-\cos\alpha)\cos^2\theta}{\cos n\theta}\quad\text{when}\quad\theta\to\alpha\equiv(2r-1)\frac{\pi}{2n},$$

hence

$$A_r=\lim\frac{-\sin\theta\cos^2\theta}{-n\sin n\theta}=\frac{\sin\alpha\cos^2\alpha}{n\sin n\alpha}=\frac{(-1)^{r-1}}{n}\sin\alpha\cos^2\alpha.$$

6. By eqn. (5),

$$\frac{1}{\sin n\theta}=\frac{K}{\sin\theta}+\sum\left\{\frac{A_r}{\sin\theta+\sin\frac{r\pi}{n}}+\frac{A_r}{\sin\theta-\sin\frac{r\pi}{n}}\right\}$$

and $K=\lim_{\theta\to0}\frac{\sin\theta}{\sin n\theta}=\frac{1}{n}$,

$$A_r=\lim_{\theta\to-\frac{r\pi}{n}}\frac{\sin\theta+\sin\frac{r\pi}{n}}{\sin n\theta}=\lim\frac{\cos\theta}{n\cos n\theta}$$

$$=\frac{\cos\frac{r\pi}{n}}{n\cos r\pi}=\frac{(-1)^r}{n}\cos\frac{r\pi}{n},$$

$$A_r'=\lim_{\theta\to\frac{r\pi}{n}}\frac{\sin\theta-\sin\frac{r\pi}{n}}{\sin n\theta}=\lim\frac{\cos\theta}{n\cos n\theta}=\frac{\cos\frac{r\pi}{n}}{n\cos r\pi};$$

$$\therefore\ \frac{n}{\sin n\theta}-\frac{1}{\sin\theta}$$

$$=\Sigma(-1)^r\cos\frac{r\pi}{n}\left(\frac{1}{\sin\theta+\sin\frac{r\pi}{n}}+\frac{1}{\sin\theta-\sin\frac{r\pi}{n}}\right)$$

$$=\Sigma(-1)^r\left(\frac{2\sin\theta\cos\frac{r\pi}{n}}{\sin^2\theta-\sin^2\frac{r\pi}{n}}\right)$$

$$=\Sigma(-1)^r\frac{\sin\left(\theta+\frac{r\pi}{n}\right)+\sin\left(\theta-\frac{r\pi}{n}\right)}{\sin\left(\theta+\frac{r\pi}{n}\right)\sin\left(\theta-\frac{r\pi}{n}\right)}.$$

7. $\frac{\sin(n-1)\theta}{\sin\theta}$ is a polynomial of degree $n-2$ in $\cos\theta$; therefore, as in Example 3,

$$\frac{n\sin(n-1)\theta}{\sin\theta\cos n\theta}=\sum_1^n\frac{A_r}{\cos\theta-\cos\alpha}\quad\text{where}\quad\alpha=\frac{(2r-1)\pi}{2n}$$

$$A_r=\lim_{\theta\to\alpha}\frac{n\sin(n-1)\theta\,.\,(\cos\theta-\cos\alpha)}{\sin\theta\cos n\theta}$$

$$=\frac{n\sin(n-1)\alpha}{\sin\alpha}\cdot\lim\frac{-\sin\theta}{-n\sin n\theta}=\frac{\sin(n-1)\alpha}{\sin n\alpha}$$

$$=\frac{\sin n\alpha\cos\alpha-\cos n\alpha\sin\alpha}{\sin n\alpha}=\cos\alpha.$$

8. From Ex. XII. b, No. 3,

$$\log(\cos n\theta-\cos n\alpha)$$

$$=(n-1)\log2+\Sigma\log\left\{\cos\theta-\cos\left(\alpha+\frac{2r\pi}{n}\right)\right\};$$

differentiate each side w.r.t. α.

9. As in No. 8, but differentiate w.r.t. θ, and divide each side by $\sin\theta$.

10. In Example 3, make $\theta\to0$, then $\frac{n\tan n\theta}{\sin\theta}\to n\,.\,n=n^2$.

11. In Example 3, put $\theta=\frac{\pi}{3}$, then

$$\frac{2n\tan\frac{n\pi}{3}}{\sqrt{3}}=\sum\frac{1}{\frac{1}{2}-\cos\frac{(2r-1)\pi}{2n}};$$

$$\therefore \text{ sum} = \frac{n}{\sqrt{3}} \cdot \tan\frac{n\pi}{3} = 0 \text{ if } n = 3p,$$
$$= n \text{ if } n = 3p+1, = -n \text{ if } n = 3p-1.$$

12. From eqn. (10), writing θ for α,

$$\log(x^{2n} - 2x^n \cos n\theta + 1) = \Sigma \log\left[x^2 - 2x\cos\left(\theta + \frac{2r\pi}{n}\right) + 1\right];$$

differentiate w.r.t. θ;

$$\therefore \frac{2nx^n \sin n\theta}{x^{2n} - 2x^n\cos n\theta + 1} = \sum \frac{2x\sin\left(\theta + \frac{2r\pi}{n}\right)}{x^2 - 2x\cos\left(\theta + \frac{2r\pi}{n}\right) + 1};$$

$$\therefore \frac{x^{n-1}}{x^{2n} - 2x^n\cos n\theta + 1} = \sum \frac{\frac{1}{n}\operatorname{cosec} n\theta \sin\left(\theta + \frac{2r\pi}{n}\right)}{x^2 - 2x\cos\left(\theta + \frac{2r\pi}{n}\right) + 1};$$

Or, expression

$$= \sum\left\{\frac{\mathsf{A}_r}{x - \operatorname{cis}\left(\theta + \frac{2r\pi}{n}\right)} + \frac{\mathsf{B}_r}{x - \operatorname{cis}\left(-\theta - \frac{2r\pi}{n}\right)}\right\}$$

and proceed as in Example 2.

13. By equation (1),

$$x^{2n} - 1 = (x^2 - 1)\prod_1^{n-1}\left(x^2 - 2x\cos\frac{r\pi}{n} + 1\right);$$

$$\therefore \log(x^{2n} - 1) = \log(x^2 - 1) + \sum_1^{n-1}\log\left(x^2 - 2x\cos\frac{r\pi}{n} + 1\right);$$

differentiate w.r.t. x.
The first result gives

$$\sum \frac{1 - \frac{1}{x}\cos\frac{r\pi}{n}}{x + \frac{1}{x} - 2\cos\frac{r\pi}{n}} = \frac{nx^{n-1}}{x^n - \frac{1}{x^n}} - \frac{1}{x - \frac{1}{x}}; \text{ put } x = \operatorname{cis}\theta;$$

$$\therefore \sum \frac{1 - \operatorname{cis}(-\theta)\cos\frac{r\pi}{n}}{2\left(\cos\theta - \cos\frac{r\pi}{n}\right)} = \frac{n \operatorname{cis}(n-1)\theta}{2i\sin n\theta} - \frac{1}{2i\sin\theta};$$

equate "second parts,"

$$\therefore \sum \frac{\sin\theta\cos\frac{r\pi}{n}}{2\left(\cos\theta - \cos\frac{r\pi}{n}\right)} = \frac{-n\cos(n-1)\theta}{2\sin n\theta} + \frac{1}{2\sin\theta};$$

multiply each side by 2 cosec θ.

[Note that this can also be deduced by differentiating logarithmically XII. b, No. 7, and then replacing

$$\sum \frac{\sin\theta}{\cos\theta - \cos\frac{r\pi}{n}} \text{ by } \frac{\sin\theta}{\cos\theta} \cdot \sum\left\{1 - \frac{\cos\frac{r\pi}{n}}{\cos\frac{r\pi}{n} - \cos\theta}\right\}.$$

14. By eqn. (10) writing $\frac{x}{a}$ for x and multiplying each side by a^{2n},

$$x^{2n} - 2x^n a^n\cos n\theta + a^{2n} = \prod\left\{x^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2\right\};$$

$$\therefore \log(x^{2n} - 2x^n a^n \cos n\theta + a^{2n})$$
$$= \Sigma\log\left\{x^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2\right\};$$

differentiate w.r.t. x.

15. By Ch. IX. eqn. 6, $\frac{\tan n\theta}{\tan\theta} = \frac{n - \binom{n}{3}\tan^2\theta + \ldots \tan^{n-2}\theta}{1 - \binom{n}{2}\tan^2\theta + \ldots \tan^n\theta}$; the denominator is zero for $n\theta = (2r+1)\frac{\pi}{2}$, and therefore for $\tan\theta = \pm\tan(2r+1)\frac{\pi}{2n}$; the factors of the denominator are given by $r = 0$ to $\frac{1}{2}n - 1$;

$$\therefore \frac{\tan n\theta}{\tan\theta}$$
$$= \sum\left\{\frac{\mathsf{A}_r}{\tan\theta + \tan(2r+1)\alpha} + \frac{\mathsf{B}_r}{\tan\theta - \tan(2r+1)\alpha}\right\},$$

where $\mathsf{A}_r = \lim \frac{\tan n\theta}{\tan\theta}\{\tan\theta + \tan(2r+1)\alpha\}$

when $\theta \to -(2r+1)\alpha$;

$$\therefore \mathsf{A}_r = \lim\frac{1 + \tan(2r+1)\alpha\cot\theta}{\cot n\theta}$$
$$= \lim\frac{-\tan(2r+1)\alpha\operatorname{cosec}^2\theta}{-n\operatorname{cosec}^2 n\theta}$$

$$=\frac{1}{n}\tan(2r+1)\alpha\,\mathrm{cosec}^2(2r+1)\alpha$$

$$=\frac{2}{n}\,\mathrm{cosec}\,2(2r+1)\alpha\,;\ \text{similarly for } B_r.$$

16. Expn. $=\dfrac{\sin\theta\,(\text{polynomial of degree 4 in }\sin\theta)}{\sin\theta\,(\text{polynomial of degree 6 in }\sin\theta)}$; and $\sin 7\theta=0$ when $\theta=\dfrac{r\pi}{7}$, hence polynomial in denominator is zero for $\sin\theta=\sin\dfrac{r\pi}{7}$, $r=\pm1,\ \pm2,\ \pm3$; hence

$$\text{expn.}=\sum_1^3\left(\frac{A_r}{\sin\theta-\sin\frac{r\pi}{7}}+\frac{B_r}{\sin\theta+\sin\frac{r\pi}{7}}\right),$$

where $A_r=\lim\dfrac{\sin 5\theta}{\sin 7\theta}\left(\sin\theta-\sin\dfrac{r\pi}{7}\right)$ for $\theta\to\dfrac{r\pi}{7}$,

$$=\lim\frac{\sin 5\theta\cos\theta}{7\cos 7\theta}=\tfrac{1}{7}\sin\frac{5r\pi}{7}\cos\frac{r\pi}{7}\cdot(-1)^r,$$

similarly $B_r=-$same;

$$\therefore\ \text{expn.}\equiv\tfrac{1}{7}\sum_1^3(-1)^r\sin\frac{5r\pi}{7}\left(\frac{\cos\frac{r\pi}{7}}{\sin\theta-\sin\frac{r\pi}{7}}-\frac{\cos\frac{r\pi}{7}}{\sin\theta+\sin\frac{r\pi}{7}}\right)$$

$$=\tfrac{1}{7}\sum_1^3(-1)^r\sin\frac{5r\pi}{7}\,\frac{2\cos\frac{r\pi}{7}\sin\frac{r\pi}{7}}{\sin^2\theta-\sin^2\frac{r\pi}{7}}$$

$$=\tfrac{1}{7}\sum_1^3(-1)^r\sin\frac{5r\pi}{7}\,\frac{\sin\left\{\left(\theta+\frac{r\pi}{7}\right)-\left(\theta-\frac{r\pi}{7}\right)\right\}}{\sin\left(\theta+\frac{r\pi}{7}\right)\sin\left(\theta-\frac{r\pi}{7}\right)}$$

$$=\tfrac{1}{7}\sum_1^3(-1)^r\sin\frac{5r\pi}{7}\left\{\cot\left(\theta-\frac{r\pi}{7}\right)-\cot\left(\theta+\frac{r\pi}{7}\right)\right\};$$

putting $r=7-s$ in the second part of this result,

$$-\sum_1^3(-1)^r\sin\frac{5r\pi}{7}\cot\left(\theta+\frac{r\pi}{7}\right)$$

$$=-\sum_4^6-(-1)^s\sin\left(5\pi-\frac{5s\pi}{7}\right)\cot\left(\theta+\pi-\frac{s\pi}{7}\right)$$

$$=+\sum_4^6(-1)^s\sin\frac{5s\pi}{7}\cot\left(\theta-\frac{s\pi}{7}\right)$$

$$=\sum_4^6(-1)^r\sin\frac{5r\pi}{7}\cot\left(\theta-\frac{r\pi}{7}\right).$$

17. Expression $=\dfrac{\sin\theta\cos\theta\ (\text{quadratic in }\sin\theta)}{\sin\theta\ (\text{sextic in }\sin\theta)}$, and the factors of the sextic are, as in No. 16, $\sin\theta-\sin\dfrac{r\pi}{7}$ for $r=\pm1,\ \pm2,\ \pm3$; hence expression

$$=\cos\theta\,.\sum_1^3\left(\frac{A_r}{\sin\theta-\sin\frac{r\pi}{7}}+\frac{B_r}{\sin\theta+\sin\frac{r\pi}{7}}\right)$$

where

$$A_r=\lim\frac{\sin 4\theta}{\cos\theta\sin 7\theta}\left(\sin\theta-\sin\frac{r\pi}{7}\right)\ \text{for}\ \theta\to\frac{r\pi}{7}$$

$$=\lim\frac{\sin 4\theta\cos\theta+4\cos 4\theta\left(\sin\theta-\sin\frac{r\pi}{7}\right)}{7\cos\theta\cos 7\theta-\sin\theta\sin 7\theta}$$

$$=\lim\frac{\sin 4\theta}{7\cos 7\theta}=\frac{(-1)^r}{7}\sin\frac{4r\pi}{7}\ \text{and}\ B_r=-A_r;$$

thus

$$\text{expn.}=\tfrac{1}{7}\sum_1^3(-1)^r\sin\frac{4r\pi}{7}\left(\frac{\cos\theta}{\sin\theta-\sin\frac{r\pi}{7}}-\frac{\cos\theta}{\sin\theta+\sin\frac{r\pi}{7}}\right)$$

$$=\tfrac{1}{7}\sum_1^3(-1)^r\sin\frac{4r\pi}{7}\,\frac{2\sin\frac{r\pi}{7}\cos\theta}{\sin^2\theta-\sin^2\frac{r\pi}{7}}$$

$$=\tfrac{1}{7}\sum_1^3(-1)^r\sin\frac{4r\pi}{7}\,\frac{\sin\left(\theta+\frac{r\pi}{7}\right)-\sin\left(\theta-\frac{r\pi}{7}\right)}{\sin\left(\theta+\frac{r\pi}{7}\right)\sin\left(\theta-\frac{r\pi}{7}\right)}$$

$$=\tfrac{1}{7}\sum_1^3(-1)^r\sin\frac{4r\pi}{7}\left\{\mathrm{cosec}\left(\theta-\frac{r\pi}{7}\right)-\mathrm{cosec}\left(\theta+\frac{r\pi}{7}\right)\right\};$$

putting $r=7-s$ the second part reduces, as in No. 16, to

$$+\tfrac{1}{7}\sum_4^6(-1)^r\sin\frac{4r\pi}{7}\,\mathrm{cosec}\left(\theta-\frac{r\pi}{7}\right).$$

Or, let $P_n(\tan\theta)$ denote a polynomial of degree n in $\tan\theta$; $\sin 7\theta = \cos^7\theta \,.\, \tan\theta \,.\, P_6(\tan\theta)$ and

$$\sin 4\theta = \cos^4\theta \,.\, \tan\theta \,.\, P_2(\tan\theta);$$

$$\therefore \frac{\cos\theta \sin 4\theta}{\sin 7\theta} = \frac{P_4(\tan\theta)}{P_6(\tan\theta)}$$

where, since $\sin 7\theta = 0$ for $\theta = \frac{r\pi}{7}$,

$$P_6(\tan\theta) = A \,.\, \prod_1^6\left(\tan\theta - \tan\frac{r\pi}{7}\right);$$

$$\therefore \frac{\cos\theta \sin 4\theta}{\sin 7\theta} = \sum_1^6 \frac{A_r}{\tan\theta - \tan\frac{r\pi}{7}},$$

where
$$A_r = \lim_{\theta\to\frac{r\pi}{7}} \frac{\cos\theta\sin 4\theta\left(\tan\theta - \tan\frac{r\pi}{7}\right)}{\sin 7\theta}$$

$$= \cos\frac{r\pi}{7}\sin\frac{4r\pi}{7} \,.\, \lim\frac{\sec^2\theta}{7\cos 7\theta} = (-1)^r \,.\, \tfrac{1}{7}\frac{\sin\frac{4r\pi}{7}}{\cos\frac{r\pi}{7}};$$

$$\therefore \frac{\sin 4\theta}{\sin 7\theta} = \tfrac{1}{7}\Sigma(-1)^r \,.\, \sin\frac{4r\pi}{7} \,.\, \frac{1}{\cos\theta\cos\frac{r\pi}{7}\left(\tan\theta - \tan\frac{r\pi}{7}\right)}$$

$$= \tfrac{1}{7}\Sigma(-1)^r \,.\, \sin\frac{4r\pi}{7} \,.\, \frac{1}{\sin\left(\theta - \frac{r\pi}{7}\right)}.$$

18. Expn. $= \dfrac{\text{polynomial of degree 5 in } \sin\theta}{\cos\theta(\text{polynomial of degree 7 in } \sin\theta)}$

$$= \frac{\cos\theta\,(\text{quartic in } \sin\theta)}{\cos^2\theta\,(\text{sextic in } \sin\theta)} = \frac{\cos\theta\,(\text{quartic})}{\text{octic}}$$

=, as in No. 17,

$$\cos\theta\sum_1^4\left(\frac{A_r}{\sin\theta - \sin\frac{r\pi}{8}} + \frac{B_r}{\sin\theta + \sin\frac{r\pi}{8}}\right);$$

the terms given by $r = 1, 2, 3$ are found, by the method of No. 17, to be equal to

$$\tfrac{1}{8}\sum_1^3(-1)^r\sin\frac{5r\pi}{8}\left\{\operatorname{cosec}\left(\theta - \frac{r\pi}{8}\right) - \operatorname{cosec}\left(\theta + \frac{r\pi}{8}\right)\right\}$$

or to $\tfrac{1}{8}\sum(-1)^r\sin\frac{5r\pi}{8}\operatorname{cosec}\left(\theta - \frac{r\pi}{8}\right)$ for $r = 1, 2, 3, 5, 6, 7$;

also $A_4 = \lim\limits_{\theta\to\frac{\pi}{2}} \dfrac{\sin 5\theta\,(\sin\theta - 1)}{\sin 8\theta\cos\theta}$

$$= \lim_{\phi\to 0}\frac{1 - \cos\phi}{\sin 8\phi\sin\phi} = \lim\frac{\tan\frac{1}{2}\phi}{\sin 8\phi} = \tfrac{1}{16},$$

and similarly $B_4 = -\tfrac{1}{16}$; $\therefore$ term given by $r = 4$ is

$$\tfrac{1}{16}\cos\theta\left(\frac{1}{\sin\theta - 1} - \frac{1}{\sin\theta + 1}\right) = -\tfrac{1}{8}\sec\theta$$

$$= \tfrac{1}{8}(-1)^4\sin\frac{5 \,.\, 4\pi}{8}\operatorname{cosec}\left(\theta - \frac{4\pi}{8}\right).$$

[The work of No. 17 does not apply to $r = 4$ in No. 18, because

$$\lim\frac{\sin 5\theta(\sin\theta - 1)}{\sin 8\theta\cos\theta} = \lim\frac{\sin 5\theta\cos\theta + 5\cos 5\theta\,(\sin\theta - 1)}{8\cos\theta\cos 8\theta - \sin\theta\sin 8\theta}$$

is not equal to $\lim\dfrac{\sin 5\theta}{8\cos 8\theta}$].

Or, using the second method of No. 17, we find that

$$\frac{\cos\theta\sin 5\theta}{\sin 8\theta} = \frac{\cos^6\theta \,.\, \tan\theta \,.\, P_4(\tan\theta)}{\cos^8\theta \,.\, \tan\theta \,.\, P_6(\tan\theta)}$$

$$= \frac{(1 + \tan^2\theta) \,.\, P_4(\tan\theta)}{P_6(\tan\theta)}.$$

The *direct* application of the partial fraction method cannot be made because the degree of the numerator = the degree of the denominator.

But $\dfrac{\sin\theta \,.\, \sin 5\theta}{\sin 8\theta} = \dfrac{\sin^6\theta \,.\, P_4(\cot\theta)}{\sin^8\theta \,.\, P_7(\cot\theta)} = \dfrac{(1 + \cot^2\theta) \,.\, P_4(\cot\theta)}{P_7(\cot\theta)}$,

where $P_7(\cot\theta) = A \cdot \prod\limits_1^7\left(\cot\theta - \cot\frac{r\pi}{8}\right)$

since $\sin 8\theta = 0$ for $\theta = \frac{r\pi}{8}$

and since $\sin\theta \,.\, \sin 5\theta \neq 0$ for $\theta = \frac{r\pi}{8}$ if $r = 1$ to 7;

$$\therefore \frac{\sin\theta\sin 5\theta}{\sin 8\theta} = \sum_1^7\frac{A_r}{\cot\theta - \cot\frac{r\pi}{8}},$$

where $$A_r = \lim_{\theta \to \frac{r\pi}{8}} \frac{\sin\theta \sin 5\theta \left(\cot\theta - \cot\frac{r\pi}{8}\right)}{\sin 8\theta}$$

$$= \text{, as in No. 17, } \sin\frac{r\pi}{8} \sin\frac{5r\pi}{8} \frac{\left(-\operatorname{cosec}^2\frac{r\pi}{8}\right)}{8\cos r\pi}$$

$$=(-1)^{r+1}\cdot\tfrac{1}{8}\,\frac{\sin\frac{5r\pi}{8}}{\sin\frac{r\pi}{8}}$$

and $\cot\theta - \cot\dfrac{r\pi}{8} = -\dfrac{\sin\left(\theta - \frac{r\pi}{8}\right)}{\sin\theta\sin\frac{r\pi}{8}}$.

19. Put $t=\tan x$, $t_1=\tan a$, etc., then

$$\sin(x-a)=\sin x\cos a-\cos x\sin a=\cos x\cos a\,.\,(t-t_1)$$

and similarly for $\sin(x-b)$, $\sin(a-b)$;

$$\therefore \text{ l.h.s.}=\frac{t\cos x}{\cos a\cos b\cos^2 x(t-t_1)(t-t_2)}$$

$=$, by partial fractions,

$$\sec a\sec b\sec x\left\{\frac{t_1}{(t-t_1)(t_1-t_2)}+\frac{t_2}{(t_2-t_1)(t-t_2)}\right\}$$

$$=\sec a\sec b\sec x\left\{\frac{\sin a\cos x\cos^2 a\cos b}{\cos a\sin(x-a)\sin(a-b)}+\frac{\sin b\cos x\cos^2 b\cos a}{\cos b\sin(b-a)\sin(x-b)}\right\}=\text{r.h.s.}$$

By the method of part (i),

$$\sin^{n-1}x\,.\prod\operatorname{cosec}(x-a)=\sec a_1\sec a_2\ldots\sec a_n\sec x\times\sum\frac{t_1^{\,n-1}}{(t-t_1)(t_1-t_2)(t_1-t_3)\ldots(t_1-t_n)}$$

$$=\sum\frac{\sin^{n-1}a_1}{\sin(x-a_1)\sin(a_1-a_2)\ldots\sin(a_1-a_n)}.$$

20. Method of No. 19; l.h.s.

$$=\frac{\sec a\sec b\sec c\sec x\,.\,t}{(t-t_1)(t-t_2)(t-t_3)}$$

$$=\sec a\sec b\sec c\sec x\,.\sum\frac{t_1}{(t-t_1)(t_1-t_2)(t_1-t_3)}$$

$$=\sec a\sec b\sec c\sec x\sum\frac{\sin a\cos x\cos^3 a\cos b\cos c}{\sin(x-a)\sin(a-b)\sin(a-c)}=\text{r.h.s.}$$

21. By No. 19,

$$\frac{\sin^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}=\sum\frac{\sin^2 a}{\sin(x-a)\sin(a-b)\sin(a-c)}$$

multiply each side of this by $\sin x$, and each side of No. 20 by $\cos x$ and add, thus,

$$\frac{\sin^3 x+\sin x\cos^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}=\sum\frac{\sin^2 a\sin x+\sin a\cos a\cos x}{\sin(x-a)\sin(a-b)\sin(a-c)}$$

but $\sin^3 x+\sin x\cos^2 x=\sin x$, and

$$\sin^2 a\sin x+\sin a\cos a\cos x=\sin a\cos(x-a).$$

22. $\angle \mathrm{POA}_{r+1}=\theta+\dfrac{2r\pi}{n}$; $\therefore$ $\mathrm{PA}_{r+1}^2=x^2-2xa\cos\left(\theta+\dfrac{2r\pi}{n}\right)+a^2$.

From No. 14,

$$\frac{nx^{n-1}(x^n-a^n\cos n\theta)}{x^{2n}-2x^n a^n\cos n\theta+a^{2n}}=\frac{1}{2x}\sum\frac{2x^2-2xa\cos\left(\theta+\frac{2r\pi}{n}\right)}{x^2-2xa\cos\left(\theta+\frac{2r\pi}{n}\right)+a^2}$$

$$=\frac{1}{2x}\sum\left\{1+\frac{x^2-a^2}{x^2-2xa\cos\left(\theta+\frac{2r\pi}{n}\right)+a^2}\right\}$$

$$=\frac{1}{2x}\left\{n+(x^2-a^2)\,.\sum\frac{1}{\mathrm{PA}_{r+1}^2}\right\};$$

$$\therefore \sum\frac{1}{\mathrm{PA}_{r+1}^2}=\left\{\frac{2nx^n(x^n-a^n\cos n\theta)}{x^{2n}-2x^n a^n\cos n\theta+a^{2n}}-n\right\}\div(x^2-a^2).$$

EXERCISE XII. e. (p. 236.)

1. Expression is zero if $x=a\operatorname{cis}\dfrac{r\pi}{n}$, factors are $\left(x-a\operatorname{cis}\dfrac{r\pi}{n}\right)$ for $r=0,\ \pm1,\ \pm2,\ \ldots\pm(n-1),\ n$, and the product of those given by $r=\pm k$ is $x^2-2xa\cos\dfrac{k\pi}{n}+a^2$, hence

$$x^{2n}-a^{2n}=(x-a\operatorname{cis}0)(x-a\operatorname{cis}\pi)\prod\left(x^2-2xa\cos\frac{k\pi}{n}+a^2\right).$$

2. $x^6 - x^3 + 1 = x^6 - 2x^3 \cos\frac{\pi}{3} + 1 =$, as on p. 226,

$$\prod\left[x - \operatorname{cis}\left\{\pm\left(\frac{\pi}{9}+\frac{2r\pi}{3}\right)\right\}\right] \text{ for } r=0,\ 1,\ 2.$$

$\Sigma(x-\alpha)^2 = 6x^2 - 2x\Sigma\alpha + \Sigma\alpha^2$; but $\Sigma\alpha = \Sigma\alpha^2 = 0$, because coeffs. of x^5, x^4 in $x^6 - x^3 + 1$ are zero.

3. Compare p. 44. To solve $x^3 - 3x + 1 = 0$, put $x = 2\cos\phi$, $x^3 - 3x = 8\cos^3\phi - 6\cos\phi = 2\cos 3\phi$; $\therefore \cos 3\phi = -\frac{1}{2}$; hence the three values of $\cos\phi$ are $\cos\frac{2\pi}{9}$, $\cos\frac{4\pi}{9}$, $\cos\frac{8\pi}{9}$.

4. To solve $(1+x)^{2n+1} - (1-x)^{2n+1} = 0$, put $x = i\tan\theta$; then as in Ex. XII. a, No. 12, $2\sin(2n+1)\theta = 0$; $\therefore \theta = \frac{r\pi}{2n+1}$, and $x = \pm i\tan\theta$ for $r = 0$ to n; hence

$$\text{expn.} = cx\prod_1^n\left(x^2 + \tan^2\frac{r\pi}{2n+1}\right) \text{ and } c = \text{coeff. of } x^{2n+1} = 2.$$

Also $$c\prod_1^n \tan^2\frac{r\pi}{2n+1} = \text{coeff. of } x = 2(2n+1).$$

5. As in Example 2,

$$\text{expn.} = 4nx\prod_1^{n-1}\left(x^2 + \tan^2\frac{r\pi}{2n}\right)$$
$$= 4nx\prod_{n-1}^{1}\left\{x^2 + \tan^2(n-s)\frac{\pi}{2n}\right\} \text{ where } s = n - r,$$
$$= 4nx\prod_1^{n-1}\left\{x^2 + \tan^2\left(\frac{\pi}{2} - \frac{\pi s}{2n}\right)\right\}.$$

Put $x = 2$; thus $3^{2n} - 1 = 8n$ (required product).

6. By IX. e, No. 21, $\frac{\cos 5\theta}{\cos\theta} = 16\sin^4\theta - \ldots$; also $\frac{\sin 2\theta}{\cos\theta} = 2\sin\theta$;
$\therefore$ given expn. is a quartic in $\sin\theta$, and it is zero when

$$\cos 5\theta = \sin 2\theta = \cos\left(\frac{\pi}{2} - 2\theta\right),\quad 5\theta = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta\right),$$

$\theta = (4n+1)\frac{\pi}{14}$ or $(4n-1)\frac{\pi}{6}$; $\therefore$ excluding values for which the denr. $(\cos\theta)$ is zero,

$$\sin\theta = \sin\frac{\pi}{14},\ \sin\frac{5\pi}{14},\ -\sin\frac{3\pi}{14},\ \text{or } -\sin\frac{\pi}{6}.$$

Hence expn. =

$$16\left(\sin\theta - \sin\frac{\pi}{14}\right)\left(\sin\theta - \sin\frac{5\pi}{14}\right)\left(\sin\theta + \sin\frac{3\pi}{14}\right)\left(\sin\theta + \tfrac{1}{2}\right).$$

7. By IX. e, No. 12,

$$\begin{aligned}\text{l.h.s.} &= 7\cos x(8\cos^6 x - 16\cos^4 x + 8\cos^2 x - 1)\\ &= 7\cos x(2\cos^2 x - 1)(4\cos^4 x - 6\cos^2 x + 1)\\ &= 7\cos x\cos 2x\{(1+\cos 2x)^2 - 3(1+\cos 2x) + 1\},\end{aligned}$$

and the last bracket

$$\begin{aligned}&= \cos^2 2x - \cos 2x - 1 = (\cos 2x - \tfrac{1}{2})^2 - \tfrac{5}{4}\\ &= (\cos 2x - \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})(\cos 2x - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}).\end{aligned}$$

8. By Ch. IX, eqn. (13), $\frac{\sin n\theta}{\sin\theta}$ is a polynomial in $\cos\theta$, equal to $A\cos^{n-1}\theta + B\cos^{n-3}\theta + \ldots$, and similarly

$$\frac{\sin n\phi}{\sin\phi} = A\cos^{n-1}\phi + B\cos^{n-3}\phi + \ldots,$$

hence given expn.

$$= A(\cos^{n-1}\theta - \cos^{n-1}\phi) + B(\cos^{n-3}\theta - \cos^{n-3}\phi) + \ldots,$$

and $\cos\theta - \cos\phi$ is a factor of each term.

9. In XII. b, No. 3, put $\theta = \pi$, $\alpha = 0$, hence

$$\begin{aligned}\cos n\pi - 1 &= 2^{n-1}\cdot\prod_1^n\left(\cos\pi - \cos\frac{2r\pi}{n}\right)\\ &= 2^{n-1}\cdot(-1)^n\cdot\prod_1^n\left(2\cos^2\frac{r\pi}{n}\right)\\ &= 2^{2n-1}\cdot(-1)^n\cdot\prod_1^n\left(\cos^2\frac{r\pi}{n}\right);\end{aligned}$$

but, since $\cos\frac{r\pi}{n} = \cos(2n-r)\frac{\pi}{n}$,

$$\begin{aligned}\prod_1^{2n-1}\cos\frac{r\pi}{n} &= \left(\prod_1^{n-1}\cos\frac{r\pi}{n}\right)^2\cdot\cos\pi = -\prod_1^n\left(\cos^2\frac{r\pi}{n}\right)\\ &= -\frac{\cos n\pi - 1}{2^{2n-1}(-1)^n} = -2^{1-2n}\cdot\frac{(-1)^n - 1}{(-1)^n}\\ &= -2^{1-2n}[1-(-1)^n].\end{aligned}$$

10. In XII. d, No. 9, make $\theta \to 0$ and write ϕ for α.

11. From eqn. (3), $\log(x^{2n}+1) = \sum_1^n \log(x^2 - 2x\cos\alpha + 1)$, where $\alpha = \frac{(2r-1)\pi}{2n}$. Differentiate w.r.t. x;

$$\therefore \frac{2nx^{2n-1}}{x^{2n}+1} = \sum\frac{2x - 2\cos\alpha}{x^2 - 2x\cos\alpha + 1}.\quad \text{Write } \frac{1}{x} \text{ for } x;$$

$$\therefore \frac{2n\left(\frac{1}{x}\right)^{2n-1}}{\left(\frac{1}{x}\right)^{2n}+1}=\sum \frac{\frac{2}{x}-2\cos\alpha}{\frac{1}{x^2}-\frac{2}{x}\cos\alpha+1};$$

$$\therefore \frac{2nx}{1+x^{2n}}=\sum \frac{2(1-x\cos\alpha)x}{1-2x\cos\alpha+x^2}.$$

12. In XII. b, No. 3, put $2n+1$ for n and $\frac{\alpha}{2n+1}$ for α, thus

$$\cos(2n+1)\theta-\cos\alpha=2^{2n}\prod_1^{2n+1}\left(\cos\theta-\cos\frac{\alpha+2r\pi}{2n+1}\right)$$

Since $\cos\frac{\alpha+2r\pi}{2n+1}=\cos\left(\frac{\alpha+2r\pi}{2n+1}-2\pi\right)$,

the factors given by $r=2n+1,\ 2n,\ \ldots\ (n+1)$ would also be given by $r=0,\ -1,\ \ldots\ -n$; thus

$$\cos(2n+1)\theta-\cos\alpha=2^{2n}\prod_{-n}^{+n}\left(\cos\theta-\cos\frac{\alpha+2r\pi}{2n+1}\right).$$

Take logarithms, and differentiate w.r.t. θ.

13. Compare Ex. XII. d, No. 15.

$$\frac{\tan n\theta}{\tan\theta}=\frac{n-\binom{n}{3}\tan^2\theta+\ldots+(-1)^{\frac{1}{2}(n-1)}\cdot\tan^{n-1}\theta}{1-\binom{n}{2}\tan^2\theta+\ldots+(-1)^{\frac{1}{2}(n-1)}n\tan^{n-1}\theta};$$

the denominator is zero for

$$n\theta=(2r+1)\frac{\pi}{2}\text{ and }\therefore\text{ for }\tan\theta=\pm\tan(2r+1)\frac{\pi}{2n};$$

the factors are given by $r=0$ to $\frac{1}{2}(n-3)$; but the expression is of the same degree in numerator and denominator, hence there is a constant term besides the partial fractions, and this is $\frac{1}{n}$ because the coeffs. of $\tan^{n-1}\theta$ in numerator and denominator are 1 and n;

$$\therefore \frac{\tan n\theta}{\tan\theta}=\frac{1}{n}+\sum\left(\frac{A_r}{\tan\theta+\tan\alpha_r}+\frac{B_r}{\tan\theta-\tan\alpha_r}\right),$$

where
$$A_r=\lim_{\theta\to-\alpha_r}\frac{\tan n\theta}{\tan\theta}(\tan\theta+\tan\alpha_r)$$
$$=\lim\frac{1+\tan\alpha_r\cot\theta}{\cot n\theta}=\lim\frac{-\tan\alpha_r\operatorname{cosec}^2\theta}{-n\operatorname{cosec}^2 n\theta}$$
$$=\frac{1}{n}\frac{\tan\alpha_r\cdot\operatorname{cosec}^2\alpha_r}{\operatorname{cosec}^2(2r+1)\frac{\pi}{2}}=\frac{1}{n}\sec\alpha_r\operatorname{cosec}\alpha_r,$$

and $B_r=-A_r$, hence the sum of the two partial fractions is
$$\frac{\tan\alpha_r(A_r-B_r)}{\tan^2\alpha_r-\tan^2\theta}=\frac{2}{n}\frac{\sec^2\alpha_r}{\tan^2\alpha_r-\tan^2\theta}.$$

14. $\sin(2n+1)\theta$ is a polynomial of degree $2n+1$ in $\sin\theta$ which is zero for $\theta=\frac{r\pi}{2n+1}$, $r=0,\ \pm1,\ \pm2,\ \ldots$ hence by p. 231,

$$\operatorname{cosec}(2n+1)\theta=\frac{C}{\sin\theta}+\sum_1^n\left(\frac{A_r}{\sin\theta+\sin\frac{r\pi}{2n+1}}+\frac{B_r}{\sin\theta-\sin\frac{r\pi}{2n+1}}\right)$$

where A_r is the limit for $\theta\to-\frac{r\pi}{2n+1}$ of $\frac{\sin\theta+\sin\frac{r\pi}{2n+1}}{\sin(2n+1)\theta}$;

$$\therefore A_r=\lim\frac{\cos\theta}{(2n+1)\cos(2n+1)\theta}=\frac{(-1)^r}{2n+1}\cos\frac{r\pi}{2n+1};$$

similarly, B_r = same; also

$$C=\lim_{\theta\to0}\frac{\sin\theta}{\sin(2n+1)\theta}=\frac{1}{2n+1};$$

$$\therefore \operatorname{cosec}(2n+1)\theta=\frac{\operatorname{cosec}\theta}{2n+1}+$$
$$\sum_1^n\frac{(-1)^r}{(2n+1)}\left\{\frac{\cos\frac{r\pi}{2n+1}}{\sin\theta+\sin\frac{r\pi}{2n+1}}+\frac{\cos\frac{r\pi}{2n+1}}{\sin\theta-\sin\frac{r\pi}{2n+1}}\right\};$$

$$\therefore (2n+1)\operatorname{cosec}(2n+1)\theta-\operatorname{cosec}\theta$$
$$=\sum_1^n(-1)^r\frac{2\sin\theta\cos\frac{r\pi}{2n+1}}{\sin^2\theta-\sin^2\frac{r\pi}{2n+1}}$$
$$=\Sigma(-1)^r\frac{\sin\left(\theta+\frac{r\pi}{2n+1}\right)+\sin\left(\theta-\frac{r\pi}{2n+1}\right)}{\sin\left(\theta+\frac{r\pi}{2n+1}\right)\sin\left(\theta-\frac{r\pi}{2n+1}\right)}.$$

15. Use the method of Ex. XII. d, No. 16; the denr. is zero for

$$\sin\theta=\sin\frac{r\pi}{n},\ r=\pm1,\ \pm2,\ldots\pm\tfrac{1}{2}(n-1);$$

$$\therefore \text{expn.}=\sum_1^{\frac{1}{2}(n-1)}\left(\frac{A_r}{\sin\theta-\sin\frac{r\pi}{n}}+\frac{B_r}{\sin\theta+\sin\frac{r\pi}{n}}\right)$$

and $$A_r = \frac{1}{n}\sin\frac{mr\pi}{n}\cos\frac{r\pi}{n}\cdot(-1)^r = -B_r, \text{ etc.}$$

For the B_r terms, put $r = n - s$.

16. If m is even, and $\therefore$ n is odd, use the method of Ex. XII. d, No. 17. Compare with No. 15. If m is odd, and $\therefore$ n is even, use the method of Ex. XII. d, No. 18.

17. By the method of Ex. XII. d. No. 19,

$$\operatorname{cosec}(x-a)\operatorname{cosec}(x-b)\operatorname{cosec}(x-c)$$
$$=\frac{\sin^2 x+\cos^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}$$
$$=\sec x\sec a\sec b\sec c\sum\frac{1+t_1^2}{(t-t_1)(t_1-t_2)(t_1-t_3)}$$
$$=\sum\frac{1}{\sin(x-a)\sin(a-b)\sin(a-c)}.$$

Similarly, if $m = 2n+1$,

$$\frac{(\sin^2 x+\cos^2 x)^n}{\sin(x-a_1)\sin(x-a_2)\ldots\sin(x-a_m)}$$
$$=\sec x\sec a_1\sec a_2\ldots\sec a_m\sum\frac{(1+t_1^2)^n}{(t-t_1)(t_1-t_2)\ldots(t_1-t_m)}$$
$$=\sec x\sec a_1\times$$
$$\sum\frac{\sec^{2n} a_1}{\sec x\sec^m a_1\sin(x-a_1)\sin(a_1-a_2)\ldots\sin(a_1-a_m)}$$
$$=\sum\frac{1}{\sin(x-a_1)\sin(a_1-a_2)\ldots\sin(a_1-a_m)}.$$

18. By Ex. XII. d, No. 19,

$$\sin x\operatorname{cosec}(x-a)\operatorname{cosec}(x-b)$$
$$=\Sigma\sin a\operatorname{cosec}(x-a)\operatorname{cosec}(a-b);$$

and, by writing $x+\frac{\pi}{2}$, $a+\frac{\pi}{2}$, $b+\frac{\pi}{2}$ for x, a, b,

$$\cos x\operatorname{cosec}(x-a)\operatorname{cosec}(x-b)$$
$$=\Sigma\cos a\operatorname{cosec}(x-a)\operatorname{cosec}(a-b);$$

multiply by $\sin x$ and $\cos x$ and add, thus

$$\operatorname{cosec}(x-a)\operatorname{cosec}(x-b)$$
$$=\Sigma(\sin a\sin x+\cos a\cos x)\operatorname{cosec}(x-a)\operatorname{cosec}(a-b)$$
$$=\Sigma\cot(x-a)\,.\operatorname{cosec}(a-b).$$

Similarly, if $m = 2n$,

$$\sin x(\sin^2 x+\cos^2 x)^{n-1}\operatorname{cosec}(x-a_1)\operatorname{cosec}(x-a_2)\ldots\operatorname{cosec}(x-a_m)$$
$$=\Sigma\sin a_1\operatorname{cosec}(x-a_1)\operatorname{cosec}(a_1-a_2)\ldots\operatorname{cosec}(a_1-a_m)$$

and $$\cos x(\sin^2 x+\cos^2 x)^{n-1}\operatorname{cosec}(x-a_1)\operatorname{cosec}(x-a_2)\ldots\times\operatorname{cosec}(x-a_m)$$
$$=\Sigma\cos a_1\operatorname{cosec}(x-a_1)\operatorname{cosec}(a_1-a_2)\ldots\operatorname{cosec}(a_1-a_m);$$

multiply by $\sin x$ and $\cos x$ and add, thus

$$(\sin^2 x+\cos^2 x)^n\operatorname{cosec}(x-a_1)\ldots\operatorname{cosec}(x-a_m)$$
$$=\Sigma(\sin a_1\sin x+\cos a_1\cos x)\operatorname{cosec}(x-a_1)\operatorname{cosec}(a_1-a_2)\ldots\times\operatorname{cosec}(a_1-a_m)$$
$$=\Sigma\cot(x-a_1)\operatorname{cosec}(a_1-a_2)\ldots\operatorname{cosec}(a_1-a_m).$$

19. $$\frac{\cos^3 x}{\sin(x-a)\sin(x-b)\sin(x-c)}=\frac{\sec a\sec b\sec c}{(t-t_1)(t-t_2)(t-t_3)}$$
$$=\sum\frac{\sec a\sec b\sec c}{(t-t_1)(t_1-t_2)(t_1-t_3)}=\sum\frac{\cos x\cos^2 a}{\sin(x-a)\sin(a-b)\sin(a-c)}$$

but $\cos x = \cos a\cos(x-a) - \sin a\sin(x-a)$;

$$\therefore\ \frac{\cos x}{\sin(x-a)}=\cos a\cot(x-a)-\sin a;$$
$$\therefore\ \text{given expn.}=\sum\frac{\cos^3 a\cot(x-a)-\sin a\cos^2 a}{\sin(a-b)\sin(a-c)}.$$

20. The perp. from A_s to PR is $\frac{PA_s\,.\,RA_s}{2a}$, see *M.G.*, p. 11;

$\therefore$ product of perps.

$$=\left(\frac{1}{2a}\right)^n.\{PA_1\,.\,PA_2\ldots PA_n\}\,.\,\{RA_1\,.\,RA_2\ldots RA_n\};$$

also $\angle POA_1=\beta-\alpha$, $\angle ROA_1=\beta+\alpha$; $\therefore$ by XII. c, No. 14,

$$\text{product}=\left(\frac{1}{2a}\right)^n.\,2a^n\sin\frac{n(\beta-\alpha)}{2}\,.\,2a^n\sin\frac{n(\beta+\alpha)}{2}$$
$$=2^{1-n}a^n\{\cos n\alpha-\cos n\beta\}.$$

EXERCISE XII. f. (p. 238.)

1. $$\cos 7\theta-\cos 8\theta=2\sin\tfrac{1}{2}\theta\sin\tfrac{15}{2}\theta=2\sin\tfrac{1}{2}\theta(3\sin\tfrac{5}{2}\theta-4\sin^3\tfrac{5}{2}\theta)$$
$$=2\sin\tfrac{1}{2}\theta\sin\tfrac{5}{2}\theta(3-4\sin^2\tfrac{5}{2}\theta)$$
$$=(\cos 2\theta-\cos 3\theta)\{3-2(1-\cos 5\theta)\}.$$

2. Put $\theta=\frac{\pi}{2}-\phi$, and $\cos\phi=c$.

$$\text{Expression}=\frac{(1-\cos 15\phi)(1-\cos\phi)}{(1-\cos 5\phi)(1-\cos 3\phi)};$$

but $$\frac{1-\cos 3\phi}{1-\cos\phi}=\frac{1-4c^3+3c}{1-c}=1+4c+4c^2=(1+2c)^2$$ and therefore, similarly, $\dfrac{1-\cos 15\phi}{1-\cos 5\phi}=(1+2\cos 5\phi)^2$;

$$\therefore \text{ expression}=\left(\frac{1+2\cos 5\phi}{1+2\cos\phi}\right)^2=\text{, by XII. b, No. 33,}$$

$$(16c^4-8c^3-16c^2+8c+1)^2=16\prod_1^4\left\{c-\cos\frac{(3r+5)\,.\,2\pi}{15}\right\};$$

but $\quad c=\sin\theta$ and $\cos\dfrac{(6r+10)\pi}{15}=-\sin\dfrac{(12r+5)\pi}{30}$.

3. $\dfrac{\sin mn\theta}{\sin\theta}$ is a polynomial of degree $(mn-1)$ in $\cos\theta$ with factors of the form $\cos\theta-\cos\dfrac{r\pi}{mn}$ for $r=1$ to $mn-1$; $\dfrac{\sin m\theta}{\sin\theta}$ similarly has factors $\cos\theta-\cos\dfrac{s\pi}{m}$ for $s=1$ to $m-1$, and $\dfrac{\sin n\theta}{\sin\theta}$ has factors $\cos\theta-\cos\dfrac{t\pi}{n}$ for $t=1$ to $n-1$, and these factors may be written $\cos\theta-\cos\dfrac{r\pi}{mn}$ by putting r for ns and for mt. Now

$$\frac{\sin mn\theta\,.\,\sin\theta}{\sin m\theta\,.\,\sin n\theta}=\frac{\sin mn\theta}{\sin\theta}\Big/\left(\frac{\sin m\theta}{\sin\theta}\,.\,\frac{\sin n\theta}{\sin\theta}\right)$$

and all the factors of the denominator are factors of the numerator; since m, n are co-prime, none of the factors of $\dfrac{\sin m\theta}{\sin\theta}$ coincide with factors of $\dfrac{\sin n\theta}{\sin\theta}$, and therefore all the factors cancel, leaving a polynomial of degree

$$mn-1-(m-1)-(n-1)$$

whose factors are given by values of r which are neither of the form ns or mt, i.e. they are not multiples of m or n.

4. r.h.s. $=\text{A}\,.\prod_1^n(\text{ch}^2x\sin^2\alpha+\text{sh}^2x\cos^2\alpha)$ where $\alpha=(2r-1)\dfrac{\pi}{2n}$ and A is independent of x;

$$\therefore \text{ r.h.s.}=\text{A}\,.\prod_1^n\{\tfrac12\text{ch}^2x(1-\cos 2\alpha)+\tfrac12\text{sh}^2x(1+\cos 2\alpha)\}$$

$$=\text{A}\,.\prod_1^n\tfrac12(\text{ch }2x-\cos 2\alpha)$$

$$=\text{A}\,.\prod_1^n\tfrac14(e^{2x}+e^{-2x}-2\cos 2\alpha)$$

$$=\text{B}e^{-2nx}\,.\prod_1^n(e^{4x}-2e^{2x}\cos 2\alpha+1)$$

$$=\text{, by eqn. 10, B}\,.\,e^{-2nx}\{e^{4nx}-2e^{2nx}\cos(-\pi)+1\}$$

$$=\text{B}e^{-2nx}(e^{2nx}+1)^2=\text{C ch}^2(nx)$$

where B and C are independent of x. Put $x=0$, thus C $=1$.

5. ch nx and ch x are always positive; $\therefore$ the $\pm$ may be omitted.

$$\cot\frac{(2k-1)\pi}{2n}=-\cot\frac{(2n-2k+1)\pi}{2n};$$

$\therefore$ the values of $\cot^2\dfrac{(2r-1)\pi}{2n}$ for $r=k$, $r=n+1-k$ are the same. If n is even, the factors consist of $\tfrac12 n$ pairs; if n is odd, there are $\tfrac12(n-1)$ pairs and the factor given by $r=\tfrac12(n+1)$ namely $1+\text{th}^2x\cot^2\dfrac{\pi}{2}=1$.

6. $2\,\text{sh}\,nx=(\text{ch}\,x+\text{sh}\,x)^n-(\text{ch}\,x-\text{sh}\,x)^n$ (see Ex. VI. e, Nos. 14, 15)

$$=\text{ch}^n x\{(1+t)^n-(1-t)^n\}\text{, where } t=\text{th}\,x,$$

$$=\text{ch}^n x\,.\,2n\,\text{th}\,x\,.\prod_1^{\frac12 n-1}\left(\text{th}^2x+\cot^2\frac{r\pi}{n}\right)$$

by Ex. XII. e, No. 5,

$$=2n\,\text{sh}\,x\,\text{ch}^{n-1}x\prod_1^{\frac12 n-1}\left(\text{th}^2x+\cot^2\frac{r\pi}{n}\right)$$

$$=\text{A sh}\,x\,\text{ch}^{n-1}x\prod_1^{\frac12 n-1}\left(1+\text{th}^2x\tan^2\frac{r\pi}{n}\right),$$

where A is independent of x; and $\text{A}=\lim\limits_{x\to 0}\dfrac{2\,\text{sh}\,nx}{\text{sh}\,x}=2n$; when n is odd, Ex. XII. e, No. 4, gives in the same way

$$2\,\text{sh}\,nx=\text{ch}^n x\,.\,2\,\text{th}\,x\prod_1^{\frac12(n-1)}\left(\text{th}^2x+\tan^2\frac{r\pi}{n}\right)$$

$$=\text{B sh}\,x\,\text{ch}^{n-1}x\prod_1^{\frac12(n-1)}\left(1+\text{th}^2x\cot^2\frac{r\pi}{n}\right);$$

where $\qquad \text{B}=\lim\limits_{x\to 0}\dfrac{2\,\text{sh}\,nx}{\text{sh}\,x}=2n.$

7. In Ex. XII. b, No. 3, put $\dfrac{\pi}{3}$ for θ, and θ for α. For

$$n^2\equiv 1\ (\text{mod } 6),\ n\equiv\pm 1\ (\text{mod } 6);$$

$$\therefore\ \cos n\theta=\cos(6m\pm 1)\frac{\pi}{3}=\cos 2m\pi\,.\,\cos\frac{\pi}{3}=\tfrac12;$$

$$\therefore\ \tfrac12-\cos n\theta=2^{n-1}\prod_0^{n-1}\left\{\cos\frac{\pi}{3}-\cos\left(\theta+\frac{2r\pi}{n}\right)\right\}.$$

8. l.h.s. $=\frac{1}{2}\{\cos\frac{1}{2}n\pi-\cos(\frac{1}{2}n\pi+2n\phi)\}$; put $\frac{1}{2}\pi$ for θ, and $\frac{1}{2}(\pi+4\phi)$ for a, in Ex. XII. b, No. 3; thus

$$\text{l.h.s.}=2^{n-2}\prod_0^{n-1}\left\{\cos\tfrac{1}{2}\pi-\cos\left(\tfrac{1}{2}\pi+2\phi+\frac{2r\pi}{n}\right)\right\}$$
$$=2^{n-2}\prod_0^{n-1}\sin\left(2\phi+\frac{2r\pi}{n}\right).$$

9. $\cos n\phi+\sin n\phi=\sqrt{2}.\ \sin\left(n\phi+\frac{\pi}{4}\right)=\sqrt{2}\,.\sin n\left(\phi+\frac{\pi}{4n}\right)=$, by eqn. 12, $\sqrt{2}\,.\,2^{n-1}.\prod_0^{n-1}\sin\left(\phi+\frac{\pi}{4n}+\frac{r\pi}{n}\right)$.

10. In Ex. XII. c, No. 1, put $\beta=\frac{a}{n}$.

11. Use No. 10; take logarithms and differentiate w.r.t. a.

12. By eqn. (12), $\prod\sin\left(\theta+\frac{r\pi}{n}\right)=2^{1-n}\sin n\theta$; by Ex. XII. c, No. 2, $\prod\cos\left(\theta+\frac{r\pi}{n}\right)=2^{1-n}.(-1)^{\frac{1}{2}(n-1)}.\cos n\theta$; square and add.

13. Put $\cos\theta+\sin\theta=y$, $\cos\theta-\sin\theta=z$, then $y^2+z^2=2$ and $yz=\cos 2\theta$; l.h.s. $=\frac{y^{2n+1}-z^{2n+1}}{2^n(y-z)}=$, by eqn. (2), putting $\frac{y}{z}$ for x, and $2n+1$ for n,

$$\frac{1}{2^n}\prod_1^n\left(y^2-2yz\cos\frac{2r\pi}{2n+1}+z^2\right)$$
$$=\frac{1}{2^n}\prod_1^n\left(2-2\cos 2\theta\cos\frac{2r\pi}{2n+1}\right).$$

14. Use Ex. XII. b, No. 3, put 2ϕ for θ and 0 for a; also put 2ϕ for θ and $\frac{\pi}{n}$ for a, and divide; thus

$$\text{l.h.s.}=\frac{\cos 2n\phi-1}{\cos 2n\phi+1}=-\tan^2 n\phi.$$

15. Let the tangent be $x=5a$, then the vertices have polar coordinates $(3a,\theta)$ where $\theta=a+(2r-1)\frac{\pi}{n}$ for $r=0$ to $n-1$. Product is thus

$$\prod_0^{n-1}\left[5a-3a\cos\left\{a+(2r-1)\frac{\pi}{n}\right\}\right]=\prod_0^{n-1}\left(5a-3a\frac{1-t^2}{1+t^2}\right)$$

where $t=\tan\frac{1}{2}\left\{a+(2r-1)\frac{\pi}{n}\right\}$.

$$=a^n\prod_0^{n-1}\frac{2+8t^2}{1+t^2}=(8a)^n\prod_0^{n-1}\frac{(\frac{1}{2})^2+t^2}{1+t^2}.$$

The equation $\left(\frac{1+x}{1-x}\right)^n=-\text{cis}\,na=\text{cis}\,\{na+(2r-1)\pi\}$ is satisfied by $\frac{1+x}{1-x}=\text{cis}\left\{a+(2r-1)\frac{\pi}{n}\right\}$, or $x=i\tan\phi$, where $\phi=\frac{1}{2}\left\{a+(2r-1)\frac{\pi}{n}\right\}$ as in the solution of Example 2; thus the factors of $(1+x)^n+(1-x)^n\,\text{cis}\,na$ are

$$\{1+(-1)^n\,\text{cis}\,na\}\prod(x-i\tan\phi);$$

change i to $-i$ and multiply, thus

$$(1+x)^{2n}+(1-x)^{2n}+(1-x^2)^n.\,2\cos na$$
$$=\{1+1+(-1)^n\,2\cos na\}\prod(x^2+\tan^2\phi);$$

put $x=\frac{1}{2}$ and $x=1$ and divide, thus

$$\prod\frac{(\frac{1}{2})^2+t^2}{1+t^2}=\{(\tfrac{3}{2})^{2n}+(\tfrac{1}{2})^{2n}+(\tfrac{3}{4})^n\,2\cos na\}\div 2^{2n}$$
$$=(3^{2n}+1+2\,.\,3^n\cos na)\div 16^n$$

Or $\prod_0^{n-1}\left[5a-3a\cos\left\{\left(a-\frac{\pi}{n}\right)+\frac{2r\pi}{n}\right\}\right]$

$$=\prod_0^{n-1}\frac{a}{2}\left[3^2-2\,.\,3\cos\left\{\left(a-\frac{\pi}{n}\right)+\frac{2r\pi}{n}\right\}+1\right]$$

$=$, by eqn. (10), $\left(\frac{a}{2}\right)^n.\,[3^{2n}-2\,.\,3^n\cos(na-\pi)+1]$.

16. If K is the projection of P and if KQR is the diameter through it,

$$PA^2=PK^2+KA^2=PK^2+(KO^2+OA^2-2KO\,.\,OA\cos KOA)$$
$$=PO^2+OQ^2-\tfrac{1}{2}(KR^2-KQ^2)\cos KOA$$
$$=\tfrac{1}{2}(r_1^2+r_2^2)-\tfrac{1}{2}(r_1^2-r_2^2)\cos KOA$$
$$=s^2+d^2-2sd\cos KOA,$$

and $\angle$ KOA $=a,\ a+\frac{2\pi}{n},\ \dots\ a+\frac{2(n-1)\pi}{n}$ for the various vertices; now use eqn. (10) with $\frac{s}{d}$ instead of x.

17. Use Ex. XII. *b*, No. 4; take logarithms and differentiate w.r.t. θ.

18. $\sum_1^n\cot^2\frac{r\pi}{2n+1}=\sum_1^n\left(\text{cosec}^2\frac{r\pi}{2n+1}\right)-n=\frac{(2n+1)^2-1}{6}-n$ by the result on p. 209 with $2n+1$ instead of n.

19. By eqn. (12) $\dfrac{\sin n\beta}{\sin\beta}=2^{n-1}\prod_1^{n-1}\sin\left(\beta+\dfrac{r\pi}{n}\right)$; make $\beta\to 0$.

Take logarithms of each side in the first part, hence

$$(n-1)\log 2+\sum_1^{n-1}\log\sin\frac{r\pi}{n}=\log n;$$

$$\therefore\ \frac{\pi}{n}\sum_1^{n-1}\log\sin\frac{r\pi}{n}=\frac{\pi\log n}{n}-\pi\left(1-\frac{1}{n}\right)\log 2$$

which $\to -\pi\log 2$ when $n\to\infty$ (see p. 68, Ex. 4 (i)). Now $\int_0^\pi(\log\sin x)dx$ is an improper integral whose existence is proved in Ex. IV. g, No. 26, and is equal to

$$\int_0^{\frac{\pi}{n}}(\log\sin x)dx+\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{n}}(\log\sin x)dx+\int_{\pi-\frac{\pi}{n}}^{\pi}(\log\sin x)dx;$$

here the first and third terms are improper integrals which $\to 0$ when $n\to\infty$ by the Ex. just quoted, and the middle term is an ordinary integral which, by the theory of integration by summation of rectangles, differs from $\dfrac{\pi}{n}\sum_1^{n-1}\log\sin\dfrac{r\pi}{n}$ by less than $-2\cdot\dfrac{\pi}{n}\log\sin\dfrac{\pi}{n}$. But

$$\lim\left[\frac{\pi}{n}\log\sin\frac{\pi}{n}\right]=\lim\left[\frac{\pi}{n}\log\frac{\pi}{n}+\frac{\pi}{n}\log\left(\sin\frac{\pi}{n}\Big/\frac{\pi}{n}\right)\right]$$

$$=\lim\left[\frac{\pi}{n}\log\frac{\pi}{n}\right]=0,\text{ by p. 68, Ex. 4 (ii).}$$

$$\therefore\ \lim(\text{middle term})=\lim\left(\frac{\pi}{n}\sum_1^{n-1}\log\sin\frac{r\pi}{n}\right)=-\pi\log 2.$$

20. By No. 19, $2^{N-1}\cdot\prod_1^{N-1}\sin\dfrac{r\pi}{N}=N$; from this result others may be deduced by writing $\dfrac{N}{p}$ for N where p is a prime factor of N, also by writing $\dfrac{N}{pp'}$ for N where p, p', are both prime factors, also by writing $\dfrac{N}{pp'p''}$ and so on. Hence

$$h\equiv\frac{N\cdot\prod\left(\frac{N}{pp'}\right)\dots}{\prod\left(\frac{N}{p}\right)\cdot\prod\left(\frac{N}{pp'p''}\right)\dots}=2^k\prod\sin\frac{r\pi}{N}$$

for those values of r which are prime to N,

$$=2^k\sin(N_1\alpha)\sin(N_2\alpha)\dots\sin(N_m\alpha),$$

also the value of k is

$$N-1-\sum\left(\frac{N}{p}-1\right)+\sum\left(\frac{N}{pp'}-1\right)-\dots$$

$$=N\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p'}\right)\dots-1+\binom{z}{1}-\binom{z}{2}+\binom{z}{3}-\dots,$$

where z is the number of prime factors; but

$$N\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p'}\right)\dots=m,\text{ and}$$

$$1-\binom{z}{1}+\binom{z}{2}-\binom{z}{3}+\dots=(1-1)^z=0;\quad\therefore\ k=m.$$

The number of N's in the numerator of h is $1+\binom{z}{2}+\dots$ and the number in the denominator is $\binom{z}{1}+\binom{z}{3}+\dots$, hence the N's cancel; so do the p's; $\therefore\ h=1$. If, however, N is itself prime, there are no numbers p, and $h=N$, see XII. b, No. 18, taking n odd and squaring.

21. In Ex. XII. b, No. 24 (i), put $2^{n+1}+1$, which is odd, instead of n, thus $\prod_1^{2n}\cos(r\alpha)=2^{-2^n}$, thus

$$\sec\alpha\sec 2\alpha\sec 3\alpha\dots\sec 2^n\alpha=2^{2^n}.$$

Also $\quad\sin\alpha\cos\alpha\cos 2\alpha\cos 2^2\alpha\dots\cos 2^n\alpha$

$$=\tfrac{1}{2}\sin 2\alpha\cos 2\alpha\cos 2^2\alpha\dots\cos 2^n\alpha$$

$$=\frac{1}{2^2}\sin 2^2\alpha\,.\,\cos 2^2\alpha\dots\cos 2^n\alpha=\dots$$

$$=\frac{1}{2^n}\sin 2^n\alpha\cos 2^n\alpha=\frac{1}{2^{n+1}}\sin 2^{n+1}\alpha.$$

Multiplication of these two results gives

$$\prod_{r=1}^{t}\sec(n_r\alpha)=2^{2^n-n-1}\frac{\sin 2^{n+1}\alpha}{\sin\alpha}\text{ and }\sin 2^{n+1}\alpha=\sin\alpha.$$

22. $\displaystyle\prod_{s=1}^{n-1}\sin\left\{\frac{(m-1)\pi}{n}-\frac{(s-1)\pi+\theta}{(n-1)}\right\}=\prod_{s=1}^{n-1}\sin\left\{\alpha-\frac{(s-1)\pi}{n-1}\right\},$

where $\alpha=\dfrac{(m-1)\pi}{n}-\dfrac{\theta}{n-1}$, $=\displaystyle\prod_{r=0}^{n-2}\sin\left(\alpha-\frac{r\pi}{n-1}\right)$

$$=\text{by eqn. 12, }\frac{(-1)^n\sin(n-1)\alpha}{2^{n-2}};$$

$\therefore$ the double product

$$=\prod_{m=1}^{n}\frac{(-1)^n}{2^{n-2}}\sin\left\{\frac{(m-1)(n-1)\pi}{n}-\theta\right\}$$

$$=\frac{(-1)^{n^2}}{2^{n(n-2)}}\prod_{m=0}^{n-1}\sin\left(m\pi-\frac{m\pi}{n}-\theta\right)$$

$$=\frac{(-1)^{n^2}}{2^{n(n-2)}}\prod_{0}^{n-1}-\cos m\pi\sin\left(\frac{m\pi}{n}+\theta\right)$$

$$=\frac{(-1)^{n^2+n}}{2^{n(n-2)}}(-1)^{\frac{1}{2}n(n-1)}\prod_{0}^{n-1}\sin\left(\frac{m\pi}{n}+\theta\right),\text{ since}$$

$$\prod_{0}^{n-1}\cos m\pi=\prod_{0}^{n-1}(-1)^m=(-1)^{1+2+\dots+(n-1)}=(-1)^{\frac{1}{2}n(n-1)},$$

and this by eqn. (12)

$$=\frac{(-1)^{\frac{1}{2}n(n-1)}}{2^{n^2-2n}}\cdot\frac{\sin n\theta}{2^{n-1}}.$$

If n is odd, $(-1)^{\frac{1}{2}n(n-1)}=(-1)^{\frac{1}{2}(n-1)}=(-1)^{[\frac{1}{2}n]}$. If n is even, $(-1)^{\frac{1}{2}n(n-1)}=(-1)^{\frac{1}{2}n}=(-1)^{[\frac{1}{2}n]}$.

23. (i) $\sin^2\theta\dfrac{d}{d\theta}(\theta\operatorname{cosec}\theta)=\sin\theta-\theta\cos\theta=\cos\theta(\tan\theta-\theta)$;

$\sin^2\theta\dfrac{d}{d\theta}(\theta\cot\theta)=\sin\theta\cos\theta-\theta=\frac{1}{2}(\sin 2\theta-2\theta)$;

but, for $0<\theta<\frac{1}{2}\pi$, $\tan\theta>\theta$ and $\sin 2\theta<2\theta$; $\therefore$ as θ increases, $\theta\operatorname{cosec}\theta$ steadily increases and $\theta\cot\theta$ steadily decreases, hence, for $0<\theta<\phi<\frac{1}{2}\pi$,

$$\theta\operatorname{cosec}\theta<\phi\operatorname{cosec}\phi\text{ and }\theta\cot\theta>\phi\cot\phi,$$

$$\therefore\ \frac{\sin\theta}{\sin\phi}>\frac{\theta}{\phi}>\frac{\tan\theta}{\tan\phi};$$

$$\therefore\ 1-\frac{\sin^2\theta}{\sin^2\phi}<1-\frac{\theta^2}{\phi^2}<1-\frac{\tan^2\theta}{\tan^2\phi}=\sec^2\theta\left(1-\frac{\sin^2\theta}{\sin^2\phi}\right);$$

(ii) Put $\theta=\dfrac{x}{n}$, $\phi=\dfrac{r\pi}{n}$; then, since $0<x<\pi$, $\theta<\phi$ even for $r=1$; also $\phi<\frac{1}{2}\pi$ even for $r=\frac{1}{2}(n-1)$; $\therefore$ conditions in (i) are satisfied; give r the various values and multiply;

(iii) l.h.s. $=\dfrac{\sin x}{n\sin(x/n)}\to\dfrac{\sin x}{x}$; p. 70 gives $\sec^n(x/n)\to 1$, $\therefore$ also $\sec^{n-1}(x/n)\to 1$, thus r.h.s. tends to the same limit.

(iv) If $0<\phi<\theta<\frac{1}{2}\pi$, $1<\dfrac{\sin\theta}{\sin\phi}<\dfrac{\theta}{\phi}<\dfrac{\tan\theta}{\tan\phi}$, as in (i);

$$\therefore\ \frac{\sin^2\theta}{\sin^2\phi}-1<\frac{\theta^2}{\phi^2}-1<\frac{\tan^2\theta}{\tan^2\phi}-1;$$

hence the result follows. For $\theta=\phi$, the signs become $=$. For an arbitrary positive value of x, to secure

$$0<(\theta,\phi)<\tfrac{1}{2}\pi,$$

choose n so large that $0<x/n<\frac{1}{2}\pi$, then although θ is not necessarily less than ϕ, the argument above can be applied to show that

$$|\sin x/x|=\prod_{1}^{\infty}\left|1-\frac{x^2}{r^2\pi^2}\right|,$$

and it can be verified that the expressions inside the mod. signs are both positive, both zero, or both negative. If x is changed to $-x$, both sides in the result of (iii) change sign; for $x=0$ the result is trivial.

24. Put $\theta=x/n$, $\phi=(2r-1)\pi/2n$ and proceed as in No. 23; we obtain

$$\left|\cos x/\cos\frac{x}{n}\right|\leqslant\prod_{1}^{\frac{1}{2}(n-1)}\left|1-\frac{4x^2}{(2r-1)^2\pi^2}\right|\leqslant\left|\sec^n\frac{x}{n}\cos x\right|;$$

make $n\to\infty$.

CHAPTER XIII

EXERCISE XIII. a. (p. 243.)

1. Use eqns. (2), (3);

(a) $\rho=1$, $\phi=2n\pi$, $\phi_1=0$; (b) $\rho=1$, $\phi=2n\pi+\dfrac{\pi}{2}$, $\phi_1=\dfrac{\pi}{2}$;

(c) $\rho=2$, $\phi=(2n+1)\pi$, $\phi_1=\pi$;

(d) $\rho=2$, $\phi=2n\pi-\dfrac{\pi}{2}$, $\phi_1=-\dfrac{\pi}{2}$.

2. Use eqns. (2), (3);

(a) $1+i=\sqrt{2}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$;

(b) $\sqrt{i}=\sqrt{\left(\operatorname{cis}\dfrac{\pi}{2}\right)}=\operatorname{cis}\dfrac{\pi}{4}$, $\operatorname{Log}\left(\operatorname{cis}\dfrac{\pi}{4}\right)=\left(2n\pi+\dfrac{\pi}{4}\right)i$;

(c) $-3+4i=5(-\frac{3}{5}+\frac{4}{5}i)$; $\rho=5$, $\cos\phi=-\frac{3}{5}$, $\sin\phi=\frac{4}{5}$.

$$\therefore\ \frac{\pi}{2}<\phi_1<\pi;\ \therefore\ \phi_1=\pi+\tan^{-1}\left(\frac{4}{-3}\right);$$

(d) $-3-4i=5(-\frac{3}{5}-\frac{4}{5}i)$, $\rho=5$, $\cos\phi=-\frac{3}{5}$, $\sin\phi=-\frac{4}{5}$;

$$\therefore\ -\pi<\phi_1<-\frac{\pi}{2};\ \therefore\ \phi_1=-\pi+\tan^{-1}\left(\frac{-4}{-3}\right).$$

3. $\text{Log}\,(e\text{ cis }\pi)=\log e+i(\pi+2n\pi)$.

4. $\text{Log}\,(\cos\alpha+i\sin\alpha)=i(\alpha+2k\pi)$; since $2n\pi-\pi<\alpha\leqslant 2n\pi+\pi$; $\therefore\ -\pi<\alpha-2n\pi\leqslant\pi$; $\therefore\ \log(\text{cis }\alpha)=i(\alpha-2n\pi)$.

5. $a+ib=\exp\left(2-\frac{3\pi i}{4}\right)=\exp(2)\,.\,\exp\left(-\frac{3\pi i}{4}\right)$

$$=e^2\text{ cis}\left(-\frac{3\pi}{4}\right)=e^2\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right).$$

6. $\text{Log}\,(1+\cos 2\theta+i\sin 2\theta)=\text{Log}\,(2\cos^2\theta+2i\sin\theta\cos\theta)$

$$=\text{Log}\,\{2\cos\theta\,.\,\text{cis }\theta\}=\text{Log}\,\{(-2\cos\theta)\,.\,\text{cis}\,(\theta+\pi)\}$$
$$=\log(-2\cos\theta)+i(\theta+\pi+2k\pi)$$

since, for $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$, $\cos\theta<0$; also $-\frac{\pi}{2}<\theta-\pi<\frac{\pi}{2}$; therefore for p.v. put $k=-1$.

7. (i) From No. 1, (b), $i\,.\left(2n\pi+\frac{\pi}{2}\right)i$;

(ii) $1+i\sqrt{3}=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

8. $\text{Log}\,(1+i\tan\alpha)=\text{Log}\,\{\sec\alpha(\cos\alpha+i\sin\alpha)\}$

$$=\text{Log}\,\{(-\sec\alpha)\,.\,\text{cis}\,(\alpha+\pi)\}$$
$$=\log(-\sec\alpha)+i(\alpha+\pi+2k\pi)$$

since, for $\frac{\pi}{2}<\alpha<\frac{3\pi}{2}$, $\sec\alpha<0$; for p.v., as in No. 6, put $k=-1$.

9. $(1+i)^3=1+3i+3i^2+i^3=-2+2i$;

$$\therefore\ a+ib=\exp\left\{\frac{\pi}{6}(-2+2i)\right\}=\exp\left(-\frac{\pi}{3}+\frac{\pi}{3}i\right)$$
$$=e^{-\frac{\pi}{3}}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=\tfrac{1}{2}e^{-\frac{\pi}{3}}(1+i\sqrt{3}).$$

10. Since $a<0<b$, $\frac{b}{a}<0$; $\therefore\ -\frac{\pi}{2}<\alpha<\frac{\pi}{2}$ implies $-\frac{\pi}{2}<\alpha<0$; put $c=+\sqrt{(a^2+b^2)}$; $\sin\alpha<0$, $\therefore\ b=-c\sin\alpha$; $\cos\alpha>0$, $\therefore\ a=-c\cos\alpha$;

(i) $\log(a+ib)=\log\{-c(\cos\alpha+i\sin\alpha)\}=\log\{c\text{ cis}\,(\alpha+\pi)\}$

$$=\log c+(\alpha+\pi)i,\text{ since }\frac{\pi}{2}<\alpha+\pi<\pi;$$

$\log(a-ib)=\log\{-c(\cos\alpha-i\sin\alpha)\}=\log\{c\text{ cis}\,(\pi-\alpha)\}$

$$=\log c+(-\pi-\alpha)i,\text{ since }-\pi<-\pi-\alpha<-\frac{\pi}{2};$$

$\therefore\ \log(a+ib)-\log(a-ib)$

$$=\{\log c+(\alpha+\pi)i\}-\{\log c+(-\pi-\alpha)i\};$$

Or, direct from eqns. (4), for $a<0<b$,

$$\log(a+ib)=\tfrac{1}{2}\log(a^2+b^2)+i\left\{\tan^{-1}\frac{b}{a}+\pi\right\}$$
$$=\tfrac{1}{2}\log(a^2+b^2)+i\{\alpha+\pi\};$$
$$\log(a-ib)=\log[a+i(-b)]$$
$$=\tfrac{1}{2}\log(a^2+b^2)+i\left\{\tan^{-1}\frac{-b}{a}-\pi\right\}$$
$$=\tfrac{1}{2}\log(a^2+b^2)+i\{-\alpha-\pi\}.$$

(ii) $\log\frac{a+ib}{a-ib}=\log\frac{c\text{ cis}\,(\alpha+\pi)}{c\text{ cis}\,(\pi-\alpha)}$

$$=\log\text{cis}\,(2\alpha)=2\alpha i\ \text{ since }-\pi<2\alpha<0.$$

11. $\text{Log}\left(2\sin^2\frac{\theta}{2}-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)=\text{Log}\left\{2\sin\frac{\theta}{2}\,.\,\text{cis}\,\frac{\theta-\pi}{2}\right\}$

$$=\text{Log}\left\{\left(-2\sin\frac{\theta}{2}\right).\,\text{cis}\,\frac{\theta+\pi}{2}\right\};$$

$\sin\frac{\theta}{2}>0$ if $2m\pi<\frac{\theta}{2}<(2m+1)\pi$ and in this case

$$-\frac{\pi}{2}<\frac{\theta-\pi}{2}-2m\pi<\frac{\pi}{2};$$

$\therefore$ expression $=\log\left(2\sin\frac{\theta}{2}\right)+i\left(\frac{\theta-\pi}{2}+2k\pi\right)$, and for p.v. put $k=-m$; $\sin\frac{\theta}{2}<0$ if $(2m+1)\pi<\frac{\theta}{2}<(2m+2)\pi$ and in this case

$$-\frac{\pi}{2}<\frac{\theta+\pi}{2}-(2m+2)\pi<\frac{\pi}{2};$$

$$\therefore\ \text{expression}=\log\left(-2\sin\frac{\theta}{2}\right)+i\left(\frac{\theta+\pi}{2}+2k\pi\right),$$

and for p.v. put $k=-(m+1)$.

12. $\log(c+di)=\frac{f+gi}{a+bi}=\frac{(f+gi)(a-bi)}{(a+bi)(a-bi)}$

$$=\frac{(af+bg)+i(ag-bf)}{a^2+b^2}=\mathsf{A}+i\mathsf{B},\text{ say};$$

then $c+di=\exp(\mathsf{A}+i\mathsf{B})=e^{\mathsf{A}}(\cos\mathsf{B}+i\sin\mathsf{B})$.

13. $\text{Log } 1 = \text{Log}(\text{cis } 0) = 2n\pi i$;

$$3 \text{ Log } w = 3 \text{ Log}\left(\text{cis}\frac{2\pi}{3}\right) = 3\left(\frac{2\pi}{3} + 2k\pi\right)i = (6k+2)\pi i;$$

therefore the values of $3 \text{ Log } w$ are only *some* of the values of $\text{Log } 1$; similarly $3 \text{ Log } w^2 = (6k+4)\pi i$ are also only some of the values of $\text{Log } 1$; the remaining values are given by $3 \text{ Log}(\sqrt[3]{1}) \equiv 3 \text{ Log}(1)$

14. $z = \exp(a+bi) = e^a \cdot \text{cis } b$; $\therefore \text{Am}(z) = b + 2k\pi$;

$$\text{Am}(p+iq) = \text{Am}\{z \log z\} = \text{Am}(z) + \text{Am}(\log z);$$

$$\therefore \tan^{-1}\left(\frac{q}{p}\right) + n\pi = b + 2k\pi + \tan^{-1}\left(\frac{b}{a}\right) + m\pi;$$

$$\therefore \text{Tan}^{-1}\left(\frac{p}{q}\right) = \frac{\pi}{2} - \text{Tan}^{-1}\left(\frac{q}{p}\right)$$

$$= \frac{\pi}{2} - b - \tan^{-1}\left(\frac{b}{a}\right) + n'\pi$$

$$= \tan^{-1}\left(\frac{a}{b}\right) - b + n''\pi = \text{Tan}^{-1}\left(\frac{a}{b}\right) - b.$$

15. $\exp[\frac{1}{2} \text{Log } a^2] = \exp[\frac{1}{2}(\log a^2 + 2n\pi i)]$

$$= \exp(\tfrac{1}{2}\log a^2) \cdot \exp(n\pi i)$$

$$= \exp(\log|a|) \cdot \text{cis } n\pi = |a| \cdot (-1)^n.$$

16. (i) By No. 1 (*b*),

$$(1+i) \text{ Log } i = (1+i) \cdot i\left(\frac{\pi}{2} + 2n\pi\right)$$

$$= -\tfrac{1}{2}(4n+1)\pi + \tfrac{1}{2}(4n+1)\pi i;$$

$$\therefore \exp\{(1+i) \text{ Log } i\}$$

$$= e^{-\frac{1}{2}(4n+1)\pi} \cdot \text{cis}\frac{(4n+1)\pi}{2} = e^{-\frac{1}{2}(4n+1)\pi} \cdot i;$$

(ii) By No. 2 (*a*),

$$(1+i) \text{ Log}(1+i) = (1+i)\left\{\tfrac{1}{2}\log 2 + i\left(\frac{\pi}{4} + 2n\pi\right)\right\}$$

$$= \tfrac{1}{2}\log 2 - \frac{\pi}{4} - 2n\pi + i\left(\tfrac{1}{2}\log 2 + \frac{\pi}{4} + 2n\pi\right);$$

$$\therefore \text{ expression} = \exp(\tfrac{1}{2}\log 2)$$

$$\times \exp\left(-\frac{\pi}{4} - 2n\pi\right) \cdot \text{cis}\left(\tfrac{1}{2}\log 2 + \frac{\pi}{4}\right)$$

but $\exp(\frac{1}{2}\log 2) = \exp(\log\sqrt{2}) = \sqrt{2}$.

17. $\exp\{i \cdot (a + 2n\pi)i\} = \exp(-a - 2n\pi)$.

18. $\text{Log}(\cos\theta + i\sin\theta) = i(\theta + 2n\pi) = (\theta + 2n\pi) \cdot \text{cis}\frac{\pi}{2}$;

$$\therefore \text{ Log}\{\text{Log}(\text{cis }\theta)\} = \log(\theta + 2n\pi) + i\left(\frac{\pi}{2} + 2m\pi\right).$$

19. $\log(z-1) = \log\{(x-1)+iy\}$; use eqn. (4). If P is to the right of the line $x=1$, $u \equiv x-1 > 0$; $\therefore k=0$; when P is to the left of the line $x=1$, (i.e. $u \equiv x-1 < 0$), if P is above Ox, $v \equiv y > 0$; $\therefore k=1$ and if P is below Ox, $v \equiv y < 0$; $\therefore k=-1$.

20. $\log(z+1) = \log\{(x+1)+iy\}$

$$= \tfrac{1}{2}\log\{(x+1)^2 + y^2\} + i\tan^{-1}\frac{y}{x+1} + ik\pi, \; x \neq -1;$$

in No. 19, replace $x-1$ by $x+1$ and $x=1$ by $x=-1$.

21. $$\log\frac{z-1}{z+1} = \log\frac{(x+yi-1)(x-yi+1)}{(x+yi+1)(x-yi+1)} = \log\frac{x^2+y^2-1+2yi}{(x+1)^2+y^2}$$

$$=, \text{ by eqn. (4)}, \log\rho + i\tan^{-1}\frac{2y}{x^2+y^2-1} + ik\pi,$$

where $\rho = \dfrac{+\sqrt{\{(x^2+y^2-1)^2+4y^2\}}}{(x+1)^2+y^2}$ and $k=0$ if $x^2+y^2-1>0$,

i.e. if P lies outside the circle $|z|=1$. If P is inside the circle (i.e. $x^2+y^2-1<0$), then $k=1$ if $y>0$, i.e. P in upper semicircle, and $k=-1$ if $y<0$, i.e. P in lower semicircle.

22. $$\frac{z-1}{z+1} = \frac{-1+\cos\theta+i\sin\theta}{1+\cos\theta+i\sin\theta} = \frac{-2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)} = i\tan\frac{\theta}{2}.$$

If P is on upper semicircle, $0 < \theta < \pi$, $\tan\frac{\theta}{2} > 0$;

$$\therefore \log\frac{z-1}{z+1} = \log\left\{\tan\frac{\theta}{2} \cdot \text{cis}\frac{\pi}{2}\right\} = \log\left(\tan\frac{\theta}{2}\right) + \frac{\pi i}{2}.$$

If P is on lower semicircle $-\pi < \theta < 0$, $\tan\frac{\theta}{2} < 0$;

$$\therefore \log\frac{z-1}{z+1} = \log\left\{\left(-\tan\frac{\theta}{2}\right)\text{cis}\left(-\frac{\pi}{2}\right)\right\}$$

$$= \log\left(-\tan\frac{\theta}{2}\right) - \frac{\pi}{2}i.$$

Or, geometrically, if A, B are the ends $(1, 0), (-1, 0)$ of the diameter of $|z|=1$, $\frac{z-1}{z+1}=\frac{\overline{AP}}{\overline{BP}}$. The modulus of $\frac{\overline{AP}}{\overline{BP}}$ $=\frac{AP}{BP}=\tan ABP=\tan\frac{\theta}{2}$ if P is on upper semicircle, and $=\tan\frac{2\pi-\theta}{2}=-\tan\frac{\theta}{2}$ if P is on lower semicircle. The amplitude of $\frac{\overline{AP}}{\overline{BP}}$ = anticlockwise rotation necessary to convert BP into AP, $=\frac{\pi}{2}$ if P is on upper semicircle, and $=-\frac{\pi}{2}$ if P is on lower semicircle.

23. $p+qi=\log\frac{z-a}{z+a}=\log\rho+\theta i$, where $\rho=\left|\frac{z-a}{z+a}\right|$, $\theta=\text{am}\left(\frac{z-a}{z+a}\right)$;

If A, B are the points $(a, 0)$, $(-a, 0)$, $\rho=\frac{|z-a|}{|z+a|}=\frac{AP}{BP}$; $\therefore$ for p = const., $\frac{AP}{BP}$ = const.; $\therefore$ P lies on a circle w.r.t. which A, B are inverse points, and for different values of p these circles form a coaxal system with A, B as limiting points. Also, as in No. 22, $q=\theta=\text{am}\left(\frac{z-a}{z+a}\right)$ = the anti-clockwise rotation necessary to convert BP into AP; $\therefore$ P lies on an arc of a circle through A, B; as q varies these arcs form the coaxal system orthogonal to the first system.

24. As in No. 22, $q=\text{am}\left(\frac{z-a}{z+a}\right)$ = anticlockwise rotation necessary to convert PB into PA. This $=\frac{\pi}{2}$ if P is on the upper semi-circle, diameter AB, and $=\frac{3\pi}{2}-2\pi=-\frac{\pi}{2}$ if P is on the lower semicircle. The function is undefined at $z=a$ and at $z=-a$.

If P moves on an arbitrary circle through A, B in which the upper segment cut off by AB contains an angle a, then as in No. 23, $q=\text{am}\left(\frac{z-a}{z+a}\right)=a$ if P moves on upper arc AB and $=a-\pi$ if P moves on lower arc AB.

25. From eqn. (6), it follows that the given equation holds unless $\text{am}(z+i)-\text{am}(z-i)>\pi$ or $\leqslant-\pi$; this cannot be so unless $\text{am}(z+i)$ and $\text{am}(z-i)$ are of opposite sign and at least one of them is obtuse or each is a right angle. Let the circle $|z|=1$ cut Oy at C (0, 1) and D(0, −1);

$$\text{am}(z+i)-\text{am}(z-i)=\text{am}(\overline{DP})-\text{am}(\overline{CP}).$$

If $x>0$, neither $\text{am}(\overline{DP})$, nor $\text{am}(\overline{CP})$ is obtuse; if $x=0$, either each is $\frac{\pi}{2}$ or each is $-\frac{\pi}{2}$ or

$$\text{am}(\overline{DP})-\text{am}(\overline{CP})=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi.$$

If $x<0$, and if $y\geqslant 1$ or if $y<-1$, $\text{am}(\overline{DP})$ and $\text{am}(\overline{CP})$ have the same sign. But if $x<0$ and if $1>y\geqslant-1$, $\text{am}(\overline{DP})$ is positive obtuse and $\text{am}(\overline{CP})$ is negative obtuse; $\therefore$ $\text{am}(\overline{DP})-\text{am}(\overline{CP})>\pi$.

EXERCISE XIII. b. (p. 250.)

1. Write $-z$ for w in eqn. (14). The relation is true, by eqn. (6), if $-\pi<\{\text{am}(1+z)-\text{am}(1-z)\}\leqslant\pi$. If

$$z=x+i\,.\,0 \text{ and } x>1;\ \text{am}(1+z)=0,\ \text{am}(1-z)=\pi;$$

$\therefore$ $\text{am}(1+z)-\text{am}(1-z)=-\pi$ and so the relation is untrue. From a figure, it is evident that the inequality holds for all other values of z, excluding $z=\pm1$, for which $\text{am}(1\mp z)$ is undefined.

2. In Example 4, p. 249, equate "first parts."

3. In No. 2, put $\cos a$ for r and β for a.

4. $\text{cis}\,A+\frac{b}{2c}\text{cis}\,2A+\frac{b^2}{3c^2}\text{cis}\,3A+\ldots=\frac{c}{b}\left\{\frac{b}{c}\text{cis}\,A+\frac{1}{2}\frac{b^2}{c^2}\text{cis}\,2A+\ldots\right\}$

$$=\frac{c}{b}\left\{-\log\left(1-\frac{b}{c}\text{cis}\,A\right)\right\} \text{ since } \left|\frac{b}{c}\text{cis}\,A\right|=\frac{b}{c}<1;$$

but $\log\left(1-\frac{b}{c}\text{cis}\,A\right)=\log\frac{c-b\cos A-ib\sin A}{c}$

$$=\log(a\cos B-ia\sin B)-\log c=\log\{a\,\text{cis}(-B)\}-\log c$$
$$=\log a-iB-\log c;$$ equate "second parts."

5. $\frac{c\,\text{cis}\,B}{a+b}+\frac{1}{2}\cdot\frac{c^2\,\text{cis}\,2B}{(a+b)^2}+\ldots=-\log\left(1-\frac{c\,\text{cis}\,B}{a+b}\right)$, since $c<a+b$,

$$=-\log\frac{a+b-c\cos B-ic\sin B}{a+b}$$
$$=\log(a+b)-\log(b+b\cos C-ib\sin C)$$

$$=\log(a+b)-\log\left\{b\left(2\cos^2\frac{C}{2}-2i\sin\frac{C}{2}\cos\frac{C}{2}\right)\right\}$$

$$=\log(a+b)-\left\{\log\left(2b\cos\frac{C}{2}\right)-i\cdot\frac{C}{2}\right\};$$

equate "second parts."

6. As in Example 4, p. 249, with $\cos\theta$ for r and θ for a,

$$\text{series}=\tan^{-1}\left(\frac{\cos\theta\sin\theta}{1-\cos^2\theta}\right)=\tan^{-1}(\cot\theta),\ \text{since}\ \sin\theta\neq 0,$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\theta\right)\right].$$

(i) For $0<\theta<\pi$, $\frac{\pi}{2}>\frac{\pi}{2}-\theta>-\frac{\pi}{2}$; $\therefore$ series $=\frac{\pi}{2}-\theta$;

(ii) For $\pi<\theta<2\pi$, $\frac{\pi}{2}>\frac{3\pi}{2}-\theta>-\frac{\pi}{2}$; $\therefore$ series $=\frac{3\pi}{2}-\theta$.

7. By eqn. (14), $\operatorname{cis}2\theta-\frac{1}{2}\operatorname{cis}4\theta+\frac{1}{3}\operatorname{cis}6\theta-\ldots=\log(1+\operatorname{cis}2\theta)$, since $\operatorname{cis}2\theta\neq-1$,

$$=\log(1+\cos2\theta+i\sin2\theta)=\log\{2\cos\theta(\cos\theta+i\sin\theta)\}$$
$$=\log\{(-2\cos\theta)\,.\,\operatorname{cis}(\pi+\theta)\}.$$

For $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, $\cos\theta>0$; $\therefore$ expression $=\log(2\cos\theta)+i\theta$; equate "first parts." For $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$, $\cos\theta<0$, also $-\frac{\pi}{2}<\theta-\pi<\frac{\pi}{2}$; $\therefore$ expression $=\log(-2\cos\theta)+i(\theta-\pi)$.

8. By the solution of No. 7, the sum is x if $-\frac{\pi}{2}<x<\frac{\pi}{2}$ and is $x-\pi$ if $\frac{\pi}{2}<x<\frac{3\pi}{2}$.

9. As in Example 4, p. 249, with $r=1$, $a=2\theta$,

$$\text{series}=\tan^{-1}\left(\frac{\sin2\theta}{1-\cos2\theta}\right)=\tan^{-1}\left(\frac{2\sin\theta\cos\theta}{2\sin^2\theta}\right)$$

$$=\tan^{-1}(\cot\theta),\ \text{since}\ \sin\theta\neq0,$$

$$=\tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\theta\right)\right\}.$$

For $n\pi<\theta<(n+1)\pi$, $\frac{\pi}{2}>\frac{\pi}{2}-\theta+n\pi>-\frac{\pi}{2}$;

$\therefore$ series $=\frac{\pi}{2}-\theta+n\pi$.

10. As in Example 4, p. 249, with $r=1$, $a=2A$,

$$\cos2A+\tfrac{1}{2}\cos4A+\ldots=-\tfrac{1}{2}\log\{(1-\cos2A)^2+\sin^2 2A\}$$
$$=-\tfrac{1}{2}\log\{2-2\cos2A\}=-\tfrac{1}{2}\log\{4\sin^2A\};$$

$$\therefore\ \text{given series}=-\tfrac{1}{2}\log\{4\sin^2A\}+\tfrac{1}{2}\log\{4\sin^2B\}$$
$$=\tfrac{1}{2}\log\frac{\sin^2B}{\sin^2A}=\tfrac{1}{2}\log\frac{b^2}{a^2}.$$

11. $\cos n\alpha\cos n\beta=\frac{1}{2}\{\cos n(\alpha+\beta)+\cos n(\alpha-\beta)\}$. By eqn. (15), since $\alpha\pm\beta\neq(2k+1)\pi$,

$$\text{series}=\tfrac{1}{2}\{\log|2\cos\tfrac{1}{2}(\alpha+\beta)|+\log|2\cos\tfrac{1}{2}(\alpha-\beta)|\}$$
$$=\tfrac{1}{2}\log|4\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)|$$
$$=\tfrac{1}{2}\log|2(\cos\alpha+\cos\beta)|.$$

12. Series $=\frac{1}{2}(1+\cos2x)-\frac{1}{4}(1-\cos4x)+\frac{1}{6}(1+\cos6x)-\ldots$

$$=\tfrac{1}{2}(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots)+\tfrac{1}{2}(\cos2x+\tfrac{1}{2}\cos4x+\tfrac{1}{3}\cos6x+\ldots)$$

=, as in No. 10, since $2x\neq2n\pi$,

$$\tfrac{1}{2}\log2-\tfrac{1}{4}\log\{4\sin^2x\}=\tfrac{1}{4}\log\left(\frac{4}{4\sin^2x}\right).$$

13. By Example 4, p. 249, for

$$|x|<1,\ y=\tan^{-1}\left(\frac{x\sin\alpha}{1-x\cos\alpha}\right);$$

$$\therefore\ \frac{\sin y}{\cos y}=\tan y=\frac{x\sin\alpha}{1-x\cos\alpha};$$

$$\therefore\ \sin y=x(\sin\alpha\cos y+\cos\alpha\sin y).$$

14. By eqn. (12),

$$x-y=t\sin2x-\tfrac{1}{2}t^2\sin4x+\tfrac{1}{3}t^3\sin6x-\ldots$$

$$=\tan^{-1}\frac{t\sin2x}{1+t\cos2x},\ \text{since}\ |t|<1;$$

$$\therefore\ \tan(x-y)=\frac{t\sin2x}{1+t\cos2x};$$

$$\therefore\ t\{\sin2x\cos(x-y)-\cos2x\sin(x-y)\}=\sin(x-y);$$

$$\therefore\ t\sin(x+y)=\sin(x-y);$$

$$\therefore\ \cos2\phi=\frac{1-t}{1+t}=\frac{\sin(x+y)-\sin(x-y)}{\sin(x+y)+\sin(x-y)}=\frac{\tan y}{\tan x}.$$

15. From No. 1, for $|z|<1$,

$$\log\frac{1+z}{1-z}=\log(1+z)-\log(1-z)=2\left(z+\frac{z^3}{3}+\frac{z^5}{5}+\ldots\right);$$

put $z=x\operatorname{cis}\theta$, $|x|<1$;

$$\frac{1+z}{1-z}=\frac{(1+x\cos\theta+ix\sin\theta)(1-x\cos\theta+ix\sin\theta)}{(1-x\cos\theta-ix\sin\theta)(1-x\cos\theta+ix\sin\theta)}$$

$$=\frac{1-x^2\cos^2\theta-x^2\sin^2\theta+2ix\sin\theta}{1+x^2\cos^2\theta+x^2\sin^2\theta-2x\cos\theta};$$

$$\therefore \log\frac{1+z}{1-z}=\log\frac{1-x^2+i\,.\,2x\sin\theta}{1-2x\cos\theta+x^2}$$

$$=\log R+i\tan^{-1}\left(\frac{2x\sin\theta}{1-x^2}\right);$$

equate "second parts."

16. In No. 15, write zi for z,

$$\log\frac{1+zi}{1-zi}=2i\left(z-\frac{z^3}{3}+\frac{z^5}{5}-\dots\right), \text{ putting } z=\tan\alpha\operatorname{cis}\theta,$$

$$=2\{\tan\alpha(i\cos\theta-\sin\theta)$$
$$-\tfrac{1}{3}\tan^3\alpha(i\cos 3\theta-\sin 3\theta)+\dots\};$$

$$\text{but } \frac{1+zi}{1-zi}=\frac{1+\tan\alpha(i\cos\theta-\sin\theta)}{1-\tan\alpha(i\cos\theta-\sin\theta)}$$

$$=\frac{\cos\alpha-\sin\alpha\sin\theta+i\sin\alpha\cos\theta}{\cos\alpha+\sin\alpha\sin\theta-i\sin\alpha\cos\theta}=\text{, as in No. 15,}$$

$$\frac{\cos^2\alpha-\sin^2\alpha\sin^2\theta-\sin^2\alpha\cos^2\theta+2i\sin\alpha\cos\alpha\cos\theta}{\cos^2\alpha+\sin^2\alpha\sin^2\theta+\sin^2\alpha\cos^2\theta+2\sin\alpha\cos\alpha\sin\theta}$$

$$=\frac{\cos 2\alpha+i\sin 2\alpha\cos\theta}{1+\sin 2\alpha\sin\theta};$$

$$\therefore \log\frac{1+zi}{1-zi}=\tfrac{1}{2}\log R^2+i\tan^{-1}\left(\frac{\sin 2\alpha\cos\theta}{\cos 2\alpha}\right),$$

since $\cos 2\alpha>0$ for $|\alpha|<\frac{\pi}{4}$ and $1+\sin 2\alpha\sin\theta>0$, where

$$R^2=\frac{\cos^2 2\alpha+\sin^2 2\alpha\cos^2\theta}{(1+\sin 2\alpha\sin\theta)^2}$$

$$=\frac{1-\sin^2 2\alpha\sin^2\theta}{(1+\sin 2\alpha\sin\theta)^2}=\frac{1-\sin 2\alpha\sin\theta}{1+\sin 2\alpha\sin\theta}.$$

Equating "first parts," -2 [series (i)] $=\frac{1}{2}\log R^2$; equating "second parts," 2 [series (ii)] $=\tan^{-1}\left(\frac{\sin 2\alpha\cos\theta}{\cos 2\alpha}\right)$.

17.
$$\frac{\cos\theta}{(1+x)\cos\phi}=\frac{\sin\theta}{(1-x)\sin\phi}$$

$$=\frac{\exp(\theta i)}{\operatorname{cis}\phi+x\operatorname{cis}(-\phi)}=\frac{\exp(-\theta i)}{\operatorname{cis}(-\phi)+x\operatorname{cis}\phi};$$

$$\therefore \exp(2\theta i)=\frac{\operatorname{cis}\phi+x\operatorname{cis}(-\phi)}{\operatorname{cis}(-\phi)+x\operatorname{cis}\phi}=\frac{\operatorname{cis}\phi\,.\,\{1+x\operatorname{cis}(-2\phi)\}}{\operatorname{cis}(-\phi)\,.\,\{1+x\operatorname{cis}(2\phi)\}};$$

$$\therefore 2\theta i=\operatorname{Log}\{\operatorname{cis}(2\phi)\}+\operatorname{Log}\{1+x\operatorname{cis}(-2\phi)\}$$
$$-\operatorname{Log}\{1+x\operatorname{cis}(2\phi)\}$$

$$=2k\pi i+2\phi i+\Sigma(-1)^{n-1}$$
$$\times\frac{1}{n}x^n\{\operatorname{cis}(-2n\phi)-\operatorname{cis}(2n\phi)\} \text{ for } |x|<1,$$

$$=2k\pi i+2\phi i+\Sigma(-1)^{n-1}\,.\,\frac{1}{n}x^n(-2i\sin 2n\phi),$$

where, as in Example 5, p. 249, $\theta-k\pi-\phi$ lies between $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

18. $\tan\lambda=\dfrac{1}{\cos^2\omega-\sin^2\omega}\,.\,\tan\alpha=\dfrac{1+\tan^2\omega}{1-\tan^2\omega}\,.\,\tan\alpha$; in No. 17, put $x=-\tan^2\omega$, $\phi=\alpha$ and $\theta=\lambda$.

19. As in No. 16, putting $z=\operatorname{cis}\theta$, $\theta\neq(2n+1)\frac{\pi}{2}$,

$$\log\frac{1+i\operatorname{cis}\theta}{1-i\operatorname{cis}\theta}=2i\{\operatorname{cis}\theta-\tfrac{1}{3}\operatorname{cis}3\theta+\tfrac{1}{5}\operatorname{cis}5\theta-\dots\}$$

$$=2\{(i\cos\theta-\sin\theta)-\tfrac{1}{3}(i\cos 3\theta-\sin 3\theta)+\dots\};$$

$$\text{but } \frac{1+i\operatorname{cis}\theta}{1-i\operatorname{cis}\theta}=\frac{1+i\cos\theta-\sin\theta}{1-i\cos\theta+\sin\theta}$$

$$=\frac{(1-\sin\theta+i\cos\theta)(1+\sin\theta+i\cos\theta)}{(1+\sin\theta)^2+\cos^2\theta}$$

$$=\frac{1-\sin^2\theta-\cos^2\theta+2i\cos\theta}{2+2\sin\theta}=\frac{i\cos\theta}{1+\sin\theta};$$

also for $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$, $\cos\theta<0$;

$$\therefore \log\frac{1+i\operatorname{cis}\theta}{1-i\operatorname{cis}\theta}=\log\left\{\frac{-\cos\theta}{1+\sin\theta}\,.\operatorname{cis}\left(-\frac{\pi}{2}\right)\right\}$$

$$=\log\left(-\frac{\cos\theta}{1+\sin\theta}\right)-\frac{\pi i}{2};$$

equate "second parts," 2 (given series) $=-\frac{\pi}{2}$.

20. For $|x|<1$, $\log(1-x^2+2ix\cos\theta)=\log R+i\tan^{-1}\left(\frac{2x\cos\theta}{1-x^2}\right)$;
but $(1-x^2+2ix\cos\theta)$

$$=(1+x\sin\theta+ix\cos\theta)(1-x\sin\theta+ix\cos\theta)$$

$$=\left\{1+x\operatorname{cis}\left(\frac{\pi}{2}-\theta\right)\right\}\left\{1+x\operatorname{cis}\left(\frac{\pi}{2}+\theta\right)\right\}$$

also for $|z|<1$, $\operatorname{am}(1+z)$ lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$;

$$\therefore \operatorname{am}\left\{1+x\operatorname{cis}\left(\frac{\pi}{2}-\theta\right)\right\}+\operatorname{am}\left\{1+x\operatorname{cis}\left(\frac{\pi}{2}+\theta\right)\right\}$$

lies between $-\pi$ and $+\pi$;

$$\therefore \log(1-x^2+2ix\cos\theta)$$
$$=\log\left\{1+x\operatorname{cis}\left(\frac{\pi}{2}-\theta\right)\right\}+\log\left\{1+x\operatorname{cis}\left(\frac{\pi}{2}+\theta\right)\right\}$$
$$=\Sigma(-1)^{n-1}\cdot\frac{1}{n}x^n\left\{\operatorname{cis}\left(\frac{n\pi}{2}-n\theta\right)+\operatorname{cis}\left(\frac{n\pi}{2}+n\theta\right)\right\};$$

equating "second parts,"

$$\tan^{-1}\left(\frac{2x\cos\theta}{1-x^2}\right)$$
$$=\Sigma(-1)^{n-1}\cdot\frac{1}{n}x^n\left\{\sin\left(\frac{n\pi}{2}-n\theta\right)+\sin\left(\frac{n\pi}{2}+n\theta\right)\right\}$$
$$=\Sigma(-1)^{n-1}\cdot\frac{1}{n}x^n\cdot 2\sin\frac{n\pi}{2}\cos n\theta;$$

if $n=2r$, $\sin\frac{n\pi}{2}=0$; if $n=2r-1$, $\sin\frac{n\pi}{2}=(-1)^{r-1}$.

Or, in No. 15, write $\frac{\pi}{2}-\theta$ for θ.

EXERCISE XIII. c. (p. 254).

Numbers 1-12 are all special cases of equation (20).
Numbers 13-16 are all special cases of equation (25).

17. $e^{u+iv}=\exp[(u+iv)\operatorname{Log} e]=\exp[(u+iv)(1+2n\pi i)]$
$=\exp[u-2n\pi v+i(v+2n\pi u)]=e^{u-2n\pi v}\cdot\operatorname{cis}(v+2n\pi u)$;
$\therefore \operatorname{Log}(e^{u+iv})=\log(e^{u-2n\pi v})+i(v+2n\pi u+2k\pi)$.

18. $\exp[i\operatorname{Log}(\cos\theta-i\sin\theta)]=\exp[i\operatorname{Log}\operatorname{cis}(-\theta)]$
$=\exp[i\cdot(-\theta+2k\pi)i]=\exp(\theta-2k\pi)$.

19. $i^2=\exp[2\operatorname{Log} i]=\exp\left[2\operatorname{Log}\operatorname{cis}\left(\frac{\pi}{2}\right)\right]$
$=\exp\left[2\left(\frac{\pi}{2}+2n\pi\right)i\right]=\operatorname{cis}(\pi+4n\pi)=\operatorname{cis}\pi$.

20. p.v. of $e^{u+iv}=\exp[(u+iv)\log e]$
$=\exp(u+iv)=e^u(\cos v+i\sin v)$.

21. p.v. $=\exp[\log(1+i)\cdot\log i]$
$=\exp\left[\log\left(\sqrt{2}\cdot\operatorname{cis}\frac{\pi}{4}\right)\cdot\log\left(\operatorname{cis}\frac{\pi}{2}\right)\right]$
$=\exp\left[\left(\log\sqrt{2}+\frac{\pi i}{4}\right)\cdot\frac{\pi i}{2}\right]$
$=\exp\left[-\frac{\pi^2}{8}+\frac{\pi i}{4}\log 2\right]=\exp\left(-\frac{\pi^2}{8}\right)\cdot\operatorname{cis}\left(\frac{\pi}{4}\log 2\right)$.

22. For $x<0$, p.v. $=\exp[x\log\{(-x)\operatorname{cis}\pi\}]$
$=\exp[x\{\log(-x)+\pi i\}]$
$=\exp[x\log(-x)]\cdot\operatorname{cis}\pi x=(-x)^x\cdot\operatorname{cis}(\pi x)$.

23. By No. 17, $e^{\cos\theta+i\sin\theta}=e^{\cos\theta-2n\pi\sin\theta}\cdot\operatorname{cis}(\sin\theta+2n\pi\cos\theta)$;
and $e^{\cos\theta-i\sin\theta}=e^{\cos\theta+2k\pi\sin\theta}\cdot\operatorname{cis}(-\sin\theta+2k\pi\cos\theta)$;
$\therefore$ product $=e^{2\cos\theta+2(k-n)\pi\sin\theta}\cdot\operatorname{cis}[2(k+n)\pi\cos\theta]$.

24. p.v. $=\exp[(a+bi)\log(x+yi)]$
$=\exp\left[(a+bi)\cdot\left\{\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{y}{x}\right\}\right]$,

since $\operatorname{am}(x+yi)=\tan^{-1}\frac{y}{x}$ for $x>0$, $=\exp[u+iv]$, where $v\equiv\frac{1}{2}b\log(x^2+y^2)+a\tan^{-1}\frac{y}{x}$, $=\exp(u)\cdot\operatorname{cis} v=p+0\cdot i$ if $v=k\pi$. If $x<0$, $y<0$, $\operatorname{am}(x+yi)=\tan^{-1}\left(\frac{y}{x}\right)-\pi$;

$\therefore v\equiv\frac{1}{2}b\log(x^2+y^2)+a\tan^{-1}\left(\frac{y}{x}\right)-a\pi$ and this $=k\pi$.

25. p.v. of $(1+i)^{1-i}=\exp\{(1-i)\log(1+i)\}$
$=\exp\left\{(1-i)\cdot\log\left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)\right\}$
$=\exp\left\{(1-i)\cdot\left(\log\sqrt{2}+i\cdot\frac{\pi}{4}\right)\right\}$
$=\exp\left\{\log\sqrt{2}+\frac{\pi}{4}+i\left(\frac{\pi}{4}-\log\sqrt{2}\right)\right\}$;

$\therefore$ p.v. of $(1-i)^{1+i}=\exp\left\{\log\sqrt{2}+\frac{\pi}{4}-i\left(\frac{\pi}{4}-\log\sqrt{2}\right)\right\}$;

$\therefore$ ratio of p.v.'s of expressions
$=\exp\left\{2i\left(\frac{\pi}{4}-\log\sqrt{2}\right)\right\}=\operatorname{cis}\left(\frac{\pi}{2}-\log 2\right)$.

26. $\operatorname{cis}(a\pi)=i^z=\exp\{(x+iy)\operatorname{Log} i\}$
$=\exp\left\{(x+iy)\cdot\left(2n\pi+\frac{\pi}{2}\right)i\right\}$
$=\exp\left\{-y\left(2n\pi+\frac{\pi}{2}\right)\right\}\cdot\operatorname{cis}\left\{x\left(2n\pi+\frac{\pi}{2}\right)\right\}$;

$\therefore y=0$ and $x\left(2n\pi+\frac{\pi}{2}\right)=a\pi+2k\pi$;

$\therefore z=x=\frac{2(a+2k)}{4n+1}$.

27. Use eqn. (20);

(i) No, because every value of $u_1 \,.\, 2k_1\pi + u_2 \,.\, 2k_2\pi$ is not a value of $(u_1+u_2) \,.\, 2k\pi$;

(ii) Yes, because every value of $(u_1+u_2) \,.\, 2k\pi$ is a value of $u_1 \,.\, 2k_1\pi + u_2 \,.\, 2k_2\pi$;

(iii) Yes, because

$$\exp(w_1 \log z) \,.\, \exp(w_2 \log z) = \exp[(w_1+w_2)\log z].$$

28. By No. 10, $e^{\cos\alpha + i\sin\alpha} = e^{\cos\alpha - 2n\pi\sin\alpha} \,.\, \text{cis}(\sin\alpha + 2n\pi\cos\alpha)$.

If the polar coordinates of the point are (r, θ),

$$r = e^{\cos\alpha - 2n\pi\sin\alpha}, \quad \theta = \sin\alpha + 2n\pi\cos\alpha;$$

$$\therefore\ r = e^{\cos\alpha - \tan\alpha(\theta - \sin\alpha)} = e^{\sec\alpha - \theta\tan\alpha} = e^{\sec\alpha} \,.\, e^{-\theta\tan\alpha}.$$

29. $\text{cis}\,\alpha + nx\,\text{cis}(\alpha+\theta) + \ldots$

$= \text{cis}\,\alpha\{1 + nx\,\text{cis}\,\theta + \ldots\} = \text{cis}\,\alpha \,.\, \{\text{p.v. of } (1 + x\,\text{cis}\,\theta)^n\}$,

since $|x\,\text{cis}\,\theta| = |x| < 1$,

$= \text{cis}\,\alpha \,.\, \exp\{n\log(1 + x\cos\theta + ix\sin\theta)\}$

$= \text{cis}\,\alpha \,.\, \exp\{n(\log r + i\phi)\}$, where

$r = +\sqrt{\{(1 + x\cos\theta)^2 + x^2\sin^2\theta\}}$

$= +\sqrt{\{1 + 2x\cos\theta + x^2\}}$ and

$\phi = \tan^{-1}\dfrac{x\sin\theta}{1 + x\cos\theta}$, since $1 + x\cos\theta > 0$,

$= \text{cis}\,\alpha \,.\, \exp(\log r^n) \,.\, \text{cis}(n\phi)$

$= r^n \,.\, \text{cis}(n\phi + \alpha)$; equate "first parts."

30. $1 + n\dfrac{a}{b}\text{cis}\,C + \dfrac{n(n+1)}{1\,.\,2} \cdot \dfrac{a^2}{b^2}\text{cis}\,2C + \ldots$

$= \text{p.v. of } \left(1 - \dfrac{a}{b}\text{cis}\,C\right)^{-n}$, since $\left|\dfrac{a}{b}\text{cis}\,C\right| < 1$,

$= \exp\left\{-n\log\dfrac{b - a\cos C - ia\sin C}{b}\right\}$

$= \exp\left\{-n\log\left[\dfrac{1}{b}(c\cos A - ic\sin A)\right]\right\}$

$= \exp\left\{-n\log\dfrac{c}{b} - n\log\text{cis}(-A)\right\}$

$= \exp\left(\log\dfrac{b^n}{c^n} + nAi\right) = \dfrac{b^n}{c^n}\text{cis}(nA)$;

equate "first parts."

31. By p. 253, if $\text{cis}\,2\theta \neq -1$, that is, if θ is not an odd multiple of $\dfrac{\pi}{2}$, if $n > -1$,

$$1 + n\,\text{cis}\,2\theta + \frac{n(n-1)}{1\,.\,2}\text{cis}\,4\theta + \ldots = \exp\{n\log(1 + \text{cis}\,2\theta)\}.$$

Also $\log(1 + \text{cis}\,2\theta) = \log\{2\cos\theta \,.\, (\cos\theta + i\sin\theta)\}$. If

$$2k\pi - \frac{\pi}{2} < \theta < 2k\pi + \frac{\pi}{2}, \cos\theta > 0, \text{ also } -\frac{\pi}{2} < \theta - 2k\pi < \frac{\pi}{2};$$

$\therefore\ \log(1 + \text{cis}\,2\theta) = \log(2\cos\theta) + i(\theta - 2k\pi)$;

$\therefore$ series $= \exp\{n\log(2\cos\theta) + i(n\theta - 2nk\pi)\}$

$= (2\cos\theta)^n \,.\, \text{cis}(n\theta - 2nk\pi)$;

equate first and second parts. If $2k\pi + \dfrac{\pi}{2} < \theta < 2k\pi + \dfrac{3\pi}{2}$,

$\cos\theta < 0$ and $\log(1 + \text{cis}\,2\theta) = \log\{(-2\cos\theta) \,.\, \text{cis}(\theta - \pi)\}$;

also $-\dfrac{\pi}{2} < \theta - \pi - 2k\pi < \dfrac{\pi}{2}$;

$\therefore$ series $= \exp\{n\log(-2\cos\theta) + i(n\theta - n\pi - 2nk\pi)\}$

$= (-2\cos\theta)^n \,.\, \text{cis}(n\theta - n\pi - 2nk\pi)$;

equate as before. If θ is an odd multiple of $\dfrac{\pi}{2}$, (ii) is zero since each term is zero. Also by p. 253, the same working holds for (i) if $n > 0$, and so its sum is zero; for $n = 0$, the sum is 1, since every term except the first is zero; and for $n < 0$, the series is divergent.

32. Use the same method as in No. 31. The results may be deduced from No. 31 (ii) by writing $-n$ for n and $\alpha + \pi$ for 2θ.

33. By p. 253, $1 - \frac{1}{2}\text{cis}\,\theta + \dfrac{1\,.\,3}{2\,.\,4}\text{cis}\,2\theta - \ldots = \exp\{-\frac{1}{2}\log(1 + \text{cis}\,\theta)\}$

since $\text{cis}\,\theta \neq -1$. Also

$\log(1 + \text{cis}\,\theta) = \log\left\{2\cos\dfrac{\theta}{2} \,.\, \text{cis}\,\dfrac{\theta}{2}\right\}$

$= \log\left(2\cos\dfrac{\theta}{2}\right) + i\dfrac{\theta}{2}$,

since, for $0 < \theta < \pi$, $\cos\dfrac{\theta}{2} > 0$;

$$\therefore\ \text{series} = \exp\left\{-\tfrac{1}{2}\log\left(2\cos\frac{\theta}{2}\right) - \frac{i\theta}{4}\right\}$$

$$=\exp\left\{\log\left(2\cos\frac{\theta}{2}\right)^{-\frac{1}{2}}\right\}.\operatorname{cis}\left(-\frac{\theta}{4}\right)$$
$$=\sqrt{\left(\tfrac{1}{2}\sec\frac{\theta}{2}\right)}.\operatorname{cis}\frac{-\theta}{4}=\tfrac{1}{2}\sqrt{\left(2\sec\frac{\theta}{2}\right)}.\operatorname{cis}\frac{-\theta}{4}.$$

Equate first parts. But for $\pi<\theta<2\pi$, $\cos\frac{\theta}{2}<0$;

$$\therefore \log(1+\operatorname{cis}\theta)=\log\left\{\left(-2\cos\frac{\theta}{2}\right).\operatorname{cis}\left(\frac{\theta}{2}-\pi\right)\right\}$$
$$=\log\left(-2\cos\frac{\theta}{2}\right)+i\left(\frac{\theta}{2}-\pi\right)$$

since $-\frac{\pi}{2}<\frac{\theta}{2}-\pi<0$; hence as before. **Or**, deduce the case $\pi<\theta<2\pi$ from the first answer by putting $\theta=2\pi-\alpha$ so that $0<\alpha<\pi$.

34. If $|z|<1$, $1+\frac{n(n+1)}{2!}z^2+\frac{n(n+1)(n+2)(n+3)}{4!}z^4+\ldots$
$$=\text{p.v. of }\tfrac{1}{2}\{(1+z)^{-n}+(1-z)^{-n}\};$$

put $z=i\tan\theta$, where $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$;

$$\text{given series}=\tfrac{1}{2}\{\exp[-n\log(1+i\tan\theta)]+\exp[-n\log(1-i\tan\theta)]\};$$
$$\log(1+i\tan\theta)=\log(\sec\theta.\operatorname{cis}\theta)$$
$$=\log(\sec\theta)+i\theta \text{ for } -\frac{\pi}{4}<\theta<\frac{\pi}{4};$$
$$\therefore \exp[-n\log(1+i\tan\theta)]$$
$$=\exp\{\log\cos^n\theta\}.\operatorname{cis}(-n\theta);$$
$$\therefore \text{series}=\tfrac{1}{2}\{(\cos\theta)^n.[\operatorname{cis}(-n\theta)+\operatorname{cis}(n\theta)]\}$$
$$=\cos^n\theta.\cos n\theta.$$

For $\frac{3\pi}{4}<\theta<\frac{5\pi}{4}$ continue as in No. 33. **Or**, put $\theta=\phi+\pi$ so that $-\frac{\pi}{4}<\phi<\frac{\pi}{4}$.

35. $\operatorname{cis}2\theta+2x\operatorname{cis}3\theta+3x^2\operatorname{cis}4\theta+\ldots=\operatorname{cis}2\theta.\{\text{p.v. of }(1-x\operatorname{cis}\theta)^{-2}\}$, since $|x|<1$; proceed as in No. 29. **Or**, more shortly, as follows: From Ch. IX., equation (9),

$$\sin\theta+x\sin2\theta+\ldots+x^{n-2}\sin(n-1)\theta$$
$$=\frac{\sin\theta-x^{n-1}\sin n\theta+x^n\sin(n-1)\theta}{1-2x\cos\theta+x^2}.$$

Differentiate w.r.t. x, then the first $n-2$ terms of the given series

$$=\frac{-\sin\theta(2x-2\cos\theta)}{(1-2x\cos\theta+x^2)^2}+\mathsf{K},$$

where $\mathsf{K}\equiv\frac{d}{dx}\left\{\frac{x^n\sin(n-1)\theta-x^{n-1}\sin n\theta}{1-2x\cos\theta+x^2}\right\}$;

since $|x|<1$, $\mathsf{K}\to0$ when $n\to\infty$.

EXERCISE XIII. d. (p. 258.)

1. (i) If $x^2+y^2<1$, then

$$\cos2\alpha>0;\ \therefore\ -\frac{\pi}{2}<2\alpha_0<\frac{\pi}{2};\ \therefore\ 2\alpha_0=\tan^{-1}\frac{2x}{1-x^2-y^2};$$

(ii) If $x^2+y^2>1$ and $x>0$, then $\cos2\alpha<0$ and $\sin2\alpha>0$;

$$\therefore\ \frac{\pi}{2}<2\alpha_0<\pi;\ \text{but}\ -\frac{\pi}{2}<\tan^{-1}\frac{2x}{1-x^2-y^2}<0;$$
$$\therefore\ 2\alpha_0=\pi+\tan^{-1}\frac{2x}{1-x^2-y^2};$$

(iii) If $x^2+y^2>1$ and $x<0$, then $\cos2\alpha<0$ and $\sin2\alpha<0$;

$$\therefore\ -\pi<2\alpha_0<-\frac{\pi}{2};\ \text{but}\ 0<\tan^{-1}\frac{2x}{1-x^2-y^2}<\frac{\pi}{2};$$
$$\therefore\ 2\alpha_0=-\pi+\tan^{-1}\frac{2x}{1-x^2-y^2}.$$

2. (i) If $\operatorname{Ch}^{-1}z=w$, $z=\operatorname{ch}w=\cos(iw)$; $\therefore\ \cos(iw)=\cos(\cos^{-1}z)$; $\therefore\ iw=2k\pi\pm\cos^{-1}z$;

(ii) If $\operatorname{Sh}^{-1}z=w$, $z=\operatorname{sh}w=\frac{1}{i}\sin(iw)$;

$$\therefore\ \sin(iw)=iz=\sin[\sin^{-1}(iz)];$$
$$\therefore\ iw=k\pi+(-1)^k.\sin^{-1}(iz);$$
$$\therefore\ w=-ik\pi-i(-1)^k.\sin^{-1}(iz)$$
$$=in\pi-i(-1)^n.\sin^{-1}(iz);$$

(iii) If $\operatorname{Th}^{-1}z=w$, $z=\operatorname{th}w=\frac{1}{i}\tan(iw)$;

$$\therefore\ \tan iw=iz=\tan[\tan^{-1}(iz)];$$
$$\therefore\ iw=k\pi+\tan^{-1}(iz);\ \therefore\ w=in\pi-i\tan^{-1}(iz).$$

3. If $\operatorname{Sh}^{-1}z=w$, $z=\operatorname{sh}w=\tfrac{1}{2}[\exp w-\exp(-w)]$;

$$\therefore\ \exp(2w)-2z.\exp(w)-1=0;$$
$$\therefore\ \exp w=z\pm\sqrt{(z^2+1)};$$

$\therefore\ w = \text{Log}\,[z + \sqrt{(z^2+1)}] = 2n\pi i + \log\,[z + \sqrt{(z^2+1)}]$

or $\quad w = \text{Log}\,[z - \sqrt{(z^2+1)}] = \text{Log}\left[\dfrac{-1}{z+\sqrt{(z^2+1)}}\right]$

$= \log(\text{cis}\,\pi) - \text{Log}\,[z+\sqrt{(z^2+1)}]$

$= (2k+1)\pi i - \log\,[z+\sqrt{(z^2+1)}]$;

the two forms are included in the single form of the question.

4. If $\text{Ch}^{-1}z = w$, $z = \text{ch}\,w = \frac{1}{2}[\exp w + \exp(-w)]$;

$\therefore\ \exp(2w) - 2z\exp w + 1 = 0$; $\therefore\ \exp w = z \pm \sqrt{(z^2-1)}$;

$\therefore\ w = \text{Log}\,[z \pm \sqrt{(z^2-1)}]$;

but $\quad \text{Log}\,[z - \sqrt{(z^2-1)}] = \text{Log}\left[\dfrac{1}{z+\sqrt{(z^2-1)}}\right]$;

$\therefore\ w = \pm\,\text{Log}\,[z+\sqrt{(z^2-1)}] = 2n\pi i \pm \log\,[z+\sqrt{(z^2-1)}]$.

5. $\alpha + i\beta = \cos(u+iv) = \cos u\,\text{ch}\,v - i\sin u\,\text{sh}\,v$;

$\therefore\ \alpha = \cos u\,.\,\text{ch}\,v,\ \beta = -\sin u\,.\,\text{sh}\,v$;

$\therefore$ (i) $\alpha^2\sec^2 u - \beta^2\text{cosec}^2 u = \text{ch}^2 v - \text{sh}^2 v$;

(ii) $\alpha^2\,\text{sech}^2 v + \beta^2\,\text{cosech}^2 v = \cos^2 u + \sin^2 u$;

(iii) The sum of $\cos^2 u$ and $\text{ch}^2 v$

$= \frac{1}{2}(1+\cos 2u) + \frac{1}{2}(1+\text{ch}\,2v) = 1 + \frac{1}{2}\{\cos 2u + \cos(2vi)\}$

$= 1 + \cos(u+vi)\cos(u-vi)$

$= 1 + (\alpha+i\beta)(\alpha-i\beta) = 1+\alpha^2+\beta^2$;

and the product is α^2, see above.

6. As in No. 5, the sum of $\sin^2 u$ and $\text{ch}^2 v$ is $1+\alpha^2+\beta^2$ and product is α^2.

7. $\tan\left(\dfrac{\pi}{4} + \dfrac{i}{2}\log\dfrac{x+y}{x-y}\right) = \dfrac{\tan\dfrac{\pi}{4}+k}{1-\tan\dfrac{\pi}{4}\,.\,k} = \dfrac{1+k}{1-k}$

where $\quad k = \tan\left(\dfrac{i}{2}\log\dfrac{x+y}{x-y}\right) = i\,\text{th}\left(\frac{1}{2}\log\dfrac{x+y}{x-y}\right)$

$= i\dfrac{y}{x}$ by eqn. (27); $\therefore\ \dfrac{1+k}{1-k} = \dfrac{x+iy}{x-iy}$.

8. If $\text{ch}^{-1}(x+iy) = p+qi$, $x+iy = \text{ch}\,(p+qi) = \cos(-q+pi)$; $\therefore$ by No. 5, $x^2 = \cos^2 q\,\text{ch}^2 p$ and $y^2 = \sin^2 q\,\text{sh}^2 p$; but $\text{ch}^{-1}a = (p+iq)+(p-iq) = 2p$;

$\therefore\ a = \text{ch}\,2p = 2\,\text{ch}^2 p - 1 = 2\text{sh}^2 p + 1$;

$\therefore\ 2x^2/(a+1) + 2y^2/(a-1) = \cos^2 q + \sin^2 q = 1$.

9. $\sin(u+iv) = 2$; $\therefore$ as in No. 6, $\sin u\,\text{ch}\,v = 2$, $\cos u\,\text{sh}\,v = 0$. If $\text{sh}\,v = 0$, then $\text{ch}\,v = 1$; $\therefore\ \sin u = 2$, which is impossible;

$\therefore\ \cos u = 0$; $\therefore\ u = (2k+1)\dfrac{\pi}{2}$; but $\text{ch}\,v > 0$; $\therefore\ \sin u$ positive; $\therefore\ u = 2n\pi + \dfrac{\pi}{2}$; also $\text{ch}\,v = 2$; $\therefore$ as in No. 4, $v = \pm\log\,[2+\sqrt{(4-1)}]$.

10. $\cos(x+iy) = \frac{5}{4}$; $\therefore$ as in No. 5, $\cos x\,\text{ch}\,y = \frac{5}{4}$, $\sin x\,\text{sh}\,y = 0$. If $\text{sh}\,y = 0$, then $\text{ch}\,y = 1$; $\therefore\ \cos x = \frac{5}{4}$, which is impossible; $\therefore\ \sin x = 0$; $\therefore\ \cos x = \pm 1$, but $\text{ch}\,y > 0$; $\therefore\ \cos x$ is positive; $\therefore\ \cos x = +1$; $\therefore\ x = 2n\pi$; also $\text{ch}\,y = \frac{5}{4}$;

$\therefore$ as in No. 4, $y = \pm\log\left[\frac{5}{4} + \sqrt{(\frac{25}{16}-1)}\right] = \pm\log\,[\frac{5}{4}+\frac{3}{4}]$.

11. $\sin(x+iy) = \frac{5}{3}$; $\therefore$ as in No. 6, $\sin x\,\text{ch}\,y = \frac{5}{3}$, $\cos x\,\text{sh}\,y = 0$, but as in No. 9, $\text{sh}\,y = 0$ is impossible; $\therefore\ \cos x = 0$, also $\sin x$ is positive since $\text{ch}\,y > 0$; $\therefore\ x = 2n\pi + \dfrac{\pi}{2}$; also $\text{ch}\,y = \frac{5}{3}$;

$\therefore$ as in No. 4, $y = \pm\log\,[\frac{5}{3} + \sqrt{(\frac{25}{9}-1)}] = \pm\log\,[\frac{5}{3}+\frac{4}{3}]$.

12. $\cos(x+iy) = -\frac{13}{5}$; use method of No. 10;

$\cos x\,\text{ch}\,y = -\frac{13}{5}$, $\sin x\,\text{sh}\,y = 0$,

also $\text{sh}\,y = 0$ is impossible; $\therefore\ \sin x = 0$; but $\text{ch}\,y > 0$; $\therefore\ \cos x$ is negative; $\therefore\ x = (2n+1)\pi$; $\text{ch}\,y = +\frac{13}{5}$ gives

$y = \pm\log\,[\frac{13}{5} + \sqrt{(\frac{169}{25}-1)}] = \pm\log\,[\frac{13}{5}+\frac{12}{5}]$.

13. $\sin(x+iy) = i$; $\therefore$ as in No. 6, $\cos x\,\text{sh}\,y = 1$, $\sin x\,\text{ch}\,y = 0$; but $\text{ch}\,y > 0$; $\therefore\ \sin x = 0$;

$\therefore\ \cos x = 1$ and $\text{sh}\,y = 1$ or $\cos x = -1$ and $\text{sh}\,y = -1$;

(i) $\cos x = 1$, $x = 2n\pi$, $e^y - e^{-y} = 2$; $\therefore\ e^{2y} - 2e^y - 1 = 0$;

$\therefore\ e^y = 1 \pm \sqrt{2}$, but $e^y > 0$; $\therefore\ e^y = 1+\sqrt{2}$;

$\therefore\ y = \log(1+\sqrt{2})$; $x+iy = 2n\pi + i\log(1+\sqrt{2})$;

(ii) $\cos x = -1$, $x = (2n+1)\pi$, $e^y - e^{-y} = -2$; $\therefore$ as before

$y = \log(\sqrt{2}-1) = \log\dfrac{1}{\sqrt{2}+1} = -\log(\sqrt{2}+1)$;

$\therefore\ x+iy = (2n+1)\pi - i\log(\sqrt{2}+1)$;

both forms included in $k\pi + i(-1)^k\log(1+\sqrt{2})$.

14. In eqns. (28), (29) put $x = \cos\theta$, $y = \sin\theta$; thus $\cos 2\alpha = 0$ and $\sin 2\alpha = \dfrac{2\cos\theta}{+\sqrt{(4\cos^2\theta)}} = +1$ in (i), but $= -1$ in (ii); hence $\alpha_0 = +\dfrac{\pi}{4}$, $-\dfrac{\pi}{4}$ in the two cases. Also

$$\frac{x^2+(1+y)^2}{x^2+(1-y)^2} = \frac{2+2y}{2-2y} = \frac{1+\sin\theta}{1-\sin\theta}$$

15. (i) See solution of No. 5;

(ii) Solving $\lambda = \frac{1}{2}[1+x^2+y^2 \pm \sqrt{\{(1+x^2+y^2)^2 - 4x^2\}}]$

$= \frac{1}{4}[\sqrt{\{(1+x)^2+y^2\}} \pm \sqrt{\{(1-x)^2+y^2\}}]^2$

$\equiv (t_1 \pm t_2)^2$, where $t_1 > t_2 > 0$ since $x > 0$;

but $\text{ch}^2 v > 1 > \cos^2 u$;

$\therefore\ \text{ch}^2 v = (t_1+t_2)^2$ and $\cos^2 u = (t_1 - t_2)^2$;

$\therefore\ \text{ch}\, v = t_1 + t_2$ and $\cos u = \pm(t_1 - t_2)$;

but $\cos u \,\text{ch}\, v = x > 0$; $\therefore\ \cos u > 0$; $\therefore\ \cos u = t_1 - t_2$;

(iii) $u = 2n\pi \pm \cos^{-1}(t_1 - t_2)$, where $0 < \cos^{-1}(t_1-t_2) < \frac{\pi}{2}$; also $v = \pm \text{ch}^{-1}(t_1+t_2)$, where $\text{ch}^{-1}(t_1+t_2) > 0$; but

$\sin u \,\text{sh}\, v = -y$;

$\therefore\ \sin u$ is opposite in sign to $\text{sh}\, v$;

$\therefore$ if $u = 2n\pi + \cos^{-1}(t_1 - t_2)$, $\sin u > 0$; $\therefore\ \text{sh}\, v < 0$;

$\therefore\ v < 0$; $\therefore\ v = -\text{ch}^{-1}(t_1+t_2)$; similarly if

$u = 2n\pi - \cos^{-1}(t_1 - t_2)$, $\sin u < 0$, $\text{sh}\, v > 0$;

$\therefore\ v = +\text{ch}^{-1}(t_1+t_2)$; hence result.

16. By No. 6, $\sin^2 u$, $\text{ch}^2 v$ are the roots of

$$\mu^2 - \mu(1+x^2+y^2) + x^2 = 0;$$

$\therefore$ as in No. 15, $\text{ch}\, v = t_1 + t_2$ and $\sin u = \pm(t_1 - t_2)$. But $\sin u \,\text{ch}\, v = x > 0$; $\therefore\ \sin u > 0$; $\therefore\ \sin u = t_1 - t_2$;

$\therefore\ u = n\pi + (-1)^n \sin^{-1}(t_1 - t_2)$,

where $0 < \sin^{-1}(t_1 - t_2) < \frac{\pi}{2}$; also $v = \pm\text{ch}^{-1}(t_1+t_2)$; but $\cos u \,\text{sh}\, v = y > 0$; $\therefore\ \text{sh}\, v$ is the same sign as $\cos u$. If n is even, $\cos u$ is $+$; $\therefore\ \text{sh}\, v$ is $+$; $\therefore\ v = +\text{ch}^{-1}(t_1+t_2)$; if n is odd, $\cos u$ is $-$; $\therefore\ \text{sh}\, v$ is $-$; $\therefore\ v = -\text{ch}^{-1}(t_1+t_2)$; hence result.

17. (i) In eqn. (28) put $x = \tan\phi$, $y = \sec\phi$; thus

$\sin 2\alpha : \cos 2\alpha : 1$

$= \tan\phi : -\tan^2\phi : +\sqrt{(\tan^2\phi + \tan^4\phi)}$

$= \tan\phi : -\tan^2\phi : \tan\phi\sec\phi$

$= \cos\phi : -\sin\phi : 1$;

$\therefore$ in eqn. (29) $2\alpha_0 = \phi + \frac{1}{2}\pi$ and

$$\alpha + i\beta = \text{Tan}^{-1}(x+iy) = n\pi + \alpha_0 + \frac{i}{4}\log(\cot^2 \tfrac{1}{2}\phi)$$

$$= n\pi + \alpha_0 + \frac{i}{2}\log(\cot\tfrac{1}{2}\phi);$$

equate "first" and "second" parts;

(ii) For $-\frac{\pi}{2} < \phi < 0$, $+\sqrt{(\tan^2\phi + \tan^4\phi)}$

$= -\tan\phi\sec\phi$, hence $\sin 2\alpha : \cos 2\alpha : 1$

$= -\cos\phi : \sin\phi : 1$; $\therefore\ 2\alpha_0 = \phi - \frac{1}{2}\pi$;

also $\log(\cot^2\frac{1}{2}\phi) = 2\log(-\cot\frac{1}{2}\phi)$.

18. If n is even, $n\pi + \frac{\pi}{2} + i\log\cot\frac{n\pi+\theta}{2} = 2k\pi + \frac{\pi}{2} + i\log\cot\frac{\theta}{2}$; if n is odd,

$$\text{expression} = (2k+1)\pi + \frac{\pi}{2} + i\log\left(-\tan\frac{\theta}{2}\right)$$

$$= (2k+1)\pi + \frac{\pi}{2} + i\left\{\log\left(\tan\frac{\theta}{2}\right) + \pi i\right\}$$

$$= 2k\pi + \frac{\pi}{2} - i\log\left(\cot\frac{\theta}{2}\right);$$

$$\therefore\ \text{given form} \equiv 2k\pi + \frac{\pi}{2} \pm i\log\left(\cot\frac{\theta}{2}\right).$$

If $\operatorname{cosec}\theta = \sin(\alpha + i\beta)$, as in No. 6,

$\sin\alpha\,\text{ch}\,\beta = \operatorname{cosec}\theta$, $\cos\alpha\,\text{sh}\,\beta = 0$;

as in No. 9, $\text{sh}\,\beta = 0$ is impossible;

$\therefore\ \cos\alpha = 0$; $\therefore\ \sin\alpha = \pm 1$, but $\text{ch}\,\beta > 0$;

$\therefore\ \sin\alpha$ is the same sign as $\operatorname{cosec}\theta$;

(i) For $2n\pi < \theta < (2n+1)\pi$, $\operatorname{cosec}\theta > 0$; $\therefore\ \sin\alpha = +1$;

$\therefore\ \alpha = 2k\pi + \frac{\pi}{2}$ and $\text{ch}\,\beta = \operatorname{cosec}\theta$;

$$\therefore\ \text{as in No. 4, } \beta = \pm\log\frac{1+\cos\theta}{\sin\theta} = \pm\log\left(\cot\frac{\theta}{2}\right);$$

$$\therefore\ \alpha + i\beta = 2k\pi + \frac{\pi}{2} \pm i\log\left(\cot\frac{\theta}{2}\right);$$

(ii) For $(2n+1)\pi < \theta < (2n+2)\pi$, $\operatorname{cosec}\theta < 0$; $\therefore\ \sin\alpha = -1$;

$\therefore\ \alpha = 2k\pi - \frac{\pi}{2}$, and $\text{ch}\,\beta = -\operatorname{cosec}\theta$;

$$\therefore\ \beta = \pm\log\left(-\cot\frac{\theta}{2}\right);$$

$$\therefore\ \alpha + i\beta = 2k\pi - \frac{\pi}{2} \pm i\log\left(-\cot\frac{\theta}{2}\right)$$

$$= 2k\pi - \frac{\pi}{2} \pm i\left\{\log\left(\cot\frac{\theta}{2}\right) + \pi i\right\}$$

$$= (2r+1)\pi - \frac{\pi}{2} \pm i\log\left(\cot\frac{\theta}{2}\right)$$

$$= 2r\pi + \frac{\pi}{2} \pm i\log\left(\cot\frac{\theta}{2}\right).$$

19. As in No. 6, $\text{ch}^2 v$, $\sin^2 u$ are the roots of $\lambda^2 - 2\lambda + \cos^2 x = 0$; $\lambda = 1 \pm \sin x$;

(i) For $0 < x < \pi$, $\sin x > 0$; $\therefore$ $\text{ch}\, v = +\sqrt{(1+\sin x)}$;

$$\sin u = \pm\sqrt{(1-\sin x)} = \pm\frac{\cos x}{\sqrt{(1+\sin x)}};$$

from $\sin u \,.\, \text{ch}\, v = \cos x$, $\sin u$ is the same sign as $\cos x$;

$$\therefore\ \sin u = +\frac{\cos x}{\sqrt{(1+\sin x)}} \text{ and, for p.v.,}$$

$$v = +\text{ch}^{-1}\{\sqrt{(1+\sin x)}\} =, \text{ as in No. 4,}$$

$$\log\{\sqrt{(1+\sin x)} + \sqrt{\sin x}\};$$

(ii) For $\pi < x < 2\pi$, $\sin x < 0$; $\therefore$ $\text{ch}\, v = +\sqrt{(1-\sin x)}$;

$$\sin u = \pm\sqrt{(1+\sin x)}$$

$$= \pm\frac{\cos x}{\sqrt{(1-\sin x)}} = +\frac{\cos x}{\sqrt{(1-\sin x)}},$$

as before; etc.

EXERCISE XIII. e. (p. 260.)

1. $\exp\left\{(2n+1)\frac{\pi i}{2} \,.\, \text{Log}\, e\right\} = \exp\left\{(2n+1)\frac{\pi i}{2} \cdot (1 + 2k\pi i)\right\}$

$$= \exp\left\{-k\pi^2(2n+1) + i \,.\, (2n+1)\frac{\pi}{2}\right\}$$

$$= e^{-k\pi^2(2n+1)} \,.\, \text{cis}\left(n\pi + \frac{\pi}{2}\right).$$

2. $\exp\{(x+yi)^2 \,.\, \text{Log}\, e\} = \exp\{(x^2 - y^2 + 2ixy) \,.\, (1 + 2k\pi i)\}$; then as in No. 1.

3. $\exp\left\{i \,.\, \text{Log cis}\left(\frac{\pi}{2} - \theta\right)\right\} = \exp\left\{i \,.\left(\frac{\pi}{2} - \theta + 2n\pi\right)i\right\}$

$$= \exp\left(\theta - 2n\pi - \frac{\pi}{2}\right).$$

4. $\text{Log}\{\text{Log}(\text{cis}\,\theta)\} = \text{Log}\{(2n\pi + \theta)i\} = \text{Log}\left\{(2n\pi + \theta) \,.\, \text{cis}\frac{\pi}{2}\right\}.$

5. $\tan(i \log t) = i\, \text{th} \log t = i\dfrac{t - \frac{1}{t}}{t + \frac{1}{t}} = i\dfrac{t^2 - 1}{t^2 + 1};$

$$\therefore\ \tan\left(i \log \tan\frac{\theta}{2}\right) = i\frac{\tan^2\frac{\theta}{2} - 1}{\tan^2\frac{\theta}{2} + 1} = -i\cos\theta;$$

$$\therefore\ \text{expression} = \frac{\tan\theta + i\cos\theta}{1 - i\cos\theta\tan\theta}$$

$$= \frac{\sin\theta + i\cos^2\theta}{\cos\theta\,(1 - i\sin\theta)} \cdot \frac{1 + i\sin\theta}{1 + i\sin\theta}.$$

6. $\exp\{(1+i) \,.\, \text{Log}\,(1+i)\} = \exp\left\{(1+i) \,.\, \text{Log}\left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)\right\}$

$$= \exp\left\{(1+i)\left[\log\sqrt{2} + \left(\frac{\pi}{4} + 2n\pi\right)i\right]\right\};$$

then as in No. 1; $\exp(\log\sqrt{2}) = \sqrt{2}$.

7. If $\sin(x+iy) \equiv r(\cos\theta + i\sin\theta)$,

$$\log\sin(x+iy) = \tfrac{1}{2}\log r^2 + i \,.\, \text{am}\,(\text{cis}\,\theta);$$

$$r^2 = \sin(x+iy) \,.\, \sin(x-iy) = \sin^2 x - \sin^2(iy)$$

$$= \sin^2 x + \text{sh}^2 y = \text{ch}^2 y - \cos^2 x.$$

Also $r\cos\theta = \sin x\,\text{ch}\, y > 0$ for $0 < x < \pi$;

$$\therefore\ \cos\theta > 0 \text{ and } r\sin\theta = \cos x\,\text{sh}\, y;$$

$$\therefore\ \text{am}\,(\text{cis}\,\theta) = \tan^{-1}(\tan\theta)$$

$$= \tan^{-1}\left(\frac{\cos x\,\text{sh}\, y}{\sin x\,\text{ch}\, y}\right) = \tan^{-1}(\cot x\,\text{th}\, y).$$

For $\pi < x < \frac{3\pi}{2}$, $\sin x < 0$, $\cos x < 0$, but $\text{sh}\, y > 0$ since $y > 0$;

$\therefore$ $\cos\theta < 0$, $\sin\theta < 0$; $\therefore$ $\text{am}\,(\text{cis}\,\theta) = -\pi + \tan^{-1}(\tan\theta)$.

8. $e^c = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \dfrac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}$; $\therefore$ $\tan\frac{\theta}{2} = \dfrac{e^c - 1}{e^c + 1} = \text{th}\frac{c}{2}$;

$$\therefore\ \frac{\exp(\theta i) - 1}{\exp(\theta i) + 1} \cdot \frac{1}{i} = \text{th}\frac{c}{2} = \frac{1}{i}\tan\frac{ic}{2};$$

$$\therefore\ \exp(\theta i) = \frac{\tan\left(\frac{ic}{2}\right) + 1}{1 - \tan\left(\frac{ic}{2}\right)} = \tan\left(\frac{ic}{2} + \frac{\pi}{4}\right);$$

$$\therefore\ \theta i = \text{Log}\tan\left(\frac{ic}{2} + \frac{\pi}{4}\right) = -\text{Log}\cot\left(\frac{ic}{2} + \frac{\pi}{4}\right).$$

9. Put $\theta = \frac{\pi}{2} + \phi$,

series $= \sin\phi \,.\, \sin\phi + \frac{1}{2}\sin 2\phi \,.\, \sin^2\phi + \frac{1}{3}\sin 3\phi \,.\, \sin^3\phi + \ldots$
$=$ by Example 4, p. 249, where $r = \sin\phi$,

$$\tan^{-1}\left(\frac{\sin^2\phi}{1 - \sin\phi\cos\phi}\right),$$

for $|\sin\phi|<1$, i.e. $\phi \neq (2n+1)\frac{\pi}{2}$; result also holds if $|\sin\phi|=1$ because $\sin\phi \operatorname{cis}\phi \neq 1$; $\therefore$ result also holds for $\phi=(2n+1)\frac{\pi}{2}$;

$$\text{sum} = \tan^{-1}\left(\frac{\sin^2\phi}{\sin^2\phi+\cos^2\phi-\sin\phi\cos\phi}\right)$$
$$=\cot^{-1}(1-\cot\phi+\cot^2\phi).$$

10. As on p. 257,

$$\operatorname{Tan}^{-1}(\cos\theta+i\sin\theta)=n\pi+\alpha+\frac{i}{4}\log\frac{\cos^2\theta+(1+\sin\theta)^2}{\cos^2\theta+(1-\sin\theta)^2},$$

where $\sin 2\alpha:\cos 2\alpha:1=2\cos\theta:0:|2\cos\theta|$ and $-\pi<2\alpha\leqslant\pi$;

$$\therefore \text{ for } \cos\theta>0,\ \sin 2\alpha=+1,\ \alpha=\frac{\pi}{4};$$

$$\tan^{-1}(\operatorname{cis}\theta)=\frac{\pi}{4}+\frac{i}{4}\log\frac{1+\sin\theta}{1-\sin\theta};$$

$$\log\frac{1+\sin\theta}{1-\sin\theta}=\log\frac{\operatorname{ch}\frac{\pi}{2}+\operatorname{sh}\frac{\pi}{2}}{\operatorname{ch}\frac{\pi}{2}-\operatorname{sh}\frac{\pi}{2}}=\log\frac{e^{\frac{\pi}{2}}}{e^{-\frac{\pi}{2}}}=\log e^{\pi}=\pi;$$

$$\therefore [\tan^{-1}(\operatorname{cis}\theta)]^m=\left\{\frac{\pi}{4}(1+i)\right\}^m=\left(\frac{\pi}{4}\right)^m\cdot\left\{\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right\}^m$$
$$=\left(\frac{\pi}{4}\right)^m\cdot 2^{\frac{1}{2}m}\cdot\operatorname{cis}\frac{m\pi}{4};$$

$$\text{expression}=\pi^m\cdot\frac{1}{2^{2m}}\cdot 2^{\frac{1}{2}m}\cdot\left[\operatorname{cis}\frac{m\pi}{4}+\operatorname{cis}\left(-\frac{m\pi}{4}\right)\right].$$

11. By Example 4, p. 249, putting $r=\frac{1}{3}$, we have series

$$=\tan^{-1}\left(\frac{\frac{1}{3}\sin\beta}{1-\frac{1}{3}\cos\beta}\right)=\tan^{-1}\frac{\sin\beta}{3-\cos\beta}$$

$$=\tan^{-1}\frac{2\tan\frac{\beta}{2}}{2+4\tan^2\frac{\beta}{2}}=\tan^{-1}\frac{\tan\frac{\alpha}{2}-\tan\frac{\beta}{2}}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$$

$$=\tan^{-1}\left\{\tan\frac{\alpha-\beta}{2}\right\}=\tfrac{1}{2}(\alpha-\beta)$$

since $\frac{\pi}{2}>\frac{1}{2}(\alpha-\beta)>-\frac{\pi}{2}$.

12. $(1-c)\tan x\,.\,(1+\tan x\tan y)=(1+c)(\tan x-\tan y)$;

$$\therefore \tan y\{(1-c)\tan^2 x+(1+c)\}=2c\tan x;$$

$$\therefore \frac{\sin y}{c\sin 2x}=\frac{\cos y}{1+c\cos 2x}$$
$$=\frac{\exp(yi)}{1+c\exp(2xi)}=\frac{\exp(-yi)}{1+c\exp(-2xi)};$$

$$\therefore \exp(2yi)=\frac{1+c\exp(2xi)}{1+c\exp(-2xi)};$$

$$\therefore 2yi=\operatorname{Log}\{1+c\exp(2xi)\}-\operatorname{Log}\{1+c\exp(-2xi)\}$$
$$=2k\pi i+\Sigma(-1)^{n-1}\cdot\frac{1}{n}c^n\{\exp(2nxi)-\exp(-2nxi)\}$$

for $|c|<1$.

13. As in XIII. b, No. 9,

$$\text{series}=\tan^{-1}\left(\frac{\sin\theta}{1-\cos\theta}\right)=\tan^{-1}\left(\cot\frac{\theta}{2}\right)$$
$$=\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right];$$

but for $2\pi<\theta<4\pi$, $-\frac{\pi}{2}<\frac{3\pi}{2}-\frac{\theta}{2}<\frac{\pi}{2}$.

14. $1+\frac{1}{2}\operatorname{cis}\theta+\frac{1\,.\,3}{2\,.\,4}\operatorname{cis}2\theta+\ldots=$ p.v. of $(1-\operatorname{cis}\theta)^{-\frac{1}{2}}$ by p. 253, for $\operatorname{cis}\theta\neq 1$, since $-\frac{1}{2}>-1$,

$$=\exp\{-\tfrac{1}{2}\log(1-\cos\theta-i\sin\theta)\};$$

$$\log(1-\cos\theta-i\sin\theta)$$
$$=\log\left\{\left(2\sin\frac{\theta}{2}\right)\cdot\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)\right\}$$
$$=\log\left(2\sin\frac{\theta}{2}\right)+i\,.\,\operatorname{am}\left[\operatorname{cis}\frac{\theta-\pi}{2}\right],$$

since $\sin\frac{\theta}{2}>0$ for $0<\theta<2\pi$,

$$=\log\left(2\sin\frac{\theta}{2}\right)+\frac{i}{2}(\theta-\pi),\ \text{since } -\frac{\pi}{2}<\frac{\theta-\pi}{2}<\frac{\pi}{2};$$

$$\therefore \text{expression}=\exp\left\{-\tfrac{1}{2}\log\left(2\sin\frac{\theta}{2}\right)-\frac{i}{4}(\theta-\pi)\right\}$$
$$=\exp\left\{\log\sqrt{\left(\tfrac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right)}\right\}\cdot\operatorname{cis}\frac{\pi-\theta}{4}$$
$$=+\sqrt{\left(\tfrac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right)}\cdot\operatorname{cis}\frac{\pi-\theta}{4}.$$

15. $1-\frac{n}{1!}\operatorname{cis}\theta+\frac{n(n+1)}{2!}\operatorname{cis}2\theta-\ldots=$ p.v. of $(1+\operatorname{cis}\theta)^{-n}$ if $-n>-1$ and if $\operatorname{cis}\theta\neq-1$, i.e. $n<1$ and $\theta\neq(2r+1)\pi$,

$=\exp\{-n\log(1+\operatorname{cis}\theta)\}=$, as in XIII. c, No. 33,

$$\exp\left\{-n\left[\log\left(2\cos\frac{\theta}{2}\right)+i\cdot\frac{\theta}{2}\right]\right\},$$

for $-\pi<\theta<\pi$, $=\exp\left[-n\log\left(2\cos\frac{\theta}{2}\right)\right]\cdot\exp\left(-\frac{in\theta}{2}\right)$

$$=\left(2\cos\frac{\theta}{2}\right)^{-n}\cdot\operatorname{cis}\left(-\frac{n\theta}{2}\right).$$

For $\pi<\theta<3\pi$,

$$\text{expression}=\exp\left\{-n\left[\log\left(-2\cos\frac{\theta}{2}\right)+i\left(\frac{\theta}{2}-\pi\right)\right]\right\};$$

by XIII. c, No. 33,

$$=\left(-2\cos\frac{\theta}{2}\right)^{-n}\cdot\operatorname{cis}n\left(\pi-\frac{\theta}{2}\right).$$

16. As in XIII. b, No. 16,

for $|\cos\theta|<1$, $\cos\theta\,.\operatorname{cis}\theta-\frac{1}{3}\cos^3\theta\,.\operatorname{cis}3\theta+\frac{1}{5}\cos^5\theta\,.\operatorname{cis}5\theta-\ldots$

$$=\frac{1}{2i}\log\frac{1+i\cos\theta\operatorname{cis}\theta}{1-i\cos\theta\operatorname{cis}\theta}$$

$$=\frac{1}{2i}\log\left\{\frac{1-\sin\theta\cos\theta+i\cos^2\theta}{1+\sin\theta\cos\theta-i\cos^2\theta}\times\frac{1+\sin\theta\cos\theta+i\cos^2\theta}{1+\sin\theta\cos\theta+i\cos^2\theta}\right\}$$

$$=\frac{1}{2i}\log\frac{\sin^2\theta+2i\cos^2\theta}{1+2\sin\theta\cos\theta+\cos^2\theta}$$

$$=\frac{1}{2i}\left\{\frac{1}{2}\log R^2+i\tan^{-1}\left(\frac{2\cos^2\theta}{\sin^2\theta}\right)\right\}.$$

If $\theta=r\pi$, series $=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\tan^{-1}1$, see p. 89, $=\frac{\pi}{4}$.

17. $\cos(x+iy)=\exp(i\alpha)$, where $\tan\alpha=\frac{3}{4}$;

$\therefore\ \cos x\operatorname{ch}y-i\sin x\operatorname{sh}y=\cos\alpha+i\sin\alpha=\frac{1}{5}(4+3i)$;

$\therefore\ \cos x\operatorname{ch}y=\frac{4}{5},\ \sin x\operatorname{sh}y=-\frac{3}{5}$:

$\therefore\ 16\sec^2x-9\operatorname{cosec}^2x=25$;

solving, $\sin^2x=\frac{3}{5}$, $\cos^2x=\frac{2}{5}$; but $\operatorname{ch}y>0$; $\therefore\ \cos x>0$;

$\therefore\ \cos x=+\sqrt{(\frac{2}{5})}$, $\operatorname{ch}y=2\sqrt{(\frac{2}{5})}$.

If $\sin x=+\sqrt{(\frac{3}{5})}$, $\operatorname{sh}y=-\sqrt{(\frac{3}{5})}$;

$\therefore\ x=2n\pi+\sin^{-1}(\sqrt{\frac{3}{5}})$, since $\cos x>0$; $y=-\operatorname{sh}^{-1}(\sqrt{\frac{3}{5}})$.

If $\sin x=-\sqrt{(\frac{3}{5})}$, $\operatorname{sh}y=+\sqrt{(\frac{3}{5})}$, etc.

18. $\exp(z\operatorname{Log}i)=e\,.\operatorname{cis}\alpha$;

$$\therefore\ \exp\left[z\,.\,(4n+1)\frac{\pi i}{2}\right]=\exp[1+(\alpha+2k\pi)i];$$

$$\therefore\ z(4n+1)\frac{\pi i}{2}=1+(\alpha+2k\pi)i;$$

$$\therefore\ z=\frac{2}{(4n+1)\pi}\left\{-i+\alpha+2k\pi\right\}.$$

19. Put $3\theta=\alpha$, then $-\frac{3\pi}{2}<\alpha<\frac{\pi}{2}$;

$$1+\frac{1}{3}\operatorname{cis}\alpha+\frac{1\,.\,4}{3\,.\,6}\operatorname{cis}2\alpha+\ldots=\text{p.v. of }(1-\operatorname{cis}\alpha)^{-\frac{1}{3}},$$

by p. 253, for $\operatorname{cis}\alpha\neq1$, i.e. $\alpha\neq2n\pi$,

$$=\exp[-\tfrac{1}{3}\log(1-\operatorname{cis}\alpha)].$$

As in No. 14, $\log(1-\operatorname{cis}\alpha)=\log\left\{2\sin\frac{\alpha}{2}\cdot\operatorname{cis}\frac{\alpha-\pi}{2}\right\}$. First let $-\frac{3\pi}{4}<\frac{\alpha}{2}<0$, then $\sin\frac{\alpha}{2}<0$;

$$\therefore\ \log(1-\operatorname{cis}\alpha)=\log\left(-2\sin\frac{\alpha}{2}\right)+i\,.\operatorname{am}\left[\operatorname{cis}\frac{\alpha+\pi}{2}\right]$$

$$=\log\left(-2\sin\frac{\alpha}{2}\right)+i\,.\,\frac{\alpha+\pi}{2},$$

since $-\frac{\pi}{4}<\frac{\alpha+\pi}{2}<\frac{\pi}{2}$;

$$\text{expression}=\exp\left[-\tfrac{1}{3}\log\left(-2\sin\frac{\alpha}{2}\right)-\frac{i}{6}(\alpha+\pi)\right]$$

$$=\left(-2\sin\frac{\alpha}{2}\right)^{-\frac{1}{3}}.\operatorname{cis}\left[-\frac{\alpha+\pi}{6}\right];$$

$$\therefore\ \text{series}=\left(-2\sin\frac{\alpha}{2}\right)^{-\frac{1}{3}}\cdot\left(-\sin\frac{\alpha+\pi}{6}\right)$$

for $-\frac{3\pi}{2}<\alpha<0$, i.e. $-\frac{\pi}{2}<\theta<0$. Next let $0<\alpha<\frac{\pi}{2}$, then $\sin\frac{\alpha}{2}>0$;

$$\therefore\ \log(1-\operatorname{cis}\alpha)=\log\left(2\sin\frac{\alpha}{2}\right)+i\left(\frac{\alpha-\pi}{2}\right),$$

since $-\frac{\pi}{2}<\frac{\alpha-\pi}{2}<-\frac{\pi}{4}$;

$$\therefore \text{ expression} = \exp\left[-\tfrac{1}{3}\log\left(2\sin\frac{\alpha}{2}\right) - \frac{i}{6}(\alpha - \pi)\right]$$
$$= \left(2\sin\frac{\alpha}{2}\right)^{-\frac{1}{3}} \cdot \text{cis}\,\frac{\pi-\alpha}{6}.$$

If $\alpha = 0$, series $= 0$.

20. $\cos 2x = \sec\frac{2}{i}(c + id) = \sec(2d - 2ic)$;

$$\therefore \cos(2d - 2ic) = \sec 2x = \frac{1+\tan^2 x}{1-\tan^2 x} = \frac{1+\text{th}^2(a+ib)}{1-\text{th}^2(a+ib)},$$

since $\tan x = \text{th}(a+ib)$, $= \text{ch}\,2(a+ib) = \cos(2ia - 2b)$;

$\therefore 2d - 2ic = 2n\pi \pm (2ia - 2b)$; equate second parts.

EXERCISE XIII. f. (p. 261.)

1. $\alpha - i\beta = \log\sin(x - iy)$; $\therefore 2\alpha = \log\sin(x+iy) + \log\sin(x-iy)$
$= \log\{\sin(x+iy)\sin(x-iy)\} = \log\{\tfrac{1}{2}(\cos 2iy - \cos 2x)\}$.

2. $\log(1 + 2h\cos\theta + h^2) = \log\{[1 + h\exp(\theta i)][1 + h\exp(-\theta i)]\}$
$= \log[1 + h\exp(\theta i)] + \log[1 + h\exp(-\theta i)]$
$=$, for $|h| < 1$, $\Sigma(-1)^{n-1}.\frac{h^n}{n}\{\exp(n\theta i) + \exp(-n\theta i)\}$
$= \Sigma(-1)^{n-1}.\frac{h^n}{n}.2\cos n\theta.$

3. $\sin^{-1}(x+iy) = \tan^{-1}\left\{\frac{x+iy}{\sqrt{[1-(x+iy)^2]}}\right\}$;

$\therefore \xi + i\eta = \frac{x+iy}{\sqrt{[1-(x+iy)^2]}}$; $\therefore \xi - i\eta = \frac{x-iy}{\sqrt{[1-(x-iy)^2]}}$;
but $\xi^2 + \eta^2 = (\xi + i\eta)(\xi - i\eta)$.

4. By Ex. XIII. c, No. 3, using $-n$ for n,
$$\text{Log}\, i^i = \text{Log}[e^{-\frac{1}{2}(4n+1)\pi}] = -\tfrac{1}{2}(4n+1)\pi + 2k\pi i;$$
$$\therefore \sin(\text{Log}\, i^i) = \sin\left(-2n\pi - \frac{\pi}{2} + 2k\pi i\right) = -\cos 2k\pi i.$$

5. If on p. 252, $z \equiv a + bi$, then
$$r = +\sqrt{(a^2+b^2)} \text{ and } \text{am}(z) = \tan^{-1}\left(\frac{b}{a}\right)$$
for $a > 0$. By eqn. (20), expression is of form vi if $Am(z^w)$ is of the form $k\pi + \frac{\pi}{2}$,
$$\text{i.e. } \beta\log r + \alpha\tan^{-1}\left(\frac{b}{a}\right) = (2k+1)\frac{\pi}{2}.$$

6. $\log(a+ib) = k\pi + \tan^{-1}(x+iy)$;

$\therefore \log(a-ib) = k\pi + \tan^{-1}(x-iy)$; $\therefore$ if $x^2+y^2 \neq 1$,
$\log(a^2+b^2) = \log(a+ib) + \log(a-ib)$
$$= n\pi + \tan^{-1}\frac{x+iy+x-iy}{1-(x+iy)(x-iy)};$$
$$\therefore \tan\{\log(a^2+b^2)\} = \frac{2x}{1-x^2-y^2}.$$

7. $\cos(u+iv) = \text{sech}(x+iy)$;

$\therefore \sin(u+iv) = \pm\sqrt{\{1-\text{sech}^2(x+iy)\}} = \pm\,\text{th}(x+iy)$;

$\therefore \cos 2u = \cos[(u+iv)+(u-iv)]$
$= \text{sech}(x+iy)\,\text{sech}(x-iy) - \text{th}(x+iy)\,\text{th}(x-iy)$
$$= \frac{1-\text{sh}(x+iy)\,\text{sh}(x-iy)}{\text{ch}(x+iy)\,\text{ch}(x-iy)} = \frac{2-(\text{ch}\,2x - \cos 2y)}{\text{ch}\,2x + \cos 2y};$$
$$\therefore \tan^2 u = \frac{1-\cos 2u}{1+\cos 2u} = \frac{\text{ch}\,2x - 1}{1+\cos 2y} = \frac{2\,\text{sh}^2 x}{2\cos^2 y};$$
similarly
$$\text{ch}\,2v = \cos\{(u+iv)-(u-iv)\} = \frac{2+\text{ch}\,2x - \cos 2y}{\text{ch}\,2x + \cos 2y};$$
$$\therefore \text{th}^2 v = \frac{\text{ch}\,2v - 1}{\text{ch}\,2v + 1} = \frac{1-\cos 2y}{1+\text{ch}\,2x} = \frac{2\sin^2 y}{2\,\text{ch}^2 x}.$$

8. If P, Q are the points (a, b), $(a, -b)$, $\text{am}\left(\frac{a+ib}{a-ib}\right) =$ rotation needed to convert OQ into OP anticlockwise or else minus the rotation needed to convert OQ into OP clockwise; $\therefore$ in all cases, $\text{am}\left(\frac{a+ib}{a-ib}\right) = 2\tan^{-1}\frac{b}{a}$; also $\left|\frac{a+ib}{a-ib}\right| = 1$;
$$\therefore \text{ expression} = \log\frac{\cos x\,\text{ch}\,y + i\sin x\,\text{sh}\,y}{\cos x\,\text{ch}\,y - i\sin x\,\text{sh}\,y}$$
$$= \log 1 + i\,.\,2\tan^{-1}\left(\frac{\sin x\,\text{sh}\,y}{\cos x\,\text{ch}\,y}\right).$$

9. $(\tfrac{1}{2}\,\text{cis}\,2\theta) - \tfrac{1}{2}(\tfrac{1}{2}\,\text{cis}\,2\theta)^2 + \tfrac{1}{3}(\tfrac{1}{2}\,\text{cis}\,2\theta)^3 - \ldots$
$$= \log(1 + \tfrac{1}{2}\,\text{cis}\,2\theta) = \log\frac{2+\cos 2\theta + i\sin 2\theta}{2}$$
$$= \tfrac{1}{2}\log R^2 + i\tan^{-1}\frac{\sin 2\theta}{2+\cos 2\theta},$$
since $2 + \cos 2\theta > 0$;
$$\therefore \text{ sum of series} = \tan^{-1}\frac{2\sin\theta\cos\theta}{3\cos^2\theta + \sin^2\theta}$$

$$=\tan^{-1}\frac{2\tan\theta}{3+\tan^2\theta}=\tan^{-1}\frac{\frac{2}{3}\tan\theta}{1+\frac{1}{3}\tan^2\theta}$$

$$=\tan^{-1}\frac{\tan\theta-\tan(\theta-\alpha)}{1+\tan\theta\tan(\theta-\alpha)},$$

where $\tan(\theta-\alpha)=\frac{1}{3}\tan\theta$,

$$=\tan^{-1}[\tan\{\theta-(\theta-\alpha)\}]=\tan^{-1}(\tan\alpha).$$

10. $1+\frac{1}{2}\operatorname{cis}2\theta-\frac{1}{2\,.\,4}\operatorname{cis}4\theta+\ldots=$, by p. 253, p.v. of $(1+\operatorname{cis}2\theta)^{\frac{1}{2}}$,

since $\operatorname{cis}2\theta\neq-1$,

$$=\exp\{\tfrac{1}{2}\log(1+\cos 2\theta+i\sin 2\theta)\}$$
$$=\exp[\tfrac{1}{2}\{\log(2\cos\theta\,.\,\operatorname{cis}\theta)\}]$$
$$=\exp\{\tfrac{1}{2}\log(2\cos\theta)+\tfrac{1}{2}\theta i\}$$

since $2\cos\theta>0$ and $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$,

$$=\exp\{\log\sqrt{(2\cos\theta)}\}\,.\,\operatorname{cis}(\tfrac{1}{2}\theta)$$
$$=+\sqrt{(2\cos\theta)}\,.\,\operatorname{cis}(\tfrac{1}{2}\theta);$$

$\therefore$ series $=+\sqrt{(2\cos\theta)}\,.\,\cos\frac{1}{2}\theta=+\sqrt{\left\{2\cos\theta\,.\,\cos^2\frac{\theta}{2}\right\}}$.

11. $\operatorname{cis}\theta+\frac{1}{3}\operatorname{cis}3\theta+\frac{1}{5}\operatorname{cis}5\theta+\ldots=\frac{1}{2}\{\log(1+\operatorname{cis}\theta)-\log(1-\operatorname{cis}\theta)\}$,

since $\operatorname{cis}\theta\neq-1$,

$$=\tfrac{1}{2}\left\{\log\left(2\cos\frac{\theta}{2}\cdot\operatorname{cis}\frac{\theta}{2}\right)-\log\left(2\sin\frac{\theta}{2}\cdot\operatorname{cis}\frac{\theta-\pi}{2}\right)\right\},$$

see XIII. e, No. 14; but $\cos\frac{\theta}{2}>0$ and $\sin\frac{\theta}{2}>0$ and

$$-\frac{\pi}{2}<\frac{\theta-\pi}{2}<0 \text{ for } 0<\theta<\pi;$$

$\therefore$ expression

$$=\tfrac{1}{2}\left\{\log\left(2\cos\frac{\theta}{2}\right)+i\frac{\theta}{2}-\log\left(2\sin\frac{\theta}{2}\right)-\frac{i}{2}(\theta-\pi)\right\}$$

$$=\tfrac{1}{2}\log\cot\frac{\theta}{2}+\frac{i\pi}{4}.$$

12. Let P, A, B be the points (x, y), $(1, 0)$, $(-1, 0)$;

$$\text{expression}=\frac{1}{i}\left\{\log\frac{\mathrm{AP}}{\mathrm{BP}}+i\,.\,\mathrm{am}\,.\,\frac{\overline{\mathrm{AP}}}{\overline{\mathrm{BP}}}\right\},$$

where P lies above the x-axis; $\therefore\ u=\mathrm{am}\,(\overline{\mathrm{AP}})-\mathrm{am}\,(\overline{\mathrm{BP}})$.
If P tends to a point Q on the x-axis,

$$\mathrm{am}\,(\overline{\mathrm{AP}})-\mathrm{am}\,(\overline{\mathrm{BP}})\to\pi-0=\pi,$$

if Q lies between A and B; $\mathrm{am}\,(\overline{\mathrm{AP}})-\mathrm{am}\,(\overline{\mathrm{BP}})\to\pi-\pi=0$, if Q lies on AB produced, and $\to 0-0$ if Q lies on BA produced.

13. Expression $=\exp\{i\operatorname{Log}\sin(x+yi)\}$

$$=\exp\{i\operatorname{Log}(\sin x\operatorname{ch}y+i\cos x\operatorname{sh}y)\}$$

$$=\exp\left\{i\left[\tfrac{1}{2}\log r^2+i\tan^{-1}\frac{\cos x\operatorname{sh}y}{\sin x\operatorname{ch}y}\right]\right\}$$

since $\sin x>0$, where

$$r^2=\sin(x+yi)\,.\,\sin(x-yi)=\sin^2x+\operatorname{sh}^2y,$$

$$=\exp\left\{-\tan^{-1}(\cot x\coth y)+\frac{i}{2}\log r^2\right\}.$$

14. As on p. 257,

$$\sin 2x:\cos 2x:1=2\cot\theta:-2\cot^2\theta:+\sqrt{(4\cot^2\theta\operatorname{cosec}^2\theta)}.$$

From $\tan 2x=-\tan\theta$, $2x=n\pi-\theta$;

(i) For $0<\theta<\frac{\pi}{2}$, $\sin 2x>0$, $\cos 2x<0$;

$$\therefore\ \frac{\pi}{2}<2x<\pi;\quad\therefore\ 2x=\pi-\theta;$$

(ii) For $-\frac{\pi}{2}<\theta<0$, $\sin 2x<0$, $\cos 2x<0$;

$$\therefore\ -\pi<2x<-\frac{\pi}{2};\quad\therefore\ 2x=-\pi-\theta;$$

$$\text{also } y=\tfrac{1}{4}\log\frac{\cot^2\theta+(1-\operatorname{cosec}\theta)^2}{\cot^2\theta+(1+\operatorname{cosec}\theta)^2}$$

$$=\tfrac{1}{4}\log\frac{2\operatorname{cosec}^2\theta-2\operatorname{cosec}\theta}{2\operatorname{cosec}^2\theta+2\operatorname{cosec}\theta}.$$

15. From XIII. d, No. 16, where $(t_1+t_2)^2$ and $(t_1-t_2)^2$ are the roots of $\mu^2-\mu\left(1+\frac{175+81}{256}\right)+\frac{25\times 7}{256}=0$,

i.e. $\mu=\frac{25}{16}$ or $\frac{7}{16}$, $t_1+t_2=\frac{5}{4}$, $t_1-t_2=\frac{\sqrt{7}}{4}$;

$$\therefore\ \text{expression}=n\pi+(-1)^n\left\{\sin^{-1}\frac{\sqrt{7}}{4}+i\operatorname{ch}^{-1}\tfrac{5}{4}\right\};$$

also $\sin^{-1}\frac{\sqrt{7}}{4}=\cos^{-1}\frac{3}{4}$, $\operatorname{ch}^{-1}\frac{5}{4}=\log\{\frac{5}{4}+\sqrt{(\frac{25}{16}-1)}\}$,

as in Ch. VI., eqn. (15), p. 110.

16. By XIII. d, No. 2 and No. 15,

$$\operatorname{Ch}^{-1}(x+iy)=2n\pi i\pm i\operatorname{Cos}^{-1}(x+iy)$$
$$=2n\pi i\pm i[\cos^{-1}(t_1-t_2)-i\operatorname{ch}^{-1}(t_1+t_2)].$$

17. The p.v. of

$$\tfrac{1}{2}\{(1+z)^n+(1-z)^n\}$$
$$=1+\frac{n(n-1)}{2!}z^2+\frac{n(n-1)(n-2)(n-3)}{4!}z^4+\dots$$

for $|z|=1,\ n>-1$; $\therefore$ series $=$ p.v. of $\tfrac{1}{2}\{(1+i)^n+(1-i)^n\}$;

$$\text{p.v. of } (1+i)^n=\exp\{n\log(1+i)\}=\exp\left\{n\log\left(\sqrt{2}\,.\,\text{cis}\,\frac{\pi}{4}\right)\right\}$$
$$=\exp\left\{\frac{n}{2}\log 2+n\,.\,\frac{\pi i}{4}\right\}$$
$$=\exp\{\log 2^{\frac{n}{2}}\}\,.\,\text{cis}\,\frac{n\pi}{4}=2^{\frac{n}{2}}\,.\,\text{cis}\,\frac{n\pi}{4};$$
$$\therefore \text{ series}=\tfrac{1}{2}\,.\,2^{\frac{n}{2}}\left[\text{cis}\,\frac{n\pi}{4}+\text{cis}\left(-\frac{n\pi}{4}\right)\right]$$
$$=2^{\frac{n}{2}}\,.\,\tfrac{1}{2}\,.\,2\cos\frac{n\pi}{4}.$$

18.
$$\frac{z}{1\,.\,2}-\frac{z^2}{2\,.\,3}+\frac{z^3}{3\,.\,4}-\dots=z(1-\tfrac{1}{2})-z^2(\tfrac{1}{2}-\tfrac{1}{3})+z^3(\tfrac{1}{3}-\tfrac{1}{4})-\dots$$
$$=(z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\dots)-(\tfrac{1}{2}z-\tfrac{1}{3}z^2+\tfrac{1}{4}z^3-\dots)$$
$$=,\ \text{for } |z|=1,\ z\neq-1,\ \log(1+z)-\frac{1}{z}\{z-\log(1+z)\}$$
$$=-1+\left(1+\frac{1}{z}\right)\log(1+z);$$

put $z=\text{cis}\,\theta,\ \theta\neq(2r+1)\pi$. Then, as in XIII. b, No. 7,

$$\text{expression}=-1+(1+\cos\theta-i\sin\theta)\,.\left\{\log\left(2\cos\frac{\theta}{2}\right)+i\frac{\theta}{2}\right\},$$

since $\cos\frac{\theta}{2}>0$ for $-\pi<\theta<\pi$;

$$\therefore \text{ given series}=-1+(1+\cos\theta)\,.\,\log\left(2\cos\frac{\theta}{2}\right)+\frac{\theta}{2}\,.\,\sin\theta.$$

Also if $\pi<\theta<3\pi$,

$$\text{expression}=-1+(1+\cos\theta-i\sin\theta)\times\left\{\log\left(-2\cos\frac{\theta}{2}\right)+i\left(\frac{\theta}{2}-\pi\right)\right\}$$

and series

$$=-1+(1+\cos\theta)\,.\,\log\left(-2\cos\frac{\theta}{2}\right)+\sin\theta\,.\left(\frac{\theta}{2}-\pi\right).$$

19.
$$\exp(y)=\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}};\quad\therefore\ \tan\frac{x}{2}=\frac{\exp(y)-1}{\exp(y)+1}=\text{th}\,\frac{y}{2};$$

$$\therefore\ \frac{\cos\frac{x}{2}}{\text{ch}\frac{y}{2}}=\frac{\sin\frac{x}{2}}{\text{sh}\frac{y}{2}}=\frac{\exp\frac{xi}{2}}{\text{ch}\frac{y}{2}+i\,\text{sh}\frac{y}{2}}=\frac{\exp\left(-\frac{xi}{2}\right)}{\text{ch}\frac{y}{2}-i\,\text{sh}\frac{y}{2}};$$
$$\therefore\ \exp(xi)=\frac{1+i\,\text{th}\frac{y}{2}}{1-i\,\text{th}\frac{y}{2}}=\frac{1+\tan\frac{iy}{2}}{1-\tan\frac{iy}{2}}=\tan\left(\frac{iy}{2}+\frac{\pi}{4}\right);$$

$\therefore\ xi=\text{Log}\left\{\tan\left(\frac{iy}{2}+\frac{\pi}{4}\right)\right\}$; $\therefore\ xi$ is the same function of yi that y is of x; $\therefore\ (2k\pi+x)i=a_1(iy)+a_3(iy)^3+\dots$; taking the value of x which vanishes with y, k is zero.

20. As on p. 248,

$$\text{cis}\,\alpha+\tfrac{1}{2}\,\text{cis}\,2\alpha+\tfrac{1}{3}\,\text{cis}\,3\alpha+\dots=-\log(1-\cos\alpha-i\sin\alpha),$$

for $\text{cis}\,\alpha\neq1,\ \alpha\neq2r\pi$,

$$=-\left[\tfrac{1}{2}\log\{(1-\cos\alpha)^2+\sin^2\alpha\}-i\tan^{-1}\frac{\sin\alpha}{1-\cos\alpha}\right]$$
$$=-\tfrac{1}{2}\log(2-2\cos\alpha)+i\tan^{-1}\left(\cot\frac{\alpha}{2}\right);$$
$$\therefore\ \sin\alpha+\tfrac{1}{2}\sin2\alpha+\dots=-\tan^{-1}\left[\tan\frac{\alpha-\pi}{2}\right];$$
$$\therefore\ \tfrac{1}{2}\{\sin2x+\tfrac{1}{2}\sin4x+\tfrac{1}{3}\sin6x+\dots\}$$
$$=-\tfrac{1}{2}\tan^{-1}\left[\tan\left(x-\frac{\pi}{2}\right)\right];$$

also $\tfrac{1}{2}\{\tfrac{1}{3}\sin6x+\tfrac{1}{6}\sin12x+\tfrac{1}{9}\sin18x+\dots\}$

$$=-\tfrac{1}{6}\tan^{-1}\left[\tan\left(3x-\frac{\pi}{2}\right)\right];$$

subtract, given series

$$=-\tfrac{1}{2}\left\{\tan^{-1}\left[\tan\left(x-\frac{\pi}{2}\right)\right]-\tfrac{1}{3}\tan^{-1}\left[\tan\left(3x-\frac{\pi}{2}\right)\right]\right\};$$

(i) $0<x<\frac{\pi}{3}$, series $=-\tfrac{1}{2}\left\{\left(x-\frac{\pi}{2}\right)-\tfrac{1}{3}\left(3x-\frac{\pi}{2}\right)\right\}$;

(ii) $\frac{\pi}{3}<x<\frac{2\pi}{3}$, series $=-\tfrac{1}{2}\left\{\left(x-\frac{\pi}{2}\right)-\tfrac{1}{3}\left(3x-\frac{\pi}{2}-\pi\right)\right\}$;

(iii) $\frac{2\pi}{3}<x<\pi$, series $=-\tfrac{1}{2}\left\{\left(x-\frac{\pi}{2}\right)-\tfrac{1}{3}\left(3x-\frac{\pi}{2}-2\pi\right)\right\}$.

21. From No. 20, $\cos\alpha + \frac{1}{2}\cos 2\alpha + \frac{1}{3}\cos 3\alpha + \ldots = -\frac{1}{2}\log\left(4\sin^2\frac{\alpha}{2}\right)$;

$$\text{twice series (i)} = \left\{\cos\left(\theta+\frac{\pi}{3}\right)+\cos\left(\theta-\frac{\pi}{3}\right)\right\}$$
$$+\frac{1}{2}\left\{\cos\left(2\theta+\frac{2\pi}{3}\right)+\cos\left(2\theta-\frac{2\pi}{3}\right)\right\}+\ldots$$
$$= -\frac{1}{2}\log\left[4\sin^2\left(\frac{\theta}{2}+\frac{\pi}{6}\right)\right]-\frac{1}{2}\log\left[4\sin^2\left(\frac{\theta}{2}-\frac{\pi}{6}\right)\right]$$
$$= -\frac{1}{2}\log\left[16\sin^2\left(\frac{\theta}{2}+\frac{\pi}{6}\right)\sin^2\left(\frac{\theta}{2}-\frac{\pi}{6}\right)\right]$$
$$= -\frac{1}{2}\log\left\{4\left(\cos\frac{\pi}{3}-\cos\theta\right)^2\right\}$$
$$= -\frac{1}{2}\log(1-2\cos\theta)^2 = -\log(1-2\cos\theta),$$

if $2\cos\theta < 1$, i.e. if $2n\pi + \frac{\pi}{3} < \theta < 2(n+1)\pi - \frac{\pi}{3}$.

Similarly, twice series (ii)

$$= \left\{\sin\left(\theta+\frac{\pi}{3}\right)+\sin\left(\theta-\frac{\pi}{3}\right)\right\}+\ldots=, \text{ by No. 20,}$$
$$-\tan^{-1}\left[\tan\left(\frac{\theta}{2}+\frac{\pi}{6}-\frac{\pi}{2}\right)\right]-\tan^{-1}\left[\tan\left(\frac{\theta}{2}-\frac{\pi}{6}-\frac{\pi}{2}\right)\right]$$
$$= -\tan^{-1}\left[\tan\left(\frac{\theta}{2}-\frac{\pi}{3}\right)\right]-\tan^{-1}\left[\tan\left(\frac{\theta}{2}-\frac{2\pi}{3}\right)\right];$$

this reduces to $-\theta$, only if it is of the form

$$-\left\{\left(\frac{\theta}{2}-\frac{\pi}{3}\right)+\left(\frac{\theta}{2}-\frac{2\pi}{3}+\pi\right)\right\};$$

these require that

$$-\frac{\pi}{2}<\left(\frac{\theta}{2}-\frac{\pi}{3}\right)<\frac{\pi}{2} \text{ and } -\frac{\pi}{2}<\left(\frac{\theta}{2}-\frac{2\pi}{3}+\pi\right)<\frac{\pi}{2},$$

i.e. $$-\frac{\pi}{3}<\theta<\frac{5\pi}{3} \text{ and } -\frac{5\pi}{3}<\theta<\frac{\pi}{3}.$$

22. The relation is true if $-\pi < \{\text{am}(z-z_1) - \text{am}(z-z_2)\} \leqslant \pi$. If the relation is untrue, am $(z-z_1)$ and am $(z-z_2)$ must be of opposite sign, i.e. the directed lines P_1P and P_2P must be such that the line through P in the sense Ox cuts P_1P_2 internally. If the *half-line* through P in the sense Ox does not cut P_1P_2, $|\text{am}(z-z_1)-\text{am}(z-z_2)| = \angle P_1PP_2$ and is $\therefore < \pi$. If the half-line through P in sense Ox cuts P_1P_2 internally, $|\text{am}(z-z_1)-\text{am}(z-z_2)| = 2\pi - \angle P_1PP_2$ and is $\therefore > \pi$.

CHAPTER XIV

EXERCISE XIV. a. (p. 264.)

1. $\text{cosec A} + \sec A = \dfrac{\cos A + \sin A}{\sin A\cos A} = (\cos A + \sin A)\dfrac{\cos^2 A + \sin^2 A}{\sin A \cos A}$
$$= (\cos A + \sin A)(\cot A + \tan A).$$

2. $(2-\cos^2 B)(2+\tan^2 B) = (2\sec^2 B - 1)(2\cos^2 B + \sin^2 B)$, by multiplying the two brackets by $\sec^2 B$ and $\cos^2 B$,
$$= (2\tan^2 B + 1)(2 - \sin^2 B).$$

3. $\sin^4 C + \cos^4 C + 2\cos^2 C\sin^2 C = (\cos^2 C + \sin^2 C)^2 = 1$. Divide by $\cos^2 C \sin^2 C$.

4. $\sin D + \cos D = \sqrt{2}\,.\sin D\cos D$, squaring
$1 + 2\sin D\cos D = 2\sin^2 D\cos^2 D$;
$$\therefore\ +1 = 2\sin D\cos D(\sin D\cos D - 1)$$
$$= \sqrt{2}(\sin D + \cos D)(\sin D\cos D - \sin^2 D - \cos^2 D)$$
$$= -\sqrt{2}(\sin^3 D + \cos^3 D).$$

5. $2(\text{l.h.s.}) = 1 + \cos 45° - 1 - \cos 135° = 2\cos 45°$.

6. $2\cos\theta = \sqrt{(4\cos^2\theta)} = \sqrt{(2+2\cos 2\theta)}$, similarly
$$= \sqrt{[2+\sqrt{\{2+2\cos 4\theta\}}]}$$
$$= \sqrt{[2+\sqrt{\{2+\sqrt{(2+2\cos 8\theta)}\}}]}.$$ Put $8\theta = 45°$.

7. l.h.s. $= 2\cos 36°(\cos 60° + \cos 108°) = \cos 36° - 2\cos 36°\cos 72°$
$= -\cos 108° = \sin 18°$.

8. The equation $\tan 5\theta = 1$ is satisfied by $\theta = 9°, 45°, 81°, 117°, 153°$; from p. 172, the equation is $t^5 - 5t^4 + \ldots = 0$;
$$\therefore \text{ sum of roots} = 5;$$
but $\tan 45° = 1$, $\tan 117° = -\tan 63°$, $\tan 153° = -\tan 27°$.

9. $\cos 12° + \cos 60° + \cos 84° - \cos 48° - \cos 24°$
$$= \cos 12° + \cos 300° + \cos 84° + \cos 228° + \cos 156°$$
$$= \cos 12° + \cos 84° + \cos 156° + \cos 228° + \cos 300°$$
$=$, by p. 128, $\cos 156°\,.\dfrac{\sin 180°}{\sin 36°} = 0$. **Or,** $\cos 5\theta = \frac{1}{2}$ is satisfied by $\theta = 12°, 60°, 84°, 132°, 156°$; the equation in $\cos\theta$ is (see p. 172) $2^4\cos^5\theta - a\cos^3\theta + b\cos\theta = \frac{1}{2}$; $\therefore$ sum of roots $= 0$; but $\cos 132° = -\cos 48°$, $\cos 156° = -\cos 24°$.

10. $\sin 40^\circ \sin 50^\circ = \frac{1}{2}(\cos 10^\circ - \cos 90^\circ) = \frac{1}{2}\cos 10^\circ = \sin 30^\circ \sin 80^\circ$.

11. By No. 10, $\dfrac{\sin 50^\circ}{\sin 30^\circ} = \dfrac{\sin 80^\circ}{\sin 40^\circ}$;

$$\therefore \frac{\sin 50^\circ + \sin 30^\circ}{\sin 50^\circ - \sin 30^\circ} = \frac{\sin 80^\circ + \sin 40^\circ}{\sin 80^\circ - \sin 40^\circ}; \quad \therefore \frac{\tan 40^\circ}{\tan 10^\circ} = \frac{\tan 60^\circ}{\tan 20^\circ}.$$

Or, $\tan 20^\circ \tan 80^\circ \tan 140^\circ$ = product of roots of eqn.

$$\tan 3\theta = \sqrt{3}, \text{ or } \frac{3t - t^3}{1 - 3t^2} = \sqrt{3}; \quad \therefore \text{ is } -\sqrt{3};$$

$$\therefore \tan 20^\circ \cot 10^\circ \tan 40^\circ = \sqrt{3} = \tan 60^\circ.$$

12. l.h.s. $= \sin^2 40^\circ - \sin^2 20^\circ : \sin^2 60^\circ - \sin^2 20^\circ : \sin^2 80^\circ - \sin^2 20^\circ$

$= \sin 60^\circ \sin 20^\circ : \sin 80^\circ \sin 40^\circ : \sin 100^\circ \sin 60^\circ$,

by the identity $\sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y)$;

1st : 3rd $= \sin 20^\circ : \sin 80^\circ = 1 : z$,

and 2nd : 3rd $= \sin 40^\circ : \sin 60^\circ = x : y$.

13. $2(\cos^2 14^\circ - \sin^2 7^\circ) = 1 + \cos 28^\circ - 1 + \cos 14^\circ = 2\cos 21^\circ \cos 7^\circ$.

14. $t = \dfrac{\sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta}$;

$$\therefore \frac{1+t}{1-t} = \frac{\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta} = \frac{(\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta)^2}{\cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta} = \frac{1 + \sin\theta}{\cos\theta}.$$

15. $1 - \tan^2 A = x \tan A$; $\therefore \tan 2A = \dfrac{2}{x}$; $\therefore \tan 4A = \dfrac{\frac{4}{x}}{1 - \frac{4}{x^2}}$.

16. $\cos A = 2\cos 2A \cos A + \cos 2A = (\cos 3A + \cos A) + \cos 2A$.

17. $\operatorname{cosec} 4\theta + \cot 4\theta = (1 + \cos 4\theta)/(\sin 4\theta)$

$= 2\cos^2 2\theta/(2\cos 2\theta \sin 2\theta) = \cot 2\theta$,

similarly $\operatorname{cosec} 2\theta + \cot 2\theta = \cot \theta$.

18. Multiply out l.h.s. and express each product as a sum; this gives

$$2(1 + \cos 2\theta + \cos 4\theta + \cos 6\theta + \dots + \cos 14\theta) = \frac{2\cos 7\theta \sin 8\theta}{\sin\theta},$$

by (13), p. 128, $= \dfrac{\sin 15\theta + \sin\theta}{\sin\theta}$.

19. l.h.s. $= (\cot\theta + \tan\theta)(\cot^2\theta - 1 + \tan^2\theta)$

$$= \frac{1}{\sin\theta \cdot \cos\theta}\left(\frac{1}{\sin^2\theta\cos^2\theta} - 3\right) = 2\operatorname{cosec} 2\theta(4\operatorname{cosec}^2 2\theta - 3).$$

20. $4 \cdot (\text{l.h.s.}) = (3\sin\theta - \sin 3\theta)\sin 3\theta + (3\cos\theta + \cos 3\theta)\cos 3\theta$

$= 3\cos(3\theta - \theta) + \cos^2 3\theta - \sin^2 3\theta$

$= 3\cos 2\theta + \cos 6\theta = 4\cos^3 2\theta$.

21. $4\sin^3(60^\circ + \theta) + 4\sin^3(60^\circ - \theta)$

$= 3\sin(60^\circ + \theta) - \sin(180^\circ + 3\theta)$

$\quad + 3\sin(60^\circ - \theta) - \sin(180^\circ - 3\theta)$

$= 6\sin 60^\circ \cos\theta$.

22. $\sin 2\theta(3\tan\theta - 2\cot\theta) = 6\sin^2\theta - 4\cos^2\theta$

$= 1 - 5(\cos^2\theta - \sin^2\theta) = 1 - 5\cos 2\theta$.

23. $$1 + \tan\frac{\alpha}{2} = 1 + \tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right) = 1 + \frac{1 - \tan\frac{\beta}{2}}{1 + \tan\frac{\beta}{2}} = \frac{2}{1 + \tan\frac{\beta}{2}}.$$

24. $$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{-2\tan^2 B}{2(1 + \tan^2 B)} = -\sin^2 B = \tfrac{1}{2}(\cos 2B - 1).$$

25. $$\tan(2\theta - \alpha) = \frac{(\mu \operatorname{cosec} 2\alpha - \cot 2\alpha) - \tan\alpha}{1 + \tan\alpha(\mu\operatorname{cosec} 2\alpha - \cot 2\alpha)}$$

$$= \frac{\mu\cos\alpha - (\cos 2\alpha\cos\alpha + \sin 2\alpha \sin\alpha)}{(\sin 2\alpha\cos\alpha - \cos 2\alpha\sin\alpha) + \mu\sin\alpha}$$

$$= \frac{\mu\cos\alpha - \cos(2\alpha - \alpha)}{\sin(2\alpha - \alpha) + \mu\sin\alpha} = \frac{(\mu - 1)\cos\alpha}{(\mu + 1)\sin\alpha}.$$

26. $$1 + 3\sin^2\phi = \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 3\sin^{\frac{2}{3}}\frac{\theta}{2}\cos^{\frac{2}{3}}\frac{\theta}{2}\left(\sin^{\frac{2}{3}}\frac{\theta}{2} + \cos^{\frac{2}{3}}\frac{\theta}{2}\right)$$

$$= 1 + 3\sin^{\frac{2}{3}}\frac{\theta}{2}\cos^{\frac{2}{3}}\frac{\theta}{2}(1 + 3\sin^2\phi)^{\frac{1}{3}};$$

$$\therefore \sin^6\phi = \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}(1 + 3\sin^2\phi) = \tfrac{1}{4}\sin^2\theta(1 + 3\sin^2\phi);$$

$\therefore 4\tan^6\phi = \sin^2\theta\sec^4\phi(\sec^2\phi + 3\tan^2\phi)$;

if $t = \tan\phi$, $4t^6 = \sin^2\theta \cdot (1 + t^2)^2(1 + t^2 + 3t^2)$

$= \sin^2\theta(1 + 6t^2 + 9t^4 + 4t^6)$;

$\therefore \cos^2\theta(1 + 6t^2 + 9t^4 + 4t^6) = 1 + 6t^2 + 9t^4 = (1 + 3t^2)^2$;

$\therefore (1 + 3t^2)^2\tan^2\theta = 4t^6$.

27. $e(\cos\alpha - \cos\beta) = e^2(\cos\alpha\cos\beta - 1)$; $\therefore e = \dfrac{\cos\alpha - \cos\beta}{\cos\alpha\cos\beta - 1}$;

$$\therefore \frac{1 - e}{1 + e} = \frac{\cos\alpha\cos\beta - 1 - \cos\alpha + \cos\beta}{\cos\alpha\cos\beta - 1 + \cos\alpha - \cos\beta}$$

$$= \frac{(\cos\alpha + 1)(\cos\beta - 1)}{(\cos\alpha - 1)(\cos\beta + 1)} = \cot^2\frac{\alpha}{2}\cdot\tan^2\frac{\beta}{2}.$$

28. $\sin\theta = k\sqrt{(1-2e\cos\theta+e^2)} = k\sqrt{x}$;

$\therefore\ (2ek\sqrt{x})^2 = (2e\sin\theta)^2$

$= (2e)^2 - (2e\cos\theta)^2 = (2e)^2 - (1+e^2-x)^2$,

which reduces to $x^2 + x(4e^2k^2 - 2 - 2e^2) + (1-e^2)^2 = 0$;

similarly if $\sin\phi = k\sqrt{y}$, y satisfies the same quadratic; xy = product of roots $=(1-e^2)^2$.

29. $2(\text{l.h.s.}) = \cos 2\alpha - \cos(2\alpha+2\beta) = 2\sin\beta\sin(2\alpha+\beta)$

$= 2\sin\beta\{\sin\beta + [\sin(2\alpha+\beta) - \sin\beta]\}$

$= 2\sin\beta\{\sin\beta + 2\cos(\alpha+\beta)\sin\alpha\}$;

Or, change α and θ of Example 1 into $\frac{\pi}{2}+\alpha$ and $\frac{\pi}{2}+\beta$.

30. $\cos 3\alpha + \cos 3\beta \equiv \cos 3\alpha + 4\cos^3\beta - 3\cos\beta$, is a cubic in $\cos\beta$ and is zero for $\beta = 180° - \alpha$, $60° - \alpha$, and $60° + \alpha$;

$\therefore$ it $\equiv 4\{\cos\beta - \cos(180° - \alpha)\}$

$\times\{\cos\beta - \cos(60° - \alpha)\}\{\cos\beta - \cos(60° + \alpha)\}$.

31. The normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at $(a\cos\beta, b\sin\beta)$ is

$$ax\sec\beta - by\operatorname{cosec}\beta = a^2 - b^2 \equiv c^2;$$

the hypothesis proves that it passes through

$$\left(\frac{c^2}{a}\cos^3\alpha,\ -\frac{c^2}{b}\sin^3\alpha\right)$$

which is the centre of curvature for $(a\cos\alpha, b\sin\alpha)$; $\therefore$ α, α, β satisfy the condition of concurrence for three normals; $\therefore$ $\sin(\alpha+\alpha) + \sin(\alpha+\beta) + \sin(\beta+\alpha) = 0$.

32. $2\sin(3\beta-\alpha) - \sin(3\beta - 2\alpha) = 3\sin\beta$;

$\therefore\ 2\{\sin(3\beta-\alpha) - \sin\beta\} = \sin\beta + \sin(3\beta - 2\alpha)$;

$\therefore\ 2\cos\left(2\beta - \frac{\alpha}{2}\right)\sin\left(\beta - \frac{\alpha}{2}\right) = \sin(2\beta - \alpha)\,.\,\cos(\beta - \alpha)$,

$2\cos\left(2\beta - \frac{\alpha}{2}\right) = 2\cos\left(\beta - \frac{\alpha}{2}\right)\cos(\beta - \alpha)$

$= \cos\frac{\alpha}{2} + \cos\left(2\beta - \frac{3\alpha}{2}\right)$;

$\therefore\ \cos\left(2\beta - \frac{\alpha}{2}\right) - \cos\frac{\alpha}{2} = \cos\left(2\beta - \frac{3\alpha}{2}\right) - \cos\left(2\beta - \frac{\alpha}{2}\right)$;

$\therefore\ -\sin\beta\,.\,\sin\left(\beta - \frac{\alpha}{2}\right) = \sin(2\beta - \alpha)\sin\frac{\alpha}{2}$;

$\therefore\ \sin\beta = -2\cos\left(\beta - \frac{\alpha}{2}\right)\sin\frac{\alpha}{2} = -[\sin\beta - \sin(\beta - \alpha)]$;

$\therefore\ 2\sin\beta = \sin\beta\cos\alpha - \cos\beta\sin\alpha$;

$\therefore\ \sin\beta(\cos\alpha - 2) = \cos\beta\sin\alpha$.

EXERCISE XIV. b. (p. 266.)

1. $-\cos A = \cos(B+C) = \cos B\cos C - \sin B\sin C$;

$\therefore\ (\cos A + \cos B\cos C)^2$

$= \sin^2 B\sin^2 C = (1-\cos^2 B)(1-\cos^2 C)$

$= 1 - \cos^2 B - \cos^2 C + \cos^2 B\cos^2 C$.

2. ABC any $\triangle$, ABX $= B + \frac{\pi}{3}$, ACX $= C + \frac{\pi}{3}$;

$\therefore\ \triangle$XBC is equilateral, XB $= a =$ XC;

$\therefore\ c^2 + a^2 - 2ca\cos\left(B + \frac{\pi}{3}\right)$

$= AX^2 = b^2 + a^2 - 2ba\cos\left(C + \frac{\pi}{3}\right)$:

but $a : b : c = \sin A : \sin B : \sin C$.

3. $\text{l.h.s.} = 2\cos^2 A + 2\cos(B+C)\cos(B-C)$

$= 2\cos A\{-\cos(B+C) - \cos(B-C)\}$

$= -4\cos A\cos B\cos C$.

4. $\text{l.h.s.} = 2\sin\frac{3A+3B}{2}\cos\frac{3A-3B}{2} + 2\sin\frac{3C}{2}\cos\frac{3C}{2}$

$= 2\cos\frac{3C}{2}\left\{-\cos\frac{3A-3B}{2} - \cos\frac{3A+3B}{2}\right\}$.

5. $\dfrac{1+\cos B}{\cos A - \cos C} = \dfrac{2\cos^2\frac{B}{2}}{2\sin\frac{A+C}{2}\sin\frac{C-A}{2}} = \dfrac{\sin\frac{C+A}{2}}{\sin\frac{C-A}{2}}$;

$\therefore\ \text{l.h.s.} = \dfrac{\sin\frac{C+A}{2} - \sin\frac{C-A}{2}}{\sin\frac{C+A}{2} + \sin\frac{C-A}{2}} = \dfrac{2\cos\frac{C}{2}\sin\frac{A}{2}}{2\sin\frac{C}{2}\cos\frac{A}{2}}$.

6. $\text{l.h.s.} = \sin\frac{B+C}{2} - 2\sin\frac{B+C}{4}\sin\frac{B-C}{4}$

$= 2\sin\frac{B+C}{4}\left(\cos\frac{B+C}{4} - \sin\frac{B-C}{4}\right)$

$$=2\sin\frac{\pi-A}{4}\left(\cos\frac{B+C}{4}+\cos\frac{2\pi+B-C}{4}\right)$$
$$=4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi-C}{4}.$$

7. $\Sigma\sin 2A\sin^2 A=\frac{1}{2}\Sigma\sin 2A(1-\cos 2A)=\frac{1}{2}\Sigma\sin 2A-\frac{1}{4}\Sigma\sin 4A$. But $\Sigma\sin 2A=4\sin A\sin B\sin C$, etc. (See No. 11 below, or *E.T.*, p. 271 and *E.T.*, XIX. b, No. 9.)

8. $\Sigma(2\sin^2 B\sin^2 C-\sin^4 A)=\frac{1}{(2R)^4}\Sigma(2b^2c^2-a^4)=\frac{1}{(2R)^4}16\Delta^2$
$$=\frac{1}{(2R)^4}\cdot 4b^2c^2\sin^2 A=4\sin^2 B\sin^2 C\sin^2 A.$$

9. l.h.s. $=2\sin\frac{5A+5B}{2}\cos\frac{5A-5B}{2}+2\sin\frac{5C}{2}\cos\frac{5C}{2}$
$$=2\cos\frac{5C}{2}\left(\cos\frac{5A-5B}{2}+\cos\frac{5A+5B}{2}\right).$$

10. $16\cos^5\theta\equiv 10\cos\theta+5\cos 3\theta+\cos 5\theta$;
$\therefore$ l.h.s. $=\frac{5}{2}\Sigma\cos A+\frac{5}{4}\Sigma\cos 3A+\frac{1}{4}\Sigma\cos 5A$.
$\Sigma\cos nA$, (n odd),
$$=2\cos\frac{nA+nB}{2}\cos\frac{nA-nB}{2}+1-2\sin^2\frac{nC}{2}$$
$$=1+2\cos\left(\frac{n\pi}{2}-\frac{nC}{2}\right)\cos\frac{nA-nB}{2}$$
$$-2\sin\frac{nC}{2}\sin\left(\frac{n\pi}{2}-\frac{nA+nB}{2}\right)$$
=, expanding and using $\cos\frac{n\pi}{2}=0$ for n odd,
$$1+2\sin\frac{n\pi}{2}\sin\frac{nC}{2}\left(\cos\frac{nA-nB}{2}-\cos\frac{nA+nB}{2}\right)$$
$$=1+(-1)^{\frac{n-1}{2}}\cdot 4\sin\frac{nA}{2}\sin\frac{nB}{2}\sin\frac{nC}{2}.$$
Take successively $n=1, 3, 5$ and substitute in the expression for l.h.s.

11. l.h.s. $=2\sin(nA+nB)\cos(nA-nB)+2\sin nC\cos nC$
$=2\sin(n\pi-nC)\cos(nA-nB)+2\sin nC\cos(n\pi-nA-nB)$
=, expanding and using $\sin n\pi=0$,
$-2\cos n\pi\sin nC\{\cos(nA-nB)-\cos(nA+nB)\}$.

12. $\cos A\sin B\sin C+\cos B\sin C\sin A+\cos C\sin A\sin B$
$-\cos A\cos B\cos C=-\cos A\cos(B+C)+\sin A\sin(B+C)$
$=-\cos(A+B+C)=1$.
Divide by $\sin A\sin B\sin C$.

13. $\Sigma(\tan A)=\prod(\tan A)$; $\therefore$ from No. 12, $\Sigma(\tan A)\Sigma(\cot A)$
$=1+\sec A\sec B\sec C$.

14. $\cot^2\alpha=\frac{\cos B\cos C}{\cos A}=\frac{\cos B\cos C}{\sin B\sin C-\cos B\cos C}$;
$\therefore\ \cos^2\alpha=\cot B\cot C$; $\therefore\ \Sigma(\cos^2\alpha)=1$.

15. $\sin A(\sin\frac{1}{2}C-\sin\frac{1}{2}B)=\sin\frac{1}{2}B\sin C-\sin\frac{1}{2}C\sin B$
$=2\sin\frac{1}{2}B\sin\frac{1}{2}C(\cos\frac{1}{2}C-\cos\frac{1}{2}B)$;
$\therefore$ B = C, (otherwise the expns. in brackets have opposite signs.
If bisector of $\angle$ABC cuts AC at K,
$$(\tfrac{1}{2}c\,.\,BK+\tfrac{1}{2}a\,.\,BK)\sin\frac{B}{2}=\triangle ABK+\triangle CBK;$$
$$\therefore\ BK(c+a)\sin\frac{B}{2}=2\Delta;$$
$$\therefore\ \text{if bisectors are equal, }(c+a)\sin\frac{B}{2}=(b+a)\sin\frac{C}{2}.$$

16. $\Sigma\{l(\tan B+\tan C)\}=0$; $\therefore$ multiplying by $\cos A\cos B\cos C$,
$\Sigma\{l\cos A(\sin B\cos C+\sin C\cos B)\}=0$;
$\therefore\ \Sigma l\sin 2A=0$; $\therefore\ (\Sigma l\sin 2A)^2=0$;
$\therefore\ \Sigma(l^2\sin^2 2A+2mn\sin 2B\sin 2C)=0$;
$\therefore$ dividing by $\sin 2A\sin 2B\sin 2C$,
$$\sum\left(\frac{l^2\sin 2A}{\sin 2B\sin 2C}+\frac{2mn}{\sin 2A}\right)=0;$$
$\therefore\ \Sigma l^2(\cot 2B+\cot 2C)=\Sigma 2mn\operatorname{cosec} 2A$;
$\therefore\ \Sigma(m^2+n^2)\cot 2A=\Sigma 2mn\operatorname{cosec} 2A$;
$\therefore\ \Sigma\{(m+n)^2(\operatorname{cosec} 2A-\cot 2A)\}$
$=\Sigma\{(m-n)^2(\operatorname{cosec} 2A+\cot 2A)\}$;
$\therefore\ \Sigma\{(m+n)^2\tan A\}=\Sigma\{(m-n)^2\cot A\}$.

17. l.h.s. $=\sin B\sin z\sin(B+x-y)-\frac{1}{2}\sin(B-y)\{\cos(A-C+z-x)$
$-\cos(A+C+z-x)-\cos(A-C+z-x)+\cos(A+C-x-z)\}$
$=\sin B\sin z\sin(B+x-y)-\sin(B-y)\sin(A+C-x)\sin z$
$=\frac{1}{2}\sin z\{\cos(x-y)-\cos(2B+x-y)$
$-\cos(A+C-B-x+y)+\cos(\pi-x-y)\}$
$=\frac{1}{2}\sin z\{\cos(x-y)-\cos(x+y)\}=\sin x\sin y\sin z$.

18. l.h.s. $=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
$$=2\sin\frac{A+B}{2}\left(\cos\frac{A-B}{2}+\cos\frac{C-D}{2}\right)$$
$$=4\sin\frac{A+B}{2}\cdot\cos\frac{(A+C)-(B+D)}{4}\cos\frac{(A+D)-(B+C)}{4};$$
use $B+D=2\pi-(A+C)$, etc.

19. l.h.s. $=\frac{1}{2}\{\cos(A+B)\cos(A-B)+\cos(C+D)\cos(C-D)\}$

$$=\cos(A+B)\cos\frac{A-B+C-D}{2}\cos\frac{A-B-C+D}{2}$$

$$=\cos(A+B)\cos(A+C-\pi)\cos(A+D-\pi).$$

Also $4(\cos A\cos B\cos C\cos D-\sin A\sin B\sin C\sin D)$

$$=\{\cos(A-B)+\cos(A+B)\}\{\cos(C-D)+\cos(C+D)\}$$
$$-\{\cos(A-B)-\cos(A+B)\}\{\cos(C-D)-\cos(C+D)\}$$
$$=2\cos(A+B)\cos(C-D)+2\cos(A-B)\cos(C+D)$$
$$=2\cos(C+D)\cos(C-D)+2\cos(A-B)\cos(A+B)$$
$$=\Sigma\cos 2A.$$

20. $\tan(A+B)\{\tan A\tan B-\tan C\tan D\}$

$$=\tan(A+B)\{(1-\tan C\tan D)-(1-\tan A\tan B)\}$$
$$=-\tan(C+D)\,.\,(1-\tan C\tan D)-\tan(A+B)\times(1-\tan A\tan B)$$
$$=-\tan C-\tan D-\tan A-\tan B,$$ and similarly.

21. $\tan\alpha=\dfrac{\sin\beta\cos\gamma+\cos\beta\sin\gamma}{\cos\beta\cos\gamma+\sin\beta\sin\gamma}=\dfrac{\sin(\beta+\gamma)}{\cos(\beta-\gamma)}$,

$$\sin 2\alpha=\frac{2\tan\alpha}{1+\tan^2\alpha}=\frac{2\sin(\beta+\gamma)\cos(\beta-\gamma)}{\cos^2(\beta-\gamma)+\sin^2(\beta+\gamma)}$$
$$=\frac{\sin 2\beta+\sin 2\gamma}{\frac{1}{2}\{1+\cos(2\beta-2\gamma)\}+\frac{1}{2}\{1-\cos(2\beta+2\gamma)\}}.$$

22. $\tan\phi(1-\cos\alpha\sec\theta)=\sin\alpha\tan\theta$;

$\therefore\ \tan^2\phi\cos^2\alpha(1+\tan^2\theta)=(\tan\phi-\sin\alpha\tan\theta)^2$;

$\therefore\ \tan^2\phi\cos^2\alpha+\tan^2\phi\cos^2\alpha\tan^2\theta$

$$=\tan^2\phi-2\sin\alpha\tan\phi\tan\theta+\tan^2\theta(1-\cos^2\alpha);$$

$\therefore\ \tan^2\theta\cos^2\alpha+\tan^2\theta\cos^2\alpha\tan^2\phi$

$$=\tan^2\theta-2\sin\alpha\tan\theta\tan\phi+\tan^2\phi(1-\cos^2\alpha);$$

$\therefore\ \tan^2\theta\cos^2\alpha(1+\tan^2\phi)=(\tan\theta-\sin\alpha\tan\phi)^2$;

$\therefore\ \tan\theta-\sin\alpha\tan\phi=\pm\tan\theta\cos\alpha\sec\phi$;

$\therefore\ \tan\theta(\cos\phi\mp\cos\alpha)=\sin\alpha\sin\phi$.

23. l.h.s. $=\frac{1}{2}\Sigma\{\cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta-\gamma)\}=0$.

If $x+y+z=0$, $x^3+y^3+z^3=3xyz$;

$\therefore$ second part follows from first.

24. $\Sigma\{t_2t_3(t_2-t_3)\}=-(t_2-t_3)(t_3-t_1)(t_1-t_2)$

and $\Sigma\{t_1^2t_2t_3(t_2-t_3)\}=t_1t_2t_3\Sigma\{t_1(t_2-t_3)\}=0$;

$\therefore\ \Sigma(1+t_1^2)t_2t_3(t_2-t_3)=-(t_2-t_3)(t_3-t_1)(t_1-t_2)$.

Put $t_1=\tan\alpha$, etc., and multiply by $\cos^2\alpha\cos^2\beta\cos^2\gamma$.

25. $C+iS\equiv\Sigma\{(\cos 3\alpha+i\sin 3\alpha)\cos(\beta-\gamma)\}$

$$=\text{cis}(\alpha+\beta+\gamma)\,.\,\Sigma\{\text{cis}(2\alpha-\beta-\gamma)\,.\,\cos(\beta-\gamma)\}$$
$$=\text{cis}(\alpha+\beta+\gamma)\,.\,\tfrac{1}{2}\Sigma\{\text{cis}(2\alpha-\beta-\gamma)\,.\,[\text{cis}(\beta-\gamma)+\text{cis}(\gamma-\beta)]\}$$
$$=\text{cis}(\alpha+\beta+\gamma)\,.\,\tfrac{1}{2}\Sigma\{\text{cis}(2\alpha-2\gamma)+\text{cis}(2\alpha-2\beta)\}$$
$$=\text{cis}(\alpha+\beta+\gamma)\,.\,\Sigma\cos(2\beta-2\gamma);$$

$\therefore\ S=\sin(\alpha+\beta+\gamma)\,.\,\Sigma\cos 2(\beta-\gamma)$

$$=\sin(\alpha+\beta+\gamma)\,.\,\{2\cos(\beta-\alpha)\cos(\beta+\alpha-2\gamma)+2\cos^2(\alpha-\beta)-1\}=\text{r.h.s.}$$

26. $D+i\,.\,N\equiv\Sigma\{(\cos 3\alpha+i\sin 3\alpha)\sin(\beta-\gamma)\}$

$=$, as in No. 25, $\text{cis}(\alpha+\beta+\gamma)\,.\,\Sigma\sin(2\gamma-2\beta)$;

$$\therefore\ \frac{N}{D}=\tan(\alpha+\beta+\gamma).$$

27. $t_1=\tan\alpha$, etc.

$$\Sigma\{t_2t_3(t_2+t_3)(1+t_1^2)\}-(t_2+t_3)(t_3+t_1)(t_1+t_2)$$
$$=\Sigma(t_2^2t_3)+t_1t_2t_3\Sigma\{t_1(t_2+t_3)\}-\Sigma(t_2^2t_3)-2t_1t_2t_3$$
$$=2t_1t_2t_3(t_2t_3+t_3t_1+t_1t_2-1);$$

multiply by $\cos^2\alpha\cos^2\beta\cos^2\gamma$, then

$$\Sigma\sin\beta\sin\gamma\sin(\beta+\gamma)-\prod\sin(\beta+\gamma)$$
$$=2\sin\alpha\sin\beta\sin\gamma(\cos\alpha\sin\beta\sin\gamma+\cos\beta\sin\gamma\sin\alpha+\cos\gamma\sin\alpha\sin\beta-\cos\alpha\cos\beta\cos\gamma)$$
$$=2\sin\alpha\sin\beta\sin\gamma\{-\cos\alpha\cos(\beta+\gamma)+\sin\alpha\sin(\beta+\gamma)\}.$$

28. r.h.s. $-$ l.h.s.

$$=\Sigma[\sin(\beta-\gamma)\{\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)+\sin 2\alpha\}]$$
$$=\Sigma\{\sin(\beta-\gamma)\sin(\beta+\gamma)\}+\Sigma[\sin(\beta-\gamma)\{\cos\alpha(\sin\gamma+\sin\beta+\sin\alpha)+\sin\alpha(\cos\gamma+\cos\beta+\cos\alpha)\}]$$
$$=\Sigma(\sin^2\beta-\sin^2\gamma)+(\sin\alpha+\sin\beta+\sin\gamma)\times\Sigma\{\cos\alpha\sin(\beta-\gamma)\}+(\cos\alpha+\cos\beta+\cos\gamma)\times\Sigma\{\sin\alpha\sin(\beta-\gamma)\}=0+0+0.$$

29. $2\sin\gamma(\sin\alpha-\sin\beta)=\cos 2\alpha-\cos 2\beta=2\sin^2\beta-2\sin^2\alpha$;

$\therefore$ unless $\sin\alpha=\sin\beta$, $\sin\gamma=-\sin\beta-\sin\alpha$;

$\therefore\ \sin\alpha=-\sin\beta-\sin\gamma$;

$\therefore\ 2\sin\alpha(\sin\beta-\sin\gamma)=2\sin^2\gamma-2\sin^2\beta=\cos 2\beta-\cos 2\gamma$;

$\therefore\ \cos 2\gamma+2\sin\alpha\sin\beta=\cos 2\beta+2\sin\alpha\sin\gamma=0$.

30. $2\sin(\alpha+\theta)\sin(\beta+\phi)=\cos(\alpha-\beta+\theta-\phi)-\cos(\alpha+\beta+\theta+\phi)$

and similarly for $2\sin(\alpha+\phi)\sin(\beta+\theta)$;

$\therefore\ \cos(\alpha-\beta+\theta-\phi)=\cos(\alpha-\beta+\phi-\theta)$;

$\therefore\ \alpha-\beta+\theta-\phi=2n\pi\pm(\alpha-\beta+\phi-\theta)$.

31. Divide by $\sin^3\theta \sin A \sin B \sin C$;

$$\operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$$
$$+(\cot A+\cot\theta)(\cot B+\cot\theta)(\cot C+\cot\theta)=0;$$
$$\therefore\ \cot^3\theta+\cot^2\theta\,.\,\Sigma\cot A+\cot\theta\,.\,\Sigma\cot A\cot B$$
$$+(\cot A\cot B\cot C+\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C)=0,$$

but $\Sigma\cot A\cot B=1$ and the term independent of $\cot\theta$ is $\Sigma\cot A$ by No. 12; $\therefore\ (\cot^2\theta+1)(\cot\theta+\Sigma\cot A)=0$.

32. $4(\text{l.h.s.})=\{\cos a-\cos(b+c)\}\{\cos(b-c)-\cos a\}$

$$+\{\cos a+\cos(b+c)\}\{\cos(b-c)+\cos a\}$$
$$=2\cos a\{\cos(b-c)+\cos(b+c)\}=4\cos a\cos b\cos c.$$

EXERCISE XIV. c. (p. 268.)

1. $\tan^2\frac{1}{2}A=\dfrac{1-\cos A}{1+\cos A}$

$$=\frac{\sin b\sin c-\cos a+\cos b\cos c}{\sin b\sin c+\cos a-\cos b\cos c}=\frac{\cos(b-c)-\cos a}{\cos a-\cos(b+c)};$$

express as products.

2. $\tan(B+C-A)+\tan(C+A-B)=\dfrac{\sin 2C}{\cos(B+C-A)\cos(C+A-B)}$,

and $\tan(A+B+C)-\tan(A+B-C)$

$$=\frac{\sin 2C}{\cos(A+B+C)\cos(A+B-C)};$$

$\therefore\ 2C=n\pi$ or $\cos(B+C-A)\cos(C+A-B)$

$$=\cos(A+B+C)\cos(A+B-C);$$

$\therefore\ \cos 2C+\cos 2(A-B)=\cos 2(A+B)+\cos 2C$;

$\therefore\ \sin 2A\sin 2B=0$.

3. $\text{r.h.s.}=2\sin\dfrac{y+z}{2}\left(\cos\dfrac{y-z}{2}-\cos\dfrac{2x+y+z}{2}\right)$

$$=\sin y+\sin z+\sin x-\sin(x+y+z);$$

$\therefore\ \sin(x+y+z)=0$.

4. $\cos 2A+\cos 2B+\cos 2C+\cos(2A+2B+2C)$

$$=2\cos(A+B)\cos(A-B)+2\cos(A+B+2C)\cos(A+B)$$
$$=4\cos(A+B)\cos(A+C)\cos(B+C),\text{ and this}=0.$$

5. $\text{l.h.s.}=\Sigma(2\cos^2 A-1)+1+4\cos A\cos B\cos C$

$$=2(\cos^2 A+\cos^2 B+\cos^2 C-1+2\cos A\cos B\cos C)$$
$$=2\{(\cos A+\cos B\cos C)^2-\cos^2 B\cos^2 C+\cos^2 B+\cos^2 C-1\}$$
$$=2\{(\cos A+\cos B\cos C)^2-\sin^2 B\sin^2 C\}$$
$$=2\{\cos A+\cos(B-C)\}\{\cos A+\cos(B+C)\}$$
$$=8\cos\tfrac{1}{2}(A+B-C)\cos\tfrac{1}{2}(A-B+C)\cos\tfrac{1}{2}(A+B+C)\times$$
$$\cos\tfrac{1}{2}(A-B-C),\text{ and this}=0.$$

6. $3+2\Sigma\cos(\beta-\gamma)=1$;

$$\therefore\ \cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)+1=0,$$
$$2\cos\frac{\beta-\alpha}{2}\cos\frac{\alpha+\beta-2\gamma}{2}+2\cos^2\frac{\beta-\alpha}{2}=0;$$
$$\therefore\ 4\cos\frac{\beta-\alpha}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\alpha-\gamma}{2}=0.$$

7. $0=\sin(\alpha+\beta+\gamma)\,.\,\Sigma(\cos\alpha)-\cos(\alpha+\beta+\gamma)\,.\,\Sigma(\sin\alpha)$

$$=\Sigma\{\sin(\alpha+\beta+\gamma)\cos\alpha-\cos(\alpha+\beta+\gamma)\sin\alpha\}=\Sigma\sin(\beta+\gamma).$$

Each fraction $=\dfrac{\cos(\alpha+\beta+\gamma)\Sigma(\cos\alpha)+\sin(\alpha+\beta+\gamma)\Sigma(\sin\alpha)}{\cos^2(\alpha+\beta+\gamma)+\sin^2(\alpha+\beta+\gamma)}$

$$=\Sigma\{\cos(\alpha+\beta+\gamma)\cos\alpha+\sin(\alpha+\beta+\gamma)\sin\alpha\}=\Sigma\cos(\beta+\gamma).$$

8. $\sin\theta\cos\dfrac{\phi+\psi}{2}\sin\dfrac{\theta+\psi}{2}=\sin\phi\cos\dfrac{\psi+\theta}{2}\sin\dfrac{\phi+\psi}{2}$;

$$\therefore\ \sin\theta\left\{\sin\left(\frac{\theta+\phi}{2}+\psi\right)+\sin\frac{\theta-\phi}{2}\right\}$$
$$=\sin\phi\left\{\sin\left(\frac{\theta+\phi}{2}+\psi\right)-\sin\frac{\theta-\phi}{2}\right\}$$
$$\therefore\ \sin\left(\frac{\theta+\phi}{2}+\psi\right).\,2\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}$$
$$=-\sin\frac{\theta-\phi}{2}\,.\,(\sin\theta+\sin\phi);$$

$\therefore$ since $\theta-\phi\neq 2n\pi$,

$$2\sin\left(\frac{\theta+\phi}{2}+\psi\right)\cos\frac{\theta+\phi}{2}+\sin\theta+\sin\phi=0$$

or $\sin(\theta+\phi+\psi)+\Sigma\sin\theta=0$;

$$\therefore\ 2\sin\left(\frac{\phi+\psi}{2}+\theta\right)\cos\frac{\phi+\psi}{2}+\sin\phi+\sin\psi=0;$$
$$\therefore\ \sin\left(\frac{\phi+\psi}{2}+\theta\right).\,2\sin\frac{\phi-\psi}{2}\cos\frac{\phi+\psi}{2}$$
$$=-\sin\frac{\phi-\psi}{2}(\sin\phi+\sin\psi);$$
$$\therefore\ \sin\phi\left\{\sin\left(\frac{\phi+\psi}{2}+\theta\right)+\sin\frac{\phi-\psi}{2}\right\}$$
$$=\sin\psi\left\{\sin\left(\frac{\phi+\psi}{2}+\theta\right)-\sin\frac{\phi-\psi}{2}\right\};$$
$$\therefore\ \sin\phi\cos\frac{\psi+\theta}{2}\sin\frac{\phi+\theta}{2}=\sin\psi\cos\frac{\theta+\phi}{2}\sin\frac{\psi+\theta}{2}.$$

9. $\sin(2\theta-\phi-\psi)\cos(2\phi+\theta+\psi)=\sin(2\phi-\theta-\psi)\cos(2\theta+\phi+\psi);$

$\therefore\ \sin(3\theta+\phi)+\sin(\theta-3\phi-2\psi)$
$=\sin(3\phi+\theta)+\sin(\phi-3\theta-2\psi)$

$\therefore\ \{\sin(3\theta+\phi)-\sin(3\phi+\theta)\}$
$+\{\sin(\theta-3\phi-2\psi)-\sin(\phi-3\theta-2\psi)\}=0;$

$\therefore\ 2\sin(\theta-\phi)\cos(2\theta+2\phi)$
$+2\sin(2\theta-2\phi)\cos(\theta+\phi+2\psi)=0;$

$\therefore$ since $\sin(\theta-\phi)\neq 0$,
$\cos(2\theta+2\phi)+2\cos(\theta-\phi)\cos(\theta+\phi+2\psi)=0;$

$\therefore\ \Sigma\cos 2(\theta+\phi)=0;$

$\therefore\ \cos(2\phi+2\psi)+2\cos(\phi-\psi)\cos(\phi+\psi+2\theta)=0,$

and reversing the argument, we arrive at
$\sin(2\phi-\psi-\theta)\cos(2\psi+\phi+\theta)$
$=\sin(2\psi-\phi-\theta)\cos(2\phi+\psi+\theta).$

10. $\dfrac{\sin 2x-\sin 2y}{\sin 2x+\sin 2y}=\dfrac{\tan\beta-\tan\alpha}{\tan\beta+\tan\alpha}=\dfrac{\sin(\beta-\alpha)}{\sin(\beta+\alpha)}=-\dfrac{\sin(x-y)}{\sin(\alpha+\beta)};$

$\therefore\ \sin(x-y)\,.\,(\sin 2x+\sin 2y)$
$=-\sin(\alpha+\beta)\,.\,2\cos(x+y)\sin(x-y);$

but $\sin(x-y)\neq 0;$

$\therefore\ \sin 2x+\sin 2y=-2\sin(\alpha+\beta)\,.\,\cos(x+y);$

$\therefore\ -2\sin(\alpha+\beta)\cos(x+y)=k\tan\alpha+k\tan\beta$
$=k\sec\alpha\sec\beta\,.\,\sin(\alpha+\beta);$

$\therefore$ since $\sin(\alpha+\beta)\neq 0$, $k=-2\cos\alpha\cos\beta\cos(x+y);$

also $\sin(\alpha+\beta)\cos(x+y)=-\frac{1}{2}(\sin 2x+\sin 2y)$
$=-\sin(x+y)\cos(x-y)=-\sin(x+y)\cos(\alpha-\beta)$
$=\sin(x+y)\{\cos(\alpha+\beta)-2\cos\alpha\cos\beta\};$

$\therefore\ \sin(\alpha+\beta)\cos(x+y)-\cos(\alpha+\beta)\sin(x+y)$
$=-2\cos\alpha\cos\beta\sin(x+y)=k\tan(x+y);$

hence result.

11. $\cos\alpha+\cos\alpha\sin\beta\sin\gamma=\cos\beta\cos\gamma;$

$\therefore\ (\cos\beta\cos\gamma-\cos\alpha)^2=\cos^2\alpha\sin^2\gamma(1-\cos^2\beta);$

$\therefore\ \cos^2\beta(\cos^2\gamma+\cos^2\alpha\sin^2\gamma)$
$-2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha(1-\sin^2\gamma)=0;$

but $\cos^2\gamma+\cos^2\alpha\sin^2\gamma=1-\sin^2\gamma+(1-\sin^2\alpha)\sin^2\gamma$
$=1-\sin^2\alpha\sin^2\gamma;$

$\therefore\ (\cos\beta-\cos\alpha\cos\gamma)^2=\cos^2\beta\sin^2\alpha\sin^2\gamma;$

$\therefore\ \cos\beta-\cos\alpha\cos\gamma=\pm\cos\beta\sin\alpha\sin\gamma.$

12. $1=\cos^2\alpha+\sin^2\alpha=(\cos\beta+\cos\gamma)^2+(\sin\beta+\sin\gamma)^2$
$=2+2\cos(\beta-\gamma);$

$\therefore\ \cos(\beta-\gamma)=-\frac{1}{2}$ and similar equations.

$$\Sigma\cos 2\alpha=\cos 2\theta+2\cos 2\theta\cos\frac{2\pi}{3}=0.$$

$$\Sigma\sin(\beta+\gamma)=\sin\left(2\theta+\frac{2\pi}{3}\right)+\sin 2\theta+\sin\left(2\theta-\frac{2\pi}{3}\right)$$
$$=\sin 2\theta+2\sin 2\theta\cos\frac{2\pi}{3}=0.$$

$$\Sigma(\cos 3\alpha)=\cos(3\theta-2\pi)+\cos 3\theta+\cos(3\theta+2\pi)$$
$$=3\cos 3\theta=3\cos(\alpha+\beta+\gamma),$$

and similarly for other results.

13. $\Sigma(\text{cis}\,\alpha)=\Sigma(\cos\alpha)+i\Sigma(\sin\alpha)=0;$

$\therefore\ \Sigma(\text{cis}^3\alpha)=3\,\text{cis}\,\alpha\,\text{cis}\,\beta\,\text{cis}\,\gamma=3\,\text{cis}(\alpha+\beta+\gamma);$

$\therefore\ \Sigma(\cos 3\alpha+i\sin 3\alpha)=3\{\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)\}.$

14. Use No. 12;

$$\Sigma\cos(2\alpha+\beta+\gamma)=\cos\left(4\theta-\frac{2\pi}{3}\right)+\cos 4\theta+\cos\left(4\theta+\frac{2\pi}{3}\right)$$
$$=\cos 4\theta+2\cos 4\theta\cos\frac{2\pi}{3}=0,\text{ and similarly.}$$

15. $\dfrac{\cos(B+C)}{\cos A}-\dfrac{\cos(A+B)}{\cos C}$
$$=\frac{\frac{1}{2}\{\cos B+\cos(B+2C)\}-\frac{1}{2}\{\cos B+\cos(B+2A)\}}{\cos A\cos C}$$
$$=\frac{\sin(A+B+C)\sin(A-C)}{\cos A\cos C}=\sin(A+B+C)(\tan A-\tan C)$$

and similarly
$$\frac{\cos(A+B)}{\cos C}-\frac{\cos(C+A)}{\cos B}=\sin(A+B+C)(\tan C-\tan B);$$

$\therefore\ \sin(A+B+C)=0$ or $\tan A-\tan C=\tan C-\tan B.$

16. $(\sin x-\sin y\sin z)^2=\sin^2 y\sin^2 z+1-\sin^2 y-\sin^2 z$
$=(1-\sin^2 y)(1-\sin^2 z)=\cos^2 y\cos^2 z;$

$\therefore\ \sin x=\cos(y-z)$ or $\sin x=-\cos(y+z);$

$\therefore\ \dfrac{\pi}{2}-x=2n\pi\pm(y-z)$ or $2n\pi\pm(\pi-y-z).$

17. The centroid is $\dfrac{a}{3}\Sigma(\cos\alpha)$, $\dfrac{b}{3}\Sigma(\sin\alpha)$.

18. By No. 17 and No. 12, α, β, γ are of the form $\theta - \frac{2\pi}{3}, \theta, \theta + \frac{2\pi}{3}$; then $\alpha + \gamma = 2\beta$ proves that the chord '$\alpha\gamma$' is parallel to the tangent at 'β.'

EXERCISE XIV. d. (p. 270.)

1. $\operatorname{Sec}\theta = \tan\beta \cot\alpha$, $\tan\theta = \tan\gamma \operatorname{cosec}\alpha$, substitute in
$$\sec^2\theta = 1 + \tan^2\theta.$$

2. $c = a\sin\theta\cos\alpha - a\cos\theta\sin\alpha$; $\therefore$ $a\cos\theta\sin\alpha = b\cos\alpha - c$, but $a\sin\theta\sin\alpha = b\sin\alpha$, square and add.

3. $a\cos^2\theta + b\sin^2\theta = c(\cos^2\theta + \sin^2\theta)$; $\therefore$ $\frac{\sin^2\theta}{c-a} = \frac{\cos^2\theta}{b-c}$;
$$\therefore (b-c)\tan^2\theta = c - a \text{ and } (c-a)\cot^2\theta = b - c.$$

4. $a\sin\theta = 2b\sin\theta\cos\theta$; $\therefore$ $\sin\theta = 0$ or $\cos\theta = \frac{a}{2b}$, substitute in $c\cos\theta = d(2\cos^2\theta - 1)$; then $\pm c = d$ or $\frac{ac}{2b} = d\left(\frac{a^2}{2b^2} - 1\right)$.

5. $a^2 + b^2 = 2 + 2\cos\theta$, substitute for $2\cos\theta$ in
$$2b = 2\cos\theta + 2\cos 2\theta \equiv (2\cos\theta - 1)(2\cos\theta + 2).$$

6. $a^{\frac{2}{3}} + b^{\frac{2}{3}} = 1$; $\therefore$ $a^2 + b^2 + 3a^{\frac{2}{3}}b^{\frac{2}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}}) = 1$;
$$\therefore a^2 + b^2 - 1 = -3a^{\frac{2}{3}}b^{\frac{2}{3}}; \text{ cube.}$$

7. $2a - 3b = -5\cos\theta$, $3a - 2b = 5\sin\theta$, square and add.

8. $a + b = 7\cos\theta$, $4a - 3b = 7\cot\theta$, substitute in
$$\cot^2\theta(1 - \cos^2\theta) = \cos^2\theta.$$

9. $x\cos\theta = 1 - \cos^2\theta = \sin^2\theta$, $y\sin\theta = 1 - \sin^2\theta = \cos^2\theta$;
$$\therefore x^2 + y^2 = \frac{\sin^4\theta}{\cos^2\theta} + \frac{\cos^4\theta}{\sin^2\theta} = \frac{\sin^6\theta + \cos^6\theta}{\cos^2\theta\sin^2\theta}$$
$$= \frac{(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)}{\cos^2\theta\sin^2\theta}$$
$$= \frac{1}{\cos^2\theta\sin^2\theta} - 3 = \frac{1}{x^2y^2} - 3.$$

10. $\frac{x}{a} = \sin\theta(2 + 2\cos 2\theta - 1) = \sin 3\theta$,
$$\frac{y}{b} = \cos\theta(2 - 2\cos 2\theta - 1) = -\cos 3\theta.$$

11. $3 = a\sin\theta + b\cos\theta$; $\therefore$ $(3 - a\sin\theta)^2 = b^2(1 - \sin^2\theta)$;
$$\therefore (a^2 + b^2)\sin^2\theta - 6a\sin\theta + (9 - b^2) = 0;$$
also $\sin^2\theta - a\sin\theta + 1 = 0$;
$$\therefore \sin^2\theta : +\sin\theta : 1$$
$$= a(b^2 - 3) : a^2 + 2b^2 - 9 : a(a^2 + b^2 - 6).$$

12. $x : y : a =$
$$-2\sin\theta\cos 2\theta + \sin 2\theta\cos\theta : 2\cos\theta\cos 2\theta + \sin\theta\sin 2\theta : 1$$
$$= \sin\theta(1 - \cos 2\theta) : \cos\theta(1 + \cos 2\theta) : 1$$
$$= 2\sin^3\theta : 2\cos^3\theta : 1;$$
$\therefore$ $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (2a)^{\frac{2}{3}}$, and see No. 6.

13. Multiply by $\cos\theta$, $\sin\theta$; add: $x = 2a\cot\theta$. Multiply by $\sin\theta$, $\cos\theta$; subtract: $y = a(1 - \cot^2\theta) = (4a^2 - x^2)/4a$.

14. Eqns. represent tangents to $x^2 + y^2 = c^2$ inclined at angle α; locus of point of intersection of such tangents is a concentric circle of radius $c\sec\frac{1}{2}\alpha$.

15. $a : b : c = \cos^2 2\theta - \cos\theta\cos 3\theta : \cos 3\theta - 2\cos\theta\cos 2\theta : 2\cos^2\theta - \cos 2\theta = \sin^2\theta : -\cos\theta : 1$.

16. Solve for ax, by. $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (c^2)^{\frac{2}{3}}$ and use the method of No. 6.

17. Put $x = \alpha\xi$, $y = \beta\eta$, $a = \alpha^2$, $b = \beta^2$. Equations become
$$\frac{\xi}{\alpha}\cos\theta + \frac{\eta}{\beta}\sin\theta = 1,\ \alpha\xi\sin\theta - \beta\eta\cos\theta$$
$$= \sqrt{(\alpha^4\sin^2\theta + \beta^4\cos^2\theta)}$$
and represent perp. tangents to $\frac{\xi^2}{\alpha^2} + \frac{\eta^2}{\beta^2} = 1$. Thus
$$\xi^2 + \eta^2 = \alpha^2 + \beta^2.$$

18. $cd = a^2 + b^2 + ab(\cot\theta + \tan\theta) = a^2 + b^2 + \frac{ab}{\sin\theta\cos\theta}$;
$$\therefore \sin\theta\cos\theta = \frac{ab}{cd - a^2 - b^2};$$
$\therefore$ from second equation,
$$a\cos\theta + b\sin\theta = d\sin\theta\cos\theta = \frac{abd}{cd - a^2 - b^2};$$
square this and square first equation and add,
$$a^2 + b^2 + 4ab \cdot \frac{ab}{cd - a^2 - b^2} = c^2 + \frac{a^2b^2d^2}{(cd - a^2 - b^2)^2};$$
$$\therefore (a^2 + b^2 - c^2)(a^2 + b^2 - cd)^2 = a^2b^2\{d^2 - 4(cd - a^2 - b^2)\}.$$

19. $\frac{c}{d} = \tan\theta\tan 2\theta = \frac{2\tan^2\theta}{1 - \tan^2\theta}$; $\therefore$ $\tan^2\theta = \frac{c}{c + 2d}$;
$$c = \tan\theta + \frac{2\tan\theta}{1 - \tan^2\theta} = \tan\theta\,\frac{3 - \tan^2\theta}{1 - \tan^2\theta};$$
$$\therefore c^2 = \frac{c}{c + 2d}\left\{\frac{3(c + 2d) - c}{(c + 2d) - c}\right\}^2.$$

20. $\dfrac{\sin(\theta-\beta)}{\sin(\theta+\beta)}=\dfrac{1-\tan a}{1+\tan a}$; $\therefore$ $\dfrac{\cos\theta\sin\beta}{\sin\theta\cos\beta}=\dfrac{\tan a}{1}$;

$$\therefore\ \tan\beta\cot a=\tan\theta=\tan[a-(a-\theta)]$$
$$=\frac{\tan a-\tan(a-\theta)}{1+\tan a\tan(a-\theta)}=\frac{\tan a(1+c^2\cos 2a)-c^2\sin 2a}{1+c^2\cos 2a+\tan a\,.\,c^2\sin 2a}$$
$$=\frac{\tan a-c^2(\sin 2a-\tan a\cos 2a)}{1+c^2(\cos 2a+\tan a\sin 2a)}$$
$$=\frac{\sin a-c^2\sin(2a-a)}{\cos a+c^2\cos(2a-a)}=\frac{\sin a\,.\,(1-c^2)}{\cos a\,.\,(1+c^2)};$$
$$\therefore\ (1+c^2)\tan\beta=(1-c^2)\tan^2 a.$$

21. $\dfrac{a}{b}=\dfrac{\tan(\theta+a)+\tan(\theta-a)}{\tan(\theta+a)-\tan(\theta-a)}=\dfrac{\sin 2\theta}{\sin 2a}$. Take $\triangle ABC$ in which $\angle A=2\theta$, $\angle B=2a$ and opp. sides a, b; then third side $=c$, because $c=a\cos B+b\cos A$; $\therefore$ $b^2=a^2+c^2-2ac\cos B$.

22. $x=\dfrac{\cos(a-3\theta)\cos\theta-\sin(a-3\theta)\sin\theta}{\cos^4\theta-\sin^4\theta}$
$$=\frac{\cos(a-2\theta)}{\cos 2\theta}=\cos a+\sin a\tan 2\theta;$$
$\therefore$ $x-\cos a=\sin a\tan 2\theta$; also
$$x=\frac{\sin(a-3\theta)\cos\theta+\cos(a-3\theta)\sin\theta}{\sin^3\theta\cos\theta+\cos^3\theta\sin\theta}$$
$$=\frac{\sin(a-2\theta)}{\frac{1}{2}\sin 2\theta}=2\sin a\cot 2\theta-2\cos a;$$
$\therefore$ $x+2\cos a=2\sin a\cot 2\theta$;
$\therefore$ $(x-\cos a)(x+2\cos a)=2\sin^2 a$.

23. $\{1+e\cos(\theta+a)\}^2-(\cos\theta+e\cos a)^2$
$$=1-\cos^2\theta+2e\{\cos(\theta+a)-\cos\theta\cos a\}$$
$$-e^2\{\cos^2 a-\cos^2(a+\theta)\}$$
$$=\sin^2\theta-2e\sin a\sin\theta-e^2\{\sin^2\theta+2\sin a\sin\theta\cos(a+\theta)\}$$
by Ex. XIV. a, No. 29,
$$=(1-e^2)\sin^2\theta-2e\sin a\sin\theta-2e^2\sin a\sin\theta\cos(a+\theta);$$
but $b^2=(1-e^2)a^2$;
$$\therefore\ \{1+e\cos(\theta+a)\}^2=(1-e^2)\sin^2\theta;$$
$$\therefore\ (\cos\theta+e\cos a)^2=2e\sin a\sin\theta\{1+e\cos(a+\theta)\};$$
$$\therefore\ x^2=2e\sin a\,.\,ab.$$

24. $\theta=\sin y$, $\tan x=\dfrac{(\theta+a)+(\theta-a)}{1-(\theta+a)(\theta-a)}=\dfrac{2\theta}{1-\theta^2+a^2}$.

EXERCISE XIV. e. (p. 272.)

1. $(2a)^2+(2b)^2=2-2\cos(\theta-\phi)=2(1-\cos 2\gamma)$.

2. $y=a\cos(\theta+\phi)+a\cos(\theta-\phi)=a\cos a+a\cos(\theta-\phi)$
$$=a\cos a+\sqrt{(a^2-x^2)}.$$

3. $a^2+b^2=2+2\cos(\theta-\phi)$, $c=\sin(\theta-\phi)$.

4. $x^2+y^2=2+2\cos(\theta-\phi)=4\cos^2\dfrac{\theta-\phi}{2}$
$$=2y\frac{\cos\frac{\theta-\phi}{2}}{\cos\frac{\theta+\phi}{2}}=2y\frac{1+\tan\frac{\theta}{2}\tan\frac{\phi}{2}}{1-\tan\frac{\theta}{2}\tan\frac{\phi}{2}}.$$

5. $s=2\sin\dfrac{\theta+\phi}{2}\cos\dfrac{\theta-\phi}{2}$,
$$t\cos\theta\cos\phi=\sin(\theta+\phi)=2\sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi}{2},$$
$$k\cos\theta\cos\phi=\cos\theta+\cos\phi=2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2};$$
$$\therefore\ \frac{st}{k}=2\sin^2\frac{\theta+\phi}{2}=1-\cos(\theta+\phi),$$
$$\frac{sk}{t}=2\cos^2\frac{\theta-\phi}{2}=1+\cos(\theta-\phi)$$
$$4\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=k\{\cos(\theta+\phi)+\cos(\theta-\phi)\}$$
$$=k\left(\frac{sk}{t}-\frac{st}{k}\right);$$
$$\therefore\ 16\left(1-\frac{st}{2k}\right)\frac{sk}{2t}=k^2\left(\frac{sk}{t}-\frac{st}{k}\right)^2.$$

6. $2a^2-b-2=2(\cos\theta+\cos\phi)^2-2\cos^2\theta-2\cos^2\phi=4\cos\theta\cos\phi$,
$$c=4(\cos^3\theta+\cos^3\phi)-3(\cos\theta+\cos\phi)$$
$$=4a(a^2-3\cos\theta\cos\phi)-3a,$$
and equate values of $4\cos\theta\cos\phi$.

7. $\tan\theta$, $\tan\phi$, are roots of $x\lambda^2-y\lambda+a=0$; $\therefore$ $\tan\theta\tan\phi=\dfrac{a}{x}$; $\therefore$ $\dfrac{a}{x}=-1$. **Or**, Eqns. represent two tangents to $y^2=4ax$ and their condition of perpendicularity.

8. Eqns. represent two tangents to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and their condition of perpendicularity.

9. $\dfrac{1-\cos\gamma}{1+\cos\gamma}=\tan^2\dfrac{\gamma}{2}=\tan^2\dfrac{\theta}{2}\tan^2\dfrac{\phi}{2}=\dfrac{(1-\cos\theta)(1-\cos\phi)}{(1+\cos\theta)(1+\cos\phi)}$;

$\therefore\ \cos\gamma=\dfrac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}=\dfrac{\cos\gamma\,(\cos\alpha+\cos\beta)}{1+\cos\alpha\cos\beta\cos^2\gamma}$;

$\therefore\ 1+\cos\alpha\cos\beta\cos^2\gamma=\cos\alpha+\cos\beta$;

$\therefore\ \sin^2\gamma=\dfrac{1+\cos\alpha\cos\beta-\cos\alpha-\cos\beta}{\cos\alpha\cos\beta}$

$=\dfrac{(1-\cos\alpha)(1-\cos\beta)}{\cos\alpha\cos\beta}=(\sec\alpha-1)(\sec\beta-1).$

10. $(x\cos\theta-2a)^2=y^2\sin^2\theta$;

$\therefore\ \left(2x\cos^2\dfrac{\theta}{2}-x-2a\right)^2=4y^2\cos^2\dfrac{\theta}{2}\left(1-\cos^2\dfrac{\theta}{2}\right)$;

hence $\cos^2\dfrac{\theta}{2}$, $\cos^2\dfrac{\phi}{2}$ are roots of

$$(2x\mathrm{T}-x-2a)^2=4y^2\mathrm{T}(1-\mathrm{T});$$

$$\tfrac{1}{4}=\text{prod. of roots}=\frac{(x+2a)^2}{4(x^2+y^2)}.$$

11. $2q=a(1+\cos 2\theta)+b(1-\cos 2\theta)$ gives $\cos 2\theta$;

$2r=(a+b-q)(1+\cos 2\phi)+c(1-\cos 2\phi)$

gives $\cos 2\phi$; substitute into

$$(1-\cos^2 2\theta)(1+\cos 2\phi)=\frac{8p^2}{(a-b)^2}.$$

12. If each $=k$, $a\sin^2\theta+b\sin^2\phi=bk(1-\sin^2\theta)+ck(1-\sin^2\phi)$, or $(a+kb)\sin^2\theta+(b+kc)\sin^2\phi=(b+c)k$, and similar equations,

$$\therefore\ \begin{vmatrix} a+kb, & b+kc, & b+c \\ b+kc, & c+ka, & c+a \\ c+ka, & a+kb, & a+b \end{vmatrix}=0.$$

which reduces to $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0.$

13. In a $\triangle\mathrm{ABC}$, $c^2=a^2+b^2-2ab\cos\mathrm{C}$, $\cos\mathrm{C}=-d$.

14. $(\cos\theta-\cos\phi\cos\psi)^2=\{\cos(\phi+\psi)-\cos\phi\cos\psi\}^2$

$=\sin^2\phi\sin^2\psi=(1-\cos^2\phi)(1-\cos^2\psi).$

Substitute for $\cos\theta$, $\cos\phi$, $\cos\psi$.

15. $p\equiv a+bi=\exp\theta i+\exp\phi i+\exp\psi i=x+y+z$, say.

$q\equiv a-bi=\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}$, $r\equiv c+di=x^2+y^2+z^2$,

$s\equiv c-di=\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}.$

Then $(p^2-r)(q^2-s)=4\Sigma yz\Sigma\left(\dfrac{1}{yz}\right)=4pq.$ Thus

$(a^2+b^2)^2-4(a^2+b^2)+(c^2+d^2)$

$=p^2s+q^2r=2\{(a^2-b^2)c+2abd\}.$

16. Put $l_1=\cos\alpha_1$, $m_1=\cos\beta_1$, $n_1=\cos\theta_1$, etc., then from

$l_1l_2+m_1m_2+n_1n_2=0=l_1l_3+m_1m_3+n_1n_3$

$$\frac{l_1}{\begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix}}=\frac{m_1}{\begin{vmatrix} n_2 & l_2 \\ n_3 & l_3 \end{vmatrix}}=\frac{n_1}{\begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix}}\quad\therefore\quad=\frac{\pm\sqrt{(l_1^2+m_1^2+n_1^2)}}{\sqrt{\Sigma\{(m_2n_3-m_3n_2)^2\}}}$$

$$=\frac{\pm 1}{\sqrt{\{(l_2^2+m_2^2+n_2^2)(l_3^2+m_3^2+n_3^2)-(l_2l_3+m_2m_3+n_2n_3)^2\}}}$$

$=\pm 1.$ Similarly for $l_2:m_2:n_2$, $l_3:m_3:n_3$. Thus

$$n_1^2+n_2^2+n_3^2=\pm\Sigma n_1\begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix}=\pm\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$=\pm\Sigma\left\{l_1\begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix}\right\}=l_1^2+m_1^2+n_1^2=1.$$

17. $-\dfrac{l}{m}=\dfrac{\cos(\beta-\gamma)-\cos(\gamma-\alpha)}{\cos\alpha-\cos\beta}=\dfrac{2\sin\dfrac{\beta-\alpha}{2}\sin\left(\gamma-\dfrac{\alpha+\beta}{2}\right)}{2\sin\dfrac{\beta-\alpha}{2}\sin\dfrac{\alpha+\beta}{2}}$

$=\dfrac{\sin\left(\gamma-\dfrac{\alpha+\beta}{2}\right)}{\sin\dfrac{\alpha+\beta}{2}}$, similarly $=\dfrac{\sin\left(\alpha-\dfrac{\beta+\gamma}{2}\right)}{\sin\dfrac{\beta+\gamma}{2}}$;

$\therefore\ \cos\left(\dfrac{\gamma-\alpha}{2}-\beta\right)-\cos\dfrac{3\gamma-\alpha}{2}=2\sin\left(\gamma-\dfrac{\alpha+\beta}{2}\right)\sin\dfrac{\beta+\gamma}{2}$

$=2\sin\left(\alpha-\dfrac{\beta+\gamma}{2}\right)\sin\dfrac{\alpha+\beta}{2}=\cos\left(\dfrac{\alpha-\gamma}{2}-\beta\right)-\cos\dfrac{3\alpha-\gamma}{2}$;

$\therefore\ \sin\beta\sin\dfrac{\gamma-\alpha}{2}=\sin\dfrac{\gamma+\alpha}{2}\sin(\alpha-\gamma)$

$=-2\sin\dfrac{\gamma+\alpha}{2}\sin\dfrac{\gamma-\alpha}{2}\cos\dfrac{\gamma-\alpha}{2}$

$=-\sin\dfrac{\gamma-\alpha}{2}(\sin\gamma+\sin\alpha)$;

$\therefore\ \sin\alpha+\sin\beta+\sin\gamma=0$; also

$\cos\alpha+\cos\beta+\cos\gamma=2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}+\cos\gamma$

$=2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}\cot\dfrac{\alpha+\beta}{2}+\cos\gamma$

$$=(\sin\alpha+\sin\beta)\cot\frac{\alpha+\beta}{2}+\cos\gamma$$

$$=-\sin\gamma\cot\frac{\alpha+\beta}{2}+\cos\gamma$$

$$=-\frac{1}{\sin\frac{\alpha+\beta}{2}}\left(\sin\gamma\cos\frac{\alpha+\beta}{2}-\cos\gamma\sin\frac{\alpha+\beta}{2}\right)$$

$$=-\frac{\sin\left(\gamma-\frac{\alpha+\beta}{2}\right)}{\sin\frac{\alpha+\beta}{2}}=\frac{l}{m}.$$

18. $2\sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi}{2}\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)$

$$=2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\cos\psi+2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\sin\psi$$
$$=(\sin\theta+\sin\phi)\cos\psi+(\cos\theta+\cos\phi)\sin\psi$$
$$=\sin(\theta+\psi)+\sin(\phi+\psi)=-\sin(\theta+\phi)$$
$$=-2\sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi}{2};\quad\therefore\ \frac{1}{\lambda}+\frac{1}{\mu}=-1.$$

19. $\begin{vmatrix}-1 & \cos\gamma & \cos\beta\\ \cos\gamma & -1 & \cos\alpha\\ \cos\beta & \cos\alpha & -1\end{vmatrix}=0;$

$\therefore\ -1+2\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha+\cos^2\beta+\cos^2\gamma=0;$

$\therefore\ (\cos\alpha+\cos\beta\cos\gamma)^2=\sin^2\beta\sin^2\gamma;\ \therefore\ \cos\alpha=-\cos(\beta\pm\gamma).$

20. m, m' are roots of $bt^2+2ht+a=0$, $m+m'=-\frac{2h}{b}$, $mm'=\frac{a}{b}$;

also $\tan^{-1}\frac{m+m'}{1-mm'}=2\tan^{-1}\frac{y}{x}=\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$;

$$\therefore\ \frac{-2h}{b-a}=\frac{2xy}{x^2-y^2}.$$

EXERCISE XIV. f. (p. 274.)

1. $\tan x>0$, expn. $=\{\sqrt{(4\tan x)}-\sqrt{(3\cot x)}\}^2+2\sqrt{12}$; $\tan x<0$, expn. $=-\{\sqrt{(-4\tan x)}-\sqrt{(-3\cot x)}\}^2-2\sqrt{12}$.

2. $(\sin x-\frac{1}{2})^2+\frac{3}{4}\geqslant\frac{3}{4}$, and is $\leqslant(-1-\frac{1}{2})^2+\frac{3}{4}$.

3. $=(\sin x-2)^2+1$, and $(\sin x-2)$ varies from -3 to -1.

4. $(5\sec\theta-3\tan\theta)^2-(3\sec\theta-5\tan\theta)^2$

$$=(5^2-3^2)(\sec^2\theta-\tan^2\theta)=16;$$

$\therefore$ square of expn. $\geqslant 16$; $\therefore$ expn. is either $\geqslant +4$ or $\leqslant -4$.

Or, put $y=5\sec\theta-3\tan\theta$, $\frac{dy}{d\theta}=\frac{5\sin\theta-3}{\cos^2\theta}$; y is stationary for $\sin\theta=\frac{3}{5}$. For $0<\theta<\frac{\pi}{2}$, $\frac{dy}{d\theta}$ changes from $-$ to $+$; $\therefore\ \sin\theta=\frac{3}{5}$, $\cos\theta=\frac{4}{5}$ gives a minimum; similarly $\sin\theta=\frac{3}{5}$, $\cos\theta=-\frac{4}{5}$ gives a maximum.

5. $=5(1-\cos2\theta)+\frac{15}{2}\sin2\theta+9(1+\cos2\theta)$

$$=14+4\cos2\theta+\tfrac{15}{2}\sin2\theta=14+\tfrac{17}{2}\sin(\alpha+2\theta),$$

where $\sin\alpha=\frac{8}{17}$, $\cos\alpha=\frac{15}{17}$.

6. $y=\tan3x\cot x=\frac{3-t^2}{1-3t^2}$ where $t=\tan x$; $\therefore\ t^2=\frac{y-3}{3y-1}$; since $t^2>0$, y is not between 3 and $\frac{1}{3}$.

7. $(a\operatorname{cosec}\theta-b\cot\theta)^2-(a\cot\theta-b\operatorname{cosec}\theta)^2=a^2-b^2$; $\therefore$ sq. of expn. $\geqslant a^2-b^2$ and the equality holds when $\cos\theta=\frac{b}{a}$. Compare No. 4.

8. $t=\tan x$, expn. $=\frac{3-t^2}{t^2(1-3t^2)}=y$ if $3yt^4-(y+1)t^2+3=0$; this has roots if $(y+1)^2>36y$, $(y-17)^2>288$, i.e. unless y is between $17\pm12\sqrt{2}$. If $y>0$ signs of terms in eqn. for t^2 are $+-+$; $\therefore$ roots are positive. If $y<0$, product of roots is $\frac{1}{y}$; $\therefore$ one root is positive; $\therefore$ at least one value of t^2 is always positive and so gives a possible value for x.

9. If $a\cos^2\theta+2b\sin\theta\cos\theta+c\sin^2\theta=x$ and $\tan\theta=t$, dividing by $\cos^2\theta$, $a+2bt+ct^2=x(1+t^2)$, and stationary values are given by the condition for equal roots.

10. $a\cos\theta+b\cos\phi=a\cos\theta+b\cos(\alpha-\theta)$

$$=(a+b\cos\alpha)\cos\theta+b\sin\alpha\sin\theta\leqslant\sqrt{\{(a+b\cos\alpha)^2+(b\sin\alpha)^2\}}.$$

11. $\tan\theta\tan\phi=\frac{\cos(\theta-\phi)-\cos(\theta+\phi)}{\cos(\theta-\phi)+\cos(\theta+\phi)}=\frac{\cos(\theta-\phi)-\cos\alpha}{\cos(\theta-\phi)+\cos\alpha}$

$$=1-\frac{2\cos\alpha}{\cos(\theta-\phi)+\cos\alpha}\leqslant1-\frac{2\cos\alpha}{1+\cos\alpha}.$$

12. $\sin^2\frac{\theta+\phi}{2}=\sin^2\frac{\theta-\phi}{2}+\sin\theta\sin\phi$; $\therefore$ if two angles are unequal the product can be increased by making them equal without altering their sum (keeping the third angle unchanged); product is greatest when $\theta=\phi=\psi$. **Or,** see Ex. 8.

13. As in No. 12, $\cos^2\frac{\theta+\phi}{2}=\sin^2\frac{\theta-\phi}{2}+\cos\theta\cos\phi$. **Or**, see Ex. 8.

14. $\tan\theta+\tan\phi=\frac{\sin(\theta+\phi)}{\cos\theta\cos\phi}>\frac{\sin(\theta+\phi)}{\cos^2\frac{\theta+\phi}{2}}$ by No. 13, $=2\tan\frac{\theta+\phi}{2}$.

Hence, as in No. 12, $\tan A+\tan B+\tan C$ has least value when $A=B=C$. Then

$$\Sigma\tan^2 A=(\Sigma\tan A)^2-2\Sigma\tan B\tan C=(\Sigma\tan A)^2-2$$

has least value $\left(3\tan\frac{\pi}{6}\right)^2-2$.

15. (i) See No. 14;

(ii) Similar method; or put $A=\frac{\pi}{2}-A'$, etc.

(iii) $\operatorname{cosec}\theta+\operatorname{cosec}\phi\equiv\dfrac{2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}}{\cos^2\frac{\theta-\phi}{2}-\cos^2\frac{\theta+\phi}{2}}$; but

$$\left(1-\cos\frac{\theta-\phi}{2}\right)\left(\cos\frac{\theta-\phi}{2}+\cos^2\frac{\theta+\phi}{2}\right)>0;$$

$$\therefore\ \cos\frac{\theta-\phi}{2}\left(1-\cos^2\frac{\theta+\phi}{2}\right)>\cos^2\frac{\theta-\phi}{2}-\cos^2\frac{\theta+\phi}{2};$$

$$\therefore\ \operatorname{cosec}\theta+\operatorname{cosec}\phi>\frac{2\sin\frac{\theta+\phi}{2}}{1-\cos^2\frac{\theta+\phi}{2}}=2\operatorname{cosec}\frac{\theta+\phi}{2};$$

$\therefore$ by the argument in No. 12 the least value occurs when the three angles are equal.

16. $\cos A+\cos B+\cos C-1=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ which varies from 0 to $4\sin^3 30°$.

17. $\cos 2A+\cos 2B+\cos 2C+1=-4\cos A\cos B\cos C$, and use No. 13 or see No. 18.

18. $OH^2=R^2(1-8\cos A\cos B\cos C)$ and is pos. (or zero only if $\triangle$ is equilateral).

19. $\Sigma 2\cos^2\frac{B-C}{2}=3+\Sigma\cos(B-C)$

$$=2+4\cos\frac{B-C}{2}\cos\frac{C-A}{2}\cos\frac{A-B}{2}>2.$$

20. See No. 15.

21. $OI^2=R^2\left(1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$ and is pos. or zero.

22. Expn. $=(x-y\cos C-z\cos B)^2+(y\sin C-z\sin B)^2$ and similar expressions, and is therefore only zero, if

$$x:y:z=\sin A:\sin B:\sin C.$$

23. The point P with coords. $(\cos\theta, \cos\phi, \cos\psi)$ lies on the plane $ax+by+cz=d$. $OP^2=\cos^2\theta+\cos^2\phi+\cos^2\psi$ and has least value when it is the square of perp. from O to the plane.

24. $2\sin^2\alpha=1-\cos 2\alpha>\cos(2\beta-2\gamma)-\cos 2\alpha$

$$=2\sin(\beta-\gamma+\alpha)\sin(\alpha-\beta+\gamma)$$

and similar results. Multiply and take square root, $\sin\alpha\sin\beta\sin\gamma$ being positive.

EXERCISE XIV. g. (p. 275.)

1. $\sin 16°+\sin 92°+\sin 20°-\sin 52°-\sin 56°$

$$=\sin 16°+\sin 88°+\sin 160°+\sin 232°+\sin 304°$$

$$=\text{, by p. 128, }\frac{\sin 160°\sin 180°}{\sin 36°}=0.$$

2. l.h.s. $=2\cos 36°\sin\theta-2\cos 72°\sin\theta$

$$=4\sin 54°\sin 18°\sin\theta=\sin\theta;$$

Or, $\sin\theta+\sin(72°+\theta)+\sin(36°-\theta)$

$-\sin(36°+\theta)-\sin(72°-\theta)=\sin\theta+\sin(72°+\theta)$

$+\sin(144°+\theta)+\sin(216°+\theta)+\sin(288°+\theta)$

$$=\text{, by p. 128, }\frac{\sin(144°+\theta)\sin 180°}{\sin 36°},\ =0.$$

3. l.h.s. $=\dfrac{2+2\cos^2 2\theta}{1-\cos 4\theta}=\dfrac{2(\cos^2\theta+\sin^2\theta)^2+2(\cos^2\theta-\sin^2\theta)^2}{2\sin^2 2\theta}$

$$=\frac{\cos^4\theta+\sin^4\theta}{2\sin^2\theta\cos^2\theta}=\tfrac{1}{2}(\cot^2\theta+\tan^2\theta).$$

4. $\frac{s}{c}=\tan\frac{1}{2}(\alpha+\beta)$; $\therefore\ \sin(\alpha+\beta)=\dfrac{2\cdot\frac{s}{c}}{1+\frac{s^2}{c^2}}$, $\cos(\alpha+\beta)=\dfrac{1-\frac{s^2}{c^2}}{1+\frac{s^2}{c^2}}$.

$$s(\sin\alpha-\sin\beta)=\sin^2\alpha-\sin^2\beta=\cos^2\beta-\cos^2\alpha$$

$$=c(\cos\beta-\cos\alpha);$$

$$\therefore\ s\sin\alpha+c\cos\alpha=s\sin\beta+c\cos\beta;$$

$\therefore \quad = \frac{1}{2}\{s(\sin\alpha + \sin\beta) + c(\cos\alpha + \cos\beta)\} = \frac{1}{2}(s^2 + c^2),$

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} = \frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}} = \frac{2\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}}$$

$$= \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} + \cos^2\frac{\alpha+\beta}{2}}$$

$$= \frac{2\sin(\alpha+\beta)}{(\cos\alpha + \cos\beta) + 1 + \cos(\alpha+\beta)}$$

$$= \frac{2cs}{c^2+s^2} \div \left\{c + 1 + \frac{c^2 - s^2}{c^2 + s^2}\right\}.$$

5. $\tan\theta = \dfrac{n\tan\phi}{1 + \tan^2\phi - n\tan^2\phi} = \dfrac{(n-1)\tan\phi + \tan\phi}{1 - (n-1)\tan\phi \,.\, \tan\phi}$

$= \tan\{\tan^{-1}\{(n-1)\tan\phi\} + \tan^{-1}(\tan\phi)\};$

$\therefore \ \theta = \tan^{-1}\{(n-1)\tan\phi\} + \phi.$

6. $2e\cos\dfrac{\phi+\phi'}{2}\sin\dfrac{\phi-\phi'}{2} = 2\sin\dfrac{\phi-\phi'}{2}\cos\dfrac{\phi-\phi'}{2};$

$$\therefore \ \cos\frac{\phi-\phi'}{2} = e\cos\frac{\phi+\phi'}{2};$$

$$\therefore \ (e^2-1)\cos^2\frac{\phi-\phi'}{2} = e^2\left(\cos^2\frac{\phi-\phi'}{2} - \cos^2\frac{\phi+\phi'}{2}\right)$$

$= \frac{1}{2}e^2\{1 + \cos(\phi-\phi') - 1 - \cos(\phi+\phi')\}$

$= e^2\sin\phi\sin\phi'.$

7. $4\sin^2(\beta-\alpha)\{\cos(2\beta-4\alpha) + 2\cos 2\alpha\}$

$= 2\{1 - \cos(2\beta-2\alpha)\}\{\cos(2\beta-4\alpha) + 2\cos 2\alpha\}$

$= 2\{\cos(2\beta-4\alpha) + 2\cos 2\alpha\} - \{\cos(4\beta-6\alpha) + \cos 2\alpha\}$

$\qquad - 2\{\cos 2\beta + \cos(2\beta-4\alpha)\}$

$= 3\cos 2\alpha - 2\cos 2\beta - \cos(4\beta-6\alpha)$

$= (3\cos 2\alpha + \cos 6\alpha) - 2\cos 2\beta - \{\cos(4\beta-6\alpha) + \cos 6\alpha\}$

$= 4\cos^3 2\alpha - 2\cos 2\beta - 2\cos 2\beta \,.\, \cos(2\beta-6\alpha)$

$= 4\cos^3 2\alpha - 2\cos 2\beta\{1 + \cos(2\beta-6\alpha)\} = 4 \,.$ Numerator.

8. See XIV. a, No. 31. $\sin\beta\cos\alpha + \sin\alpha\cos\beta = -\cos\beta\sin\beta;$ $\therefore \ \sin(\alpha+\beta) = -\frac{1}{2}\sin 2\beta;$ $\therefore$ the normals at α, β, β are concurrent; $\therefore$ the centre of curvature for β lies on the normal at α.

9. As in Ex. XIV. b, No. 19, (ii),

r.h.s. $= 2\cos(A+B)\cos(C-D) - 2\cos(A-B)\cos(C+D)$

$= \cos 2C + \cos 2D - \cos 2A - \cos 2B.$

10. $\cos(A+B) = \cos(C+D);$

$\therefore \ c_1c_2 - \sqrt{\{(1-c_1^2)(1-c_2^2)\}} = c_3c_4 - \sqrt{\{(1-c_3^2)(1-c_4^2)\}};$

$\therefore \ (c_1c_2 - c_3c_4)^2 = (1-c_1^2)(1-c_2^2) + (1-c_3^2)(1-c_4^2)$

$\qquad - 2\sqrt{\{(1-c_1^2)(1-c_2^2)(1-c_3^2)(1-c_4^2)\}};$

$\therefore \ 4(1-c_1^2)(1-c_2^2)(1-c_3^2)(1-c_4^2) = (2 - \Sigma c^2 + 2c_1c_2c_3c_4)^2.$

11. r.h.s. $= 2\sin\alpha\{\cos(\beta-\gamma) - \cos(\beta+\gamma)\}$

$= 2\sin\alpha\cos(\beta-\gamma) - 2\sin\alpha\cos(\beta+\gamma) =$ l.h.s.

12. $\sin(\gamma+\alpha) - \sin(\beta+\delta) = \sin(\gamma+\delta) - \sin(\alpha+\beta);$

$\therefore \ 2\cos\frac{1}{2}(\alpha+\beta+\gamma+\delta)\sin\frac{1}{2}(\gamma+\alpha-\beta-\delta)$

$= 2\cos\frac{1}{2}(\alpha+\beta+\gamma+\delta)\sin\frac{1}{2}(\gamma+\delta-\alpha-\beta);$

$\therefore \ \cos\frac{1}{2}(\alpha+\beta+\gamma+\delta) = 0,$

i.e. $\alpha+\beta+\gamma+\delta = (2n+1)\pi$

or $\frac{1}{2}(\gamma+\alpha-\beta-\delta) = n\pi + (-1)^n\frac{1}{2}(\gamma+\delta-\alpha-\beta),$

i.e. $\alpha - \delta = 2k\pi$ or $\gamma - \beta = (2k+1)\pi.$

13. $\cos\alpha, \cos\beta, \cos\gamma$ are roots of $x^3 - ax^2 + bx + a = 0;$ put

$y = \dfrac{1}{1-x^2} = \operatorname{cosec}^2\alpha, \beta$ or γ, then $x(x^2+b) + \dfrac{a}{y} = 0;$

$$\therefore \ \left(1 - \frac{1}{y}\right)\left(1 - \frac{1}{y} + b\right)^2 = x^2(x^2+b)^2 = \frac{a^2}{y^2};$$

$\therefore \ \operatorname{cosec}^2\alpha, \operatorname{cosec}^2\beta, \operatorname{cosec}^2\gamma$ are the roots of

$(b+1)^2y^3 - \{(b+1)^2 + 2(b+1)\}y^2 + \ldots - 1 = 0;$

$\therefore \ \Sigma\operatorname{cosec}^2\alpha = 1 + \dfrac{2}{b+1}$ and $\operatorname{cosec}^2\alpha\operatorname{cosec}^2\beta\operatorname{cosec}^2\gamma$

$= \dfrac{1}{(b+1)^2};$ hence result.

14. $\sin(x+a)\cos(c+a) - \sin(x+b)\cos(b+c)$

$= \sin(b+c)\cos(c+a) - \cos(b+c)\sin(c+a)$

$= \sin\{(b+c) - (c+a)\};$

$\therefore \ \sin(x+2a+c) + \sin(x-c) - \sin(x+2b+c) - \sin(x-c)$

$= 2\sin(b-a);$

$$\therefore\ 2\cos(x+a+b+c)\sin(a-b)=2\sin(b-a);$$
$$\therefore \text{ if } \sin(a-b)\neq 0,\ \cos(x+a+b+c)=-1;$$
$$\therefore \text{ for } a-b\neq n\pi,\ x+a+b+c=(2n+1)\pi;$$
$$\therefore\ \frac{\sin(x+a)-\sin[(2n+1)\pi-(x+a)]}{\cos(b+c)}=0$$

and similarly

$$\frac{\sin(x+c)-\sin(a+b)}{\cos(a+b)}=0 \text{ (assuming } a+b\neq n\pi+\tfrac{1}{2}\pi).$$

15. l.h.s. $=\sin 3(s-a)\sin(b-c)+\frac{1}{2}\{\cos(3s-3b-c+a)$
$-\cos(3s-3b+c-a)+\cos(3s-3c-a+b)$
$-\cos(3s-3c+a-b)\}$
$=\sin 3(s-a)\sin(b-c)+\sin(3s+a-2b-2c)\sin(b-c)$
$+\sin(3s-a-b-c)\sin 2(c-b)$
$=\sin(b-c)\{\sin 3(s-a)+\sin(3s+a-2b-2c)$
$-2\cos(b-c)\sin(3s-a-b-c)\}$
$=\sin(b-c)$
$\times 2\sin(3s-a-b-c)\{\cos(2a-b-c)-\cos(b-c)\}=$ r.h.s.

16. Numerator $\times 4\cos(y-z)\cos(z-x)\cos(x-y)$
$=\Sigma\{2\sin 2x(\sin 2y-\sin 2z)\cos(z-x)\cos(x-y)\cos(y+z)\}$
$=2\Sigma[\sin 2y\sin 2z\{\cos(x-y)\cos(z+x)$
$-\cos(z-x)\cos(x+y)\}\cos(y-z)]$
$=\Sigma[\sin 2y\sin 2z\{\cos(2x-y+z)+\cos(y+z)$
$-\cos(y+z)-\cos(2x+y-z)\}\cos(y-z)]$
$=\sin 2x\sin 2y\sin 2z\,\Sigma[\sin 2(y-z)]$.

Denominator in a similar way; the italicised sines become cosines and cosines become sines, with slight differences in signs.

17. Square and add.

18. $a+b=2+\sin\theta+\cos\theta$,
$$a-b=(\cos\theta-\sin\theta)\{2(\cos\theta+\sin\theta)-1\}$$
$$=(\cos\theta-\sin\theta)(2a+2b-5);$$
square and add the values of $\sin\theta+\cos\theta$ and $\cos\theta-\sin\theta$.

19. $x:y:1$
$=3\sin 3\theta\sin\theta+\cos 3\theta\cos\theta:\sin\theta\cos 3\theta-3\cos\theta\sin 3\theta:1;$
$\therefore\ x=2\cos 2\theta-\cos 4\theta,\ -y=2\sin 2\theta+\sin 4\theta;$
$\therefore\ a\equiv x+iy=\frac{2}{z}-z^2$ where

$z=\operatorname{cis}2\theta$, and $b\equiv x-iy=2z-\frac{1}{z^2}$;
$\therefore\ z^3+az-2=0$ and $2z^3-bz^2-1=0$;
eliminate z^3, $bz^2+2az-3=0$;
also $2(2z^3-bz^2)=2=z^3+az$; $\therefore\ 3z^2-2bz-a=0$;
$\therefore\ z^2:2z:1=a^2+3b:9-ab:b^2+3a$;
$\therefore\ (ab-9)^2=4(a^2+3b)(b^2+3a)$;
$\therefore\ a^2b^2+4(a^3+b^3)+18ab=27$;
but $ab=x^2+y^2$ and $a^3+b^3=(a+b)\{(a+b)^2-3ab\}$
$=2x\{4x^2-3(x^2+y^2)\}=2x(x^2-3y^2)$.

20. $(a\cos\alpha+b\cos\beta+c\cos\gamma)\sin\theta$
$+(a\sin\alpha+b\sin\beta+c\sin\gamma)\cos\theta=0$ gives $\tan\theta\equiv t$, say;
$$\Sigma a\sec(\theta-\alpha)=0;\ \therefore\ \sum\frac{a}{\cos\alpha+t\sin\alpha}=0.$$
But $(a\cos\alpha+b\cos\beta+c\cos\gamma)(\cos\alpha+t\sin\alpha)$
$=(a\cos\alpha+b\cos\beta+c\cos\gamma)\cos\alpha$
$-(a\sin\alpha+b\sin\beta+c\sin\gamma)\sin\alpha$
$=a\cos 2\alpha+b\cos(\alpha+\beta)+c\cos(\alpha+\gamma)$;
$\therefore\ \Sigma a\{a\cos(\beta+\alpha)+b\cos 2\beta+c\cos(\beta+\gamma)\}\times$
$\{a\cos(\gamma+\alpha)+b\cos(\gamma+\beta)+c\cos 2\gamma\}=0$.

21. Put $\cos\theta\sin\theta=x$.
$b^2+c^2=\cos^6\theta+\sin^6\theta+2a(\cos^4\theta+\sin^4\theta)+a^2$
$=(1-3x^2)+2a(1-2x^2)+a^2$
$=(1+a)^2-x^2(3+4a)$;
also $bc=x(\cos^2\theta+a)(\sin^2\theta+a)$
$$=x(x^2+a+a^2)=x\left\{\frac{(1+a)^2-b^2-c^2}{3+4a}+a+a^2\right\}$$
$$=\frac{x\{(1+a)(1+2a)^2-b^2-c^2\}}{3+4a};$$
$\therefore\ b^2c^2(4a+3)^3$
$=\{(1+a)^2-b^2-c^2\}\{(1+a)(1+2a)^2-b^2-c^2\}^2$.

22. $5x=20\sin^3\theta-5\sin\theta$, $5y=20\cos^3\theta-5\cos\theta$. In No. 21, put
$a=-\frac{1}{4},\ b=\frac{y}{4},\ c=\frac{x}{4}$.

23. $p\cos^2\alpha+q\sin^2\alpha=\sin\alpha\cos\alpha+\sin^2\alpha\cos^2\alpha(\tan\theta+\cot\theta)$
$$=\sin\alpha\cos\alpha+\frac{\sin^2\alpha\cos^2\alpha}{\sin\theta\cos\theta},\text{ and}$$

$$pq = \sin^2\theta\cos^2\theta + \sin^2\alpha\cos^2\alpha + 2\sin\theta\cos\theta\sin\alpha\cos\alpha$$
$$= (\sin\theta\cos\theta + \sin\alpha\cos\alpha)^2.$$

Substitute for $\sin\theta\cos\theta$ from first result in second.

24. $x = \dfrac{\cos(3\theta-\alpha)\pm\cos(3\theta+\alpha)}{\cos(\theta-\beta)\pm\cos(\theta+\beta)} = \dfrac{\cos 3\theta\cos\alpha}{\cos\theta\cos\beta} = \dfrac{\sin 3\theta\sin\alpha}{\sin\theta\sin\beta}$;

$$\therefore \frac{x\cos\beta}{\cos\alpha} = 4\cos^2\theta - 3 = 2\cos 2\theta - 1,$$

$$\text{and } \frac{x\sin\beta}{\sin\alpha} = 3 - 4\sin^2\theta = 1 + 2\cos 2\theta;$$

equate the two values of $2\cos 2\theta$.

25. $t = \tan\theta$;

$$\text{expn.} = \frac{1+\dfrac{1}{t^2}-t^2}{\dfrac{1}{t^2}+t^2-1} = \frac{t^2+1-t^4}{1+t^4-t^2}$$

$$= \frac{2}{1-t^2+t^4} - 1 = \frac{2}{(t^2-\frac{1}{2})^2+\frac{3}{4}} - 1.$$

26. $\cot\theta - \cot 2\theta = \dfrac{2\cos^2\theta - \cos 2\theta}{\sin 2\theta} = \operatorname{cosec} 2\theta \geqslant 1$, similarly

$\cot 2\theta - \cot 4\theta \geqslant 1$; add.

The first is an equality if $2\theta = (4n+1)\dfrac{\pi}{2}$;

the second if $4\theta = (4m+1)\dfrac{\pi}{2}$;

$\therefore$ one at least is an inequality.

27. $x^2 - 2xy\cos\theta + y^2 = k(x^2 - 2xy\cos\phi + y^2)$ can be solved for $x : y$ if $(\cos\theta - k\cos\phi)^2 - (1-k)^2 > 0$, i.e. i

$$\{(1+\cos\theta) - k(1+\cos\phi)\}\{(1-\cos\theta) - k(1-\cos\phi)\} < 0,$$

i.e. if k is between the values stated.

28. See Ex. XIV. f. No. 14.

29. $\triangle\text{DEF} = 2\Delta\cos A\cos B\cos C \leqslant 2\Delta\cos^3 60°$.

30. $\sin\theta = \mu\sin\phi > \sin\phi$; but θ, ϕ are acute; $\therefore \theta > \phi$;

$\therefore \cos\theta < \cos\phi$; but $\cos\theta\dfrac{d\theta}{d\phi} = \mu\cos\phi > \cos\phi > \cos\theta$;

$\therefore \dfrac{d\theta}{d\phi} > 1$; $\therefore \dfrac{d}{d\phi}(\theta - \phi)$ is positive; $\therefore \theta - \phi$ increases with ϕ.

Or, Take a circle of unit radius OA, and a point P in OA produced so that $\text{OP} = \mu$. Let Q be a point on circle so that

$\angle\text{OPQ} = \phi$. As Q moves on the circle from A, ϕ increases from zero to its maximum $\sin^{-1}\dfrac{1}{\mu}$. $\angle\text{OQP}$ is obtuse and

$$\sin\text{OQP} = \mu\sin\text{OPQ} = \sin\theta;$$
$$\therefore \text{OQP} = \pi - \theta; \quad \therefore \angle\text{POQ} = \theta - \phi$$

and $\angle$POQ evidently increases.

MISCELLANEOUS EXAMPLES.

EXERCISE XV. (p. 278.)

1. Objects A, B; the man starts at K and walks to P and then on to Q; $\text{AB} = x$, $\angle\text{APB} = 2\alpha$, $\angle\text{AQB} = \alpha$;

by eqn. (16) Ch. I, applied to PKB,

$(x + \text{KA})\cot\text{KAP} = x\cot 2\alpha - \text{KA}\cot\text{KPA}$;

$\therefore \text{KB} . \text{KA}/a = x\cot 2\alpha - a$.

Similarly from QKB,

$\text{KB} . \text{KA}/3a = x\cot\alpha - 3a$;

$\therefore 3x\cot\alpha - 9a = x\cot 2\alpha - a = x(1-\tan^2\alpha)/(2\tan\alpha) - a$;

$\therefore 6x - 18a\tan\alpha = x - x\tan^2\alpha - 2a\tan\alpha$;

$\therefore x(5 + \tan^2\alpha) = 16a\tan\alpha$.

2. $4\text{R}\Delta = 4\cdot\dfrac{a}{2\sin A}\cdot\frac{1}{2}bc\sin A = abc$;

$$r^2 + s^2 + 4\text{R}r = \frac{\Delta^2}{s^2} + s^2 + \frac{abc}{\Delta}\cdot\frac{\Delta}{s} = \frac{(s-a)(s-b)(s-c)}{s} + s^2 + \frac{abc}{s}$$
$$= s^2 - s(a+b+c) + (bc+ca+ab) - \frac{abc}{s} + s^2 + \frac{abc}{s}$$
$$= s^2 - s\,.\,2s + (bc+ca+ab) + s^2 = bc+ca+ab;$$

also $2s = a+b+c$; $\therefore a, b, c$ are the roots of

$$x^3 - 2sx^2 + (r^2+s^2+4\text{R}r)x - 4\text{R}\Delta = 0.$$

3. Put $\sin\theta = s$, $3s - 4s^3 - 2s = 4(1-s^2) - 3$; $\therefore 4s^3 - 4s^2 - s + 1 = 0$;

$\therefore (s-1)(4s^2-1) = 0$; $\therefore s = 1$ or $\pm\frac{1}{2}$;

$$\therefore \theta = 2n\pi + \frac{\pi}{2} \text{ or } n\pi \pm \frac{\pi}{6}.$$

4. If $y = \frac{1}{3}x^3 - x + (1-x^2)\tan^{-1}x + x\log(1+x^2)$,

$$\frac{dy}{dx} = x^2 - 1 + \frac{1-x^2}{1+x^2} - 2x\tan^{-1}x + \log(1+x^2) + \frac{2x^2}{1+x^2}$$
$$= x^2 - 2x\tan^{-1}x + \log(1+x^2);$$

$\therefore \frac{d^2y}{dx^2}=2(x-\tan^{-1}x)>0$ for $x>0$;

now $\frac{dy}{dx}=0$ if $x=0$ and $\frac{d^2y}{dx^2}>0$ for $x>0$;

$\therefore \frac{dy}{dx}$ increases as x increases;

$\therefore \frac{dy}{dx}>0$ for $x>0$; repeating the argument, it follows that $y>0$ for $x>0$.

5. If $Z=a \operatorname{cis}\theta, \frac{1}{Z}=\frac{1}{a}\operatorname{cis}(-\theta)$; $\therefore$ if $z=x+iy$, $x=\frac{1}{2}\cos\theta\left(a+\frac{1}{a}\right)$ and $y=\frac{1}{2}\sin\theta.\left(a-\frac{1}{a}\right)$; these are parametric eqns. to ellipses centre the origin and foci $(\pm c, 0)$ where

$$c^2=\tfrac{1}{4}\left(a+\frac{1}{a}\right)^2-\tfrac{1}{4}\left(a-\frac{1}{a}\right)^2=1.$$

Or, Let O be origin, S, S′ the points (1, 0), (−1, 0) and let P which represents Z move on circle $|Z|=a$.

$$z-1=\tfrac{1}{2}\left(Z+\frac{1}{Z}\right)-1=\frac{(Z-1)^2}{2Z};\quad \therefore |z-1|=\frac{PS^2}{2a};$$

similarly $|z+1|=\frac{PS'^2}{2a}$; $\therefore$ if Q represents z,

$$QS+QS'=\frac{PS^2+PS'^2}{2a}=\frac{OP^2+OS^2}{a}=a+\frac{1}{a}=\text{const.};$$

$\therefore$ Q moves on ellipse, foci S, S′.

6. The *principal value* of $i^i=\exp(i\log i)=\exp\left[i\left(\frac{\pi i}{2}\right)\right]$, see p. 252,

$$=e^{-\frac{\pi}{2}}\simeq\frac{1}{(2{\cdot}718)^{1{\cdot}571}}\simeq\frac{1}{4{\cdot}81}.\ [£1\simeq 4{\cdot}8 \text{ dollars.}]$$

7. From $\tan 7\theta=0$, see p. 172, removing the root $\tan\theta=0$ and putting $\tan^2\theta=\frac{1}{x}$, it follows that $\cot^2\frac{\pi}{7}$, $\cot^2\frac{2\pi}{7}$, $\cot^2\frac{3\pi}{7}$ are the roots of $7x^3-35x^2+21x-1=0$;

$$\therefore \Sigma\cot^4\frac{r\pi}{7}=\left(\frac{35}{7}\right)^2-2\left(\frac{21}{7}\right)=25-6.$$

8. expn. $=1-\cos^2x-\cos^2y+\cos^2x\cos^2y-(\cos z-\cos x\cos y)^2$
$=(1-\cos^2x)(1-\cos^2y)-(\cos z-\cos x\cos y)^2$
$=\sin^2x\sin^2y-(\cos z-\cos x\cos y)^2$

$=(\sin x\sin y+\cos z-\cos x\cos y)$
$\quad\times(\sin x\sin y-\cos z+\cos x\cos y)$
$=[\cos z-\cos(x+y)][\cos(x-y)-\cos z]$
$=2\sin\frac{1}{2}(x+y+z)\sin\frac{1}{2}(x+y-z)$
$\quad\times 2\sin\frac{1}{2}(x-y+z)\sin\frac{1}{2}(y+z-x).$

9. From AL, BM cut off $AL'=p-r$ and $BM'=q-r$, then the plane CL′M′ is parallel to LMN and therefore makes the same angle with ABC. If M′L′, BA produced meet at Z, CZ is the common line of the planes CL′M′, ABC; draw AU, BV perp. to ZC, ZC produced. Then

$$\tan\theta=\frac{AL'}{AU} \text{ and } =\frac{BM'}{BV};$$

$$\therefore \tan^2\theta=\frac{(p-r)^2}{AU^2}=\frac{(q-r)^2}{BV^2}=\frac{(p-r)^2+(q-r)^2}{AU^2+BV^2};$$

but BCV, CAU are congruent triangles;

$$\therefore BV=CU;\quad \therefore AU^2+BV^2=l^2.$$

10. Since △ABD, PBC are similar, $\frac{AB}{BD}=\frac{PB}{BC}$; $\therefore \frac{AB}{BP}=\frac{DB}{BC}$; also ∠ABP = ∠DBC since ∠ABD = ∠PBC; ∴ △ABP is directly similar to △DBC. Also

$$\angle APB=\angle DCB=C,\quad \angle BPC=\angle BAD=A;$$

$$\therefore \angle APC=2\pi-A-C \text{ or } A+C;$$

$$\therefore AC^2=AP^2+PC^2-2AP.PC\cos(A+C).$$

Also from similar triangles, $\frac{AP}{a}=\frac{c}{BD}$ and $\frac{PC}{b}=\frac{d}{BD}$;

$$\therefore AC^2=\frac{a^2c^2}{BD^2}+\frac{b^2d^2}{BD^2}-\frac{2acbd}{BD^2}.\cos(A+C).$$

11. $\cos x=-\cos 2y$; $\therefore x=\pi-2y$ or $2y-\pi$, and similars;

$\therefore$ (i) $x=\pi-2y$, $y=\pi-2z$, $z=\pi-2x$ whence $x=y=z=\frac{\pi}{3}$,
or (ii) $x=\pi-2y$, $y=\pi-2z$, $z=2x-\pi$ whence

$$x=\frac{5\pi}{7},\ y=\frac{\pi}{7},\ z=\frac{3\pi}{7},$$

or (iii) $x=\pi-2y$, $y=2z-\pi$, $z=2x-\pi$ whence

$$x=\frac{7\pi}{9},\ y=\frac{\pi}{9},\ z=\frac{5\pi}{9},$$

or (iv) $x=2y-\pi$, $y=2z-\pi$, $z=2x-\pi$, whence $x=y=z=\pi$, or results like those in (ii) and (iii).

12. For $|x| < 1$, $1 - x + x^2 - x^3 + \dots = (1+x)^{-1}$;

$$-x + \frac{x^2}{2} - \frac{x^3}{3} + \dots = -\log(1+x);\ \text{add.}$$

13. $\log(1-at) + \log(1-bt) = \log[(1-at)(1-bt)] = \log[1 - pt + qt^2]$, where $p = a+b$, $q = ab$. For values of t for which $|at| < 1$ and $|bt| < 1$ and $|pt - qt^2| < 1$, we have

$$-\sum \frac{t^n}{n}(a^n + b^n) \equiv \log[1 - t(p - qt)] \equiv -\sum \frac{t^r}{r}(p-qt)^r;$$

$\therefore\ \frac{1}{n}(a^n + b^n)$ is the coefficient of t^n in

$$\frac{1}{n}t^n(p-qt)^n + \frac{1}{n-1}t^{n-1}(p-qt)^{n-1}$$
$$+ \frac{1}{n-2}t^{n-2}(p-qt)^{n-2} + \dots,$$

and this is

$$\frac{1}{n}\cdot p^n - \frac{1}{n-1}\binom{n-1}{1}p^{n-2}q + \frac{1}{n-2}\binom{n-2}{2}p^{n-4}q^2 - \dots;$$

$$\therefore\ a^n + b^n = p^n - np^{n-2}q$$
$$+ \frac{n(n-3)}{2!}p^{n-4}q^2 - \frac{n(n-4)(n-5)}{3!}p^{n-6}q^3 + \dots;$$

put $a = \sin^2\theta$, $b = \cos^2\theta$, so that $p = 1$. The least value of r in $-\sum \frac{t^r}{r}(p-qt)^r$ which gives a term t^n is $r = \frac{1}{2}(n+1)$ if n is odd and is $r = \frac{1}{2}n$ if n is even.

For the justification of the process of "equating coefficients," see *Bromwich, Infinite Series*, pp. 65-68 and p. 134 (1st edn.).

Or, proceed as follows:

By Ex. VI. e, No. 15, ch $n\alpha$ is a polynomial of degree n in ch α, and the polynomial may be obtained either by the method of p. 178 or by writing αi for θ in Ex. IX. e, No. 12.

Hence, $2\text{ch}\, n\alpha = (2\text{ch}\,\alpha)^n - n(2\text{ch}\,\alpha)^{n-2} + \dots$, the last term being $2(-1)^{\frac{1}{2}n}$ if n is even, Ex. IX. e, No. 14, and being $(-1)^{\frac{1}{2}(n-1)}\,2n\,\text{ch}\,\alpha$ if n is odd, Ex. IX. e, No. 15.

Put ch $\alpha = \operatorname{cosec} 2\theta$ and divide by $(2\text{ch}\,\alpha)^n$.

Then sh $\alpha = \pm\cot 2\theta$; $\therefore$ the values of ch $\alpha \pm$ sh α are $\cot\theta$ and $\tan\theta$; $\therefore$ by Ex. VI. e, No. 15,

$$2\text{ch}\, n\alpha = \cot^n\theta + \tan^n\theta.$$

14. $\cos\alpha\,\text{ch}\,\beta - i\sin\alpha\,\text{sh}\,\beta = \cos\gamma + i\sin\gamma$;

$$\therefore\ \cos\alpha\,\text{ch}\,\beta = \cos\gamma,\ \sin\alpha\,\text{sh}\,\beta = -\sin\gamma;$$

$$\therefore\ \frac{\cos^2\gamma}{\cos^2\alpha} - \frac{\sin^2\gamma}{\sin^2\alpha} = \text{ch}^2\beta - \text{sh}^2\beta = 1;$$

$$\frac{1-\sin^2\gamma}{\cos^2\alpha} - \frac{\sin^2\gamma}{\sin^2\alpha} = 1;$$

$$\therefore\ \sin^2\gamma\left(\frac{1}{\cos^2\alpha} + \frac{1}{\sin^2\alpha}\right) = \sec^2\alpha - 1 = \tan^2\alpha;$$

$$\therefore\ \sin^2\gamma = \sin^2\alpha\cos^2\alpha\tan^2\alpha = \sin^4\alpha;$$

$$\therefore\ \text{sh}^2\beta = \frac{\sin^2\gamma}{\sin^2\alpha} = \sin^2\alpha.$$

Or, $1 = \text{cis}\,\gamma\ \text{cis}\,(-\gamma) = \cos(\alpha + i\beta)\cos(\alpha - i\beta)$
$= \frac{1}{2}(\cos 2\alpha + \text{ch}\,2\beta) = \cos^2\alpha + \text{sh}^2\beta$;

$\therefore\ \sin^2\alpha = \text{sh}^2\beta$; but as above, $\sin\alpha\,\text{sh}\,\beta = -\sin\gamma$.

15. Put $\frac{2\pi}{21} = \alpha$ and use $\cos r\alpha = \cos(21 - r)\alpha$; thus sum of

$$\text{expressions} = (\cos\alpha + \cos 4\alpha + \cos 16\alpha)$$
$$+ (\cos 19\alpha + \cos 13\alpha + \cos 10\alpha)$$
$$= -\cos 7\alpha + \sum_0^6 \cos(3r+1)\alpha$$
$$=,\ \text{by p. 128},\ -\cos 7\alpha + \frac{\cos 10\alpha\,.\,\sin\pi}{\sin\frac{3}{2}\alpha}$$
$$= -\cos 7\alpha = \tfrac{1}{2},$$

$$\text{and product} = \Sigma(\cos r\alpha\,.\,\cos s\alpha)$$
$$= \tfrac{1}{2}\Sigma\{\cos(r-s)\alpha + \cos(r+s)\alpha\}$$
$$= \tfrac{1}{2}\Sigma\cos t\alpha$$
$$= \cos 7\alpha + \tfrac{3}{2}(\cos 3\alpha + \cos 6\alpha + \cos 9\alpha)$$
$$+ \tfrac{1}{2}\sum_0^6\{\cos(3r+1)\alpha\}$$
$$= -\tfrac{1}{2} + \tfrac{3}{2}\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right) + 0$$
$$= -\tfrac{1}{2} + \tfrac{3}{2}(-\tfrac{1}{2})$$

by Example 1 of Ch. X, $= -\frac{5}{4}$; hence expressions are the roots of $x^2 - \frac{1}{2}x - \frac{5}{4} = 0$, viz. $\frac{1 \pm \sqrt{21}}{4}$, but the first expression is the sum of 3 positive numbers and is $\therefore\ \frac{1+\sqrt{21}}{4}$.

16. Put $\theta+\phi=\alpha$; then $a:b:c=\sin(\alpha-\phi):\sin\alpha:\sin(\alpha+\phi)$;

$\therefore\ c-a:2b:c+a=\sin\phi\cos\alpha:\sin\alpha:\sin\alpha\cos\phi$; thus

$$\sin\alpha\sin\phi:\cos\alpha:\cos\alpha\cos\phi$$
$$=(c-a)\tan^2\alpha:2b:c+a,$$

also $(2b)^2-(c+a)^2:(c-a)^2$

$$=\sin^2\alpha(1-\cos^2\phi):\sin^2\phi\cos^2\alpha=\tan^2\alpha:1,$$

thus $\sin\alpha\sin\phi:\cos\alpha:\cos\alpha\cos\phi$

$$=(2b)^2-(c+a)^2:2b(c-a):c^2-a^2;$$

$\therefore\ \cos(\alpha-\phi):\cos\alpha:\cos(\alpha+\phi)$

$$=c^2-a^2+(2b)^2-(c+a)^2:2b(c-a):(c^2-a^2)-(2b)^2+(c+a)^2.$$

17. A is point in plain; P is top of hill PQR, PQ $=\frac{3}{4}$PR $=3$QR; slope of hill $=\theta°$; AQ bisects $\angle$PAR;

$$\therefore\ \frac{PA}{RA}=\frac{PQ}{QR}=3;\quad\therefore\ \frac{\sin\theta}{\sin(\theta-30°)}=3;$$

$\therefore\ 2\sin\theta=3(\sin\theta\,.\sqrt{3}-\cos\theta)$; $\therefore\ \cot\theta=\sqrt{3}-\frac{2}{3}$.

18. Make CBX $=$ A $=20°$ with X on AC, then BX $=$ BC $=$ BF, but FBX $=60°$; $\therefore$ FBX is equilateral.

EXF $=180°-$FXB$-$BXC$=40°$, also BEX $=40°$;

$\therefore$ XE $=$ XB $=$ XF; $\therefore$ X is centre of circle EBF;

$\therefore$ FEB $=\frac{1}{2}$FXB $=30°$.

Or, Let FEB $=\theta$; $\dfrac{\sin(\theta+20°)}{\sin\theta}=\dfrac{BE}{BF}=\dfrac{BE}{BC}=\dfrac{\sin 80°}{\sin 40°}=2\cos 40°$;

$\therefore\ \cos 20°+\cot\theta\sin 20°=2\cos 40°=2\cos(60°-20°)$

$$=\cos 20°+\sqrt{3}\sin 20°;\quad\therefore\ \cot\theta=\sqrt{3}.$$

19. Let $y=\cos 2\theta+b^2\cos\theta+c$

$$=2\cos^2\theta+b^2\cos\theta+c-1\equiv 2x^2+b^2x+c-1,$$

where $x=\cos\theta$, so that $-1\leqslant x\leqslant 1$. The equation gives two possible values of $\cos\theta$ if, and only if, the *arc* of the parabola, $y=2x^2+b^2x+c-1$, corresponding to $-1\leqslant x\leqslant 1$, crosses the x-axis twice; the vertex of the parabola is downwards; $\therefore$ the vertex must be on this arc and below the x-axis, and each extremity of the parabolic arc must be above the x-axis. For the vertex, $\dfrac{dy}{dx}=0$;

$$\therefore\ 4x+b^2=0;\quad\therefore\ -1<-\frac{b^2}{4}<1;$$

$\therefore\ b^2<4$; also for $x=-\frac{1}{4}b^2$, $y=c-1-\frac{1}{8}b^4$; $\therefore\ c-1-\frac{1}{8}b^4<0$; $\therefore\ c<\frac{1}{8}b^4+1<3$. And for

$x=\pm 1$, $y=c+1\pm b^2$; $\therefore\ c+1\pm b^2>0$; $\therefore\ b^2-1<c$.

20. $e^x\,.\,\mathrm{cis}\,x=\exp(x)\,.\,\exp(xi)=\exp(x+xi)$

$$=\exp\{x(1+i)\}=1+\sum\frac{x^n}{n!}(1+i)^n;$$

but $(1+i)^n=\left(\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}\right)^n=2^{\frac{1}{2}n}\,\mathrm{cis}\,\frac{n\pi}{4}$; equate second parts;

$$\therefore\ e^x\sin x=\sum\left(\frac{x^n}{n!}2^{\frac{1}{2}n}\,.\,\sin\frac{n\pi}{4}\right).$$

21. If the points are concyclic, then points $z_2-z_1, z_3-z_1, z_4-z_1$, are concyclic with the origin; $\therefore$ their inverses w.r.t. $|z|=1$ are collinear, and so are the images in OX of those inverses;

$\therefore$ the points $\dfrac{1}{z_2-z_1}, \dfrac{1}{z_3-z_1}, \dfrac{1}{z_4-z_1}$ are collinear. The point which divides the join of w_1 and w_2 in ratio $\mu:\lambda$ is w_3, where

$w_3=\dfrac{\lambda w_1+\mu w_2}{\lambda+\mu}$; $\therefore\ w_1, w_2, w_3$ are collinear if

$$\lambda w_1+\mu w_2-(\lambda+\mu)w_3=0.$$

If $\lambda=b-c$ and $\mu=c-a$, $-(\lambda+\mu)=a-b$;

$$\therefore\ \frac{b-c}{z_2-z_1}+\frac{c-a}{z_3-z_1}+\frac{a-b}{z_4-z_1}=0,$$

which gives required result.

22. $\mathrm{ch}(ax)\cos bx=\mathrm{ch}(ax)\,.\,\mathrm{ch}(ibx)$

$=\frac{1}{2}[\mathrm{ch}(a+ib)x+\mathrm{ch}(a-ib)x]=$, by eqn. (22), p. 198,

$$1+\tfrac{1}{2}\sum\frac{x^{2n}}{(2n)!}[(a+ib)^{2n}+(a-ib)^{2n}];$$

if $a=c\cos\alpha$, $b=c\sin\alpha$, so that $c^2=a^2+b^2$ and $\tan\alpha=\dfrac{b}{a}$,

$$(a+ib)^{2n}+(a-ib)^{2n}=c^{2n}\,\mathrm{cis}\,(2n\alpha)+c^{2n}\,\mathrm{cis}\,(-2n\alpha)=2c^{2n}\cos 2n\alpha.$$

23. The values of $\cos x$ which satisfy $\cos(2n+1)x=\cos(2n+1)\theta$ are $\cos\left(\theta+\dfrac{2r\pi}{2n+1}\right)$ for $r=0$ to $2n$. By IX. e, No. 15, p. 183, this eqn. may be written

$$\cos(2n+1)\theta-(-1)^n\,.\,(2n+1)\cos x+a\cos^3x\ldots+b\cos^{2n+1}x=0;$$

expression $=$ sum of reciprocals of the roots

$$=(-1)^n\,.\,\frac{2n+1}{\cos(2n+1)\theta}.$$

24. Put $a=\tan A$, $b=\tan B$, etc., then

$$\frac{\tan A-\tan B}{1+\tan A\tan B}=\frac{\tan C-\tan D}{1+\tan C\tan D};$$

$$\therefore\ \tan(A-B)=\tan(C-D);\quad \therefore\ A-B=C-D+r\pi;$$

$$\therefore\ A-C=B-D+r\pi;\quad \therefore\ \cos(A-C)=\pm\cos(B-D),$$

but $\cos(A-C)=\dfrac{1+\tan A\tan C}{\sec A\sec C}$

$$=\pm\frac{1+ac}{\sqrt{(1+a^2)}\,.\,\sqrt{(1+c^2)}};\ \text{etc.}$$

25. A, B are centres of circles, radii a, b; N is mid-point of common chord; O is centre of sphere;

$$\therefore\ AN=\sqrt{(a^2-c^2)}=e,\ \text{say};\ BN=\sqrt{(b^2-c^2)}=f,\ \text{say};$$

$$\angle ANB=\theta;\ R^2=ON^2+c^2;$$

ON is diameter of circle ANBO;

$$\therefore\ ON=\frac{AB}{\sin ANB}=\frac{\sqrt{(e^2+f^2-2ef\cos\theta)}}{\sin\theta};$$

$$\therefore\ (R^2-c^2)\sin^2\theta=ON^2\,.\,\sin^2\theta=e^2+f^2-2ef\cos\theta.$$

26. Suppose $AB<DC$.

Let BA′, parallel to AD, cut DC at A′.

Suppose a', b', c', A′, B′, C′, s' refer to triangle A′BC.

Then $\cos\frac{1}{2}(A+C)=\sin\frac{1}{2}(A'-C')$.

As on p. 19, $\sin\frac{1}{2}(A'-C')=\dfrac{a'-c'}{b'}\cos\frac{1}{2}B'$

$$=\frac{a'-c'}{b'}\sqrt{\left\{\frac{s'(s'-b')}{a'c'}\right\}}.$$

But $a'=b$, $c'=d$, $b'=c-a$.

Hence $2s'=b+d+c-a$;

but $2s=a+b+c+d$;

$$\therefore\ s'=s-a\ \text{and}\ s'-b'=s-c;$$

$$\therefore\ \cos^2\tfrac{1}{2}A=\frac{(b-d)^2}{(c-a)^2}\cdot\frac{(s-a)(s-c)}{bd}.$$

27. $s_n=\sin x+\sin 3x+\ldots+\sin(2n-1)x=$ by p. 128,

$$\frac{\sin nx\,.\,\sin nx}{\sin x}=\frac{1-\cos 2nx}{2\sin x};$$

$$\therefore\ \frac{1}{n}\sum_1^n s_r=\frac{1}{2n}\operatorname{cosec}x\left\{n-\frac{\sin nx\cos(n+1)x}{\sin x}\right\}$$

$$=\tfrac{1}{2}\operatorname{cosec}x-\tfrac{1}{2}\operatorname{cosec}^2x\,.\,\frac{\sin nx\cos(n+1)x}{n},$$

but $|\sin nx\cos(n+1)x|\leqslant 1$;

$$\therefore\ \frac{\sin nx\cos(n+1)x}{n}\to 0\ \text{when}\ n\to\infty;$$

$\therefore$ if $x\neq k\pi$, so that $\operatorname{cosec}x$ exists, expression$\to\frac{1}{2}\operatorname{cosec}x$. If $x=k\pi$, $s_n=0$ and the limit is 0.

28. By pp. 104, 80, $x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\ldots=\operatorname{sh}x$; $\quad x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots=\sin x$; subtract.

29. Put $x=\operatorname{cis}2\theta$, $a=\operatorname{cis}2\alpha$, $b=\operatorname{cis}2\beta$, $c=\operatorname{cis}2\gamma$; then, as in Ex. VIII. g, No. 10 (ii),

$$x-b=2i\sin(\theta-\beta)\,.\,\operatorname{cis}(\theta+\beta);\ \text{etc.};\ \therefore\ \frac{a^2(x-b)(x-c)}{(a-b)(a-c)}$$

$$=\frac{\operatorname{cis}4\alpha\,.\,2i\sin(\theta-\beta)\operatorname{cis}(\theta+\beta)\,.\,2i\sin(\theta-\gamma)\operatorname{cis}(\theta+\gamma)}{2i\sin(\alpha-\beta)\operatorname{cis}(\alpha+\beta)\,.\,2i\sin(\alpha-\gamma)\operatorname{cis}(\alpha+\gamma)}$$

$$=\frac{\operatorname{cis}(2\theta+2\alpha)\,.\,\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)};$$

etc.; also $x^2=\operatorname{cis}4\theta$. Equate first parts.

30. Put $\dfrac{r}{2c}=p$; $p[\operatorname{ch}(\xi+i\eta)-1]=\operatorname{cis}(-\theta)$;

$$\therefore\ p[\operatorname{ch}\xi\cos\eta+i\operatorname{sh}\xi\sin\eta-1]=\cos\theta-i\sin\theta;$$

$$\therefore\ p(\operatorname{ch}\xi\cos\eta-1)=\cos\theta\ \text{and}\ p\operatorname{sh}\xi\sin\eta=-\sin\theta;$$

$$\therefore\ \left(\frac{p+\cos\theta}{p\operatorname{ch}\xi}\right)^2+\left(\frac{-\sin\theta}{p\operatorname{sh}\xi}\right)^2=\cos^2\eta+\sin^2\eta=1;$$

$$\operatorname{sh}^2\xi(p^2+2p\cos\theta+\cos^2\theta)+\operatorname{ch}^2\xi(1-\cos^2\theta)=p^2\operatorname{ch}^2\xi\operatorname{sh}^2\xi;$$

$$\therefore\ p^2\operatorname{sh}^2\xi(\operatorname{ch}^2\xi-1)-2p\operatorname{sh}^2\xi\cos\theta$$

$$=\operatorname{sh}^2\xi\cos^2\theta+\operatorname{ch}^2\xi-\operatorname{ch}^2\xi\cos^2\theta=\operatorname{ch}^2\xi-\cos^2\theta;$$

$$\therefore\ p^2\operatorname{sh}^4\xi-2p\operatorname{sh}^2\xi\cos\theta+\cos^2\theta=\operatorname{ch}^2\xi;$$

$$\therefore\ (p\operatorname{sh}^2\xi-\cos\theta)^2=\operatorname{ch}^2\xi;\ \therefore\ p\operatorname{sh}^2\xi-\cos\theta=\pm\operatorname{ch}\xi.$$

31. $\alpha^5=\operatorname{cis}2\pi=1$; if $a=\alpha+\alpha^4$, $b=\alpha^2+\alpha^3$,

$$a+b=\alpha+\alpha^2+\alpha^3+\alpha^4=-1+\frac{1-\alpha^5}{1-\alpha}=-1;$$

$$ab=(\alpha+\alpha^4)(\alpha^2+\alpha^3)=\alpha^3+\alpha^4+\alpha^6+\alpha^7=\alpha^3+\alpha^4+\alpha+\alpha^2,$$

since $a^5=1$, $=$as before -1;

$$\therefore\ a,\ b \text{ are roots of } x^2+x-1=0;$$

$$\therefore \text{ the values of } a,\ b \text{ are } \frac{-1\pm\sqrt{5}}{2};$$

$$a=\alpha+\alpha^4=\alpha+\frac{1}{\alpha}=\text{cis}\,\frac{2\pi}{5}+\text{cis}\left(-\frac{2\pi}{5}\right)=2\cos\frac{2\pi}{5},$$

which is positive;

$$\therefore\ 2\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{2};$$

$$b=\alpha^2+\alpha^3=\alpha^2+\frac{1}{\alpha^2}=,\text{ as before, } 2\cos\frac{4\pi}{5},$$

which is negative;

$$\therefore\ 2\cos\frac{4\pi}{5}=\frac{-1-\sqrt{5}}{2}.$$

32. Substitute $\theta-\frac{\pi}{3}$, $\theta+\frac{\pi}{3}$ for x and subtract; then

$$a\left\{\cos\left(2\theta-\frac{2\pi}{3}\right)-\cos\left(2\theta+\frac{2\pi}{3}\right)\right\}+b\left\{\sin\left(2\theta-\frac{2\pi}{3}\right)-\sin\left(2\theta+\frac{2\pi}{3}\right)\right\}=c\left\{\cos\left(\theta-\frac{\pi}{3}\right)-\cos\left(\theta+\frac{\pi}{3}\right)\right\}+d\left\{\sin\left(\theta-\frac{\pi}{3}\right)-\sin\left(\theta+\frac{\pi}{3}\right)\right\};$$

$$\therefore\ a\sin 2\theta\sin\frac{2\pi}{3}-b\cos 2\theta\sin\frac{2\pi}{3}=c\sin\theta\sin\frac{\pi}{3}-d\cos\theta\sin\frac{\pi}{3};$$

$\therefore\ a\sin 2\theta-b\cos 2\theta=c\sin\theta-d\cos\theta$; also, since θ is a solution, $a\cos 2\theta+b\sin 2\theta=c\cos\theta+d\sin\theta$. Square and add these two results.

33. $\tan(\theta+h)+\tan(\theta-h)=\dfrac{\sin[(\theta+h)+(\theta-h)]}{\cos(\theta+h)\cos(\theta-h)}$;

$$\therefore\ \text{l.h.s.}=\frac{2\sin\theta\cos\theta}{\cos(\theta+h)\cos(\theta-h)}-\frac{2\sin\theta}{\cos\theta}$$

$$=\frac{2\sin\theta[\cos^2\theta-\cos(\theta+h)\cos(\theta-h)]}{\cos\theta\cos(\theta+h)\cos(\theta-h)}$$

$$=\frac{\tan\theta[1+\cos 2\theta-\cos 2\theta-\cos 2h]}{\cos(\theta+h)\cos(\theta-h)}$$

$$=\frac{2\sin^2 h\tan\theta}{\cos(\theta+h)\cos(\theta-h)};$$

$$\therefore\ \tfrac{1}{2}\{\tan(\theta+h)+\tan(\theta-h)\}-\tan\theta=\frac{\tan\theta\,.\,\sin^2 h}{\cos(\theta+h)\cos(\theta-h)};$$

using 4-figure tables, $\tan 82°=7{\cdot}115$, $\tan 82°\,6'=7{\cdot}207$, $\sin^2 3'\simeq\left(\dfrac{\pi\times 3}{180\times 60}\right)^2$; $\therefore$ the interpolation value, $7{\cdot}161$, exceeds the true value by approximately

$$\frac{7{\cdot}161\pi^2\,.\,\sec 82°\,.\,\sec 82°\,6'}{3600^2}\simeq 0{\cdot}000285.$$

Since $\tan 82°\simeq 7{\cdot}115370$ and $\tan 82°\,6'\simeq 7{\cdot}206612$, the interpolation value$\simeq 7{\cdot}160991$, and so the true value $\simeq 7{\cdot}160991-0{\cdot}000285$.

34. From $\triangle$AWB, $\dfrac{\mathrm{AW}}{\sin\frac{1}{3}\mathrm{B}}=\dfrac{c}{\sin\frac{1}{3}(\mathrm{A}+\mathrm{B})}=\dfrac{c}{\sin(60°-\frac{1}{3}\mathrm{C})}$,

similarly for AV;

$$\therefore\ \frac{\mathrm{AW}}{\mathrm{AV}}=\frac{c\sin\frac{1}{3}\mathrm{B}}{\sin(60°-\frac{1}{3}\mathrm{C})}\times\frac{\sin(60°-\frac{1}{3}\mathrm{B})}{b\sin\frac{1}{3}\mathrm{C}},\ \text{ but } \frac{c}{b}=\frac{\sin\mathrm{C}}{\sin\mathrm{B}};$$

$\therefore$ using the identity

$$4\sin\tfrac{1}{3}\theta\sin(60°-\tfrac{1}{3}\theta)\sin(60°+\tfrac{1}{3}\theta)=\sin\theta,$$

$$\frac{\mathrm{AW}}{\mathrm{AV}}=\frac{\sin(60°+\frac{1}{3}\mathrm{C})}{\sin(60°+\frac{1}{3}\mathrm{B})}.$$

The sum of AWV, AVW is $60°+\frac{1}{3}\mathrm{B}+60°+\frac{1}{3}\mathrm{C}$ and the ratio of their sines is that of $\sin(60°+\frac{1}{3}\mathrm{B})$ to $\sin(60°+\frac{1}{3}\mathrm{C})$;

$\therefore$ these angles are $60°+\frac{1}{3}\mathrm{B}$, $60°+\frac{1}{3}\mathrm{C}$;

$$\therefore\ \frac{\mathrm{VW}}{\sin\frac{1}{3}\mathrm{A}}=\frac{\mathrm{AW}}{\sin(60+\frac{1}{3}\mathrm{C})}=\frac{c\sin\frac{1}{3}\mathrm{B}}{\sin(60°+\frac{1}{3}\mathrm{C})\sin(60°-\frac{1}{3}\mathrm{C})}$$

$$=\frac{4c\sin\frac{1}{3}\mathrm{B}\sin\frac{1}{3}\mathrm{C}}{\sin\mathrm{C}}=8\mathrm{R}\sin\tfrac{1}{3}\mathrm{B}\sin\tfrac{1}{3}\mathrm{C}.$$

The facts that $\angle\mathrm{AWV}=60°+\frac{1}{3}\mathrm{B}$, etc., show at once that $\triangle$UVW is equilateral, because

$$\angle\mathrm{UWV}=360°-\angle\mathrm{VWA}-\angle\mathrm{UWB}-\angle\mathrm{BWA}$$
$$=360°-(60°+\tfrac{1}{3}\mathrm{B})-(60°+\tfrac{1}{3}\mathrm{A})-(180°-\tfrac{1}{3}\mathrm{A}-\tfrac{1}{3}\mathrm{B})=60°.$$

35. Since $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\dfrac{x-y}{1+xy}$,

$$\text{expression}=\left(\tan^{-1}a_1-\tan^{-1}\frac{y}{x}\right)+(\tan^{-1}a_2-\tan^{-1}a_1)+\dots+(\tan^{-1}a_n-\tan^{-1}a_{n-1})=\tan^{-1}a_n-\tan^{-1}\frac{y}{x}.$$

For given special values,

$$\tan^{-1}0+\tan^{-1}\frac{2}{1+1\,.\,3}+\tan^{-1}\frac{2}{1+3\,.\,5}$$

$$+\ldots+\tan^{-1}\frac{2}{1+(2n-3)(2n-1)}=\tan^{-1}(2n-1)-\tan^{-1}1\,;$$

when $n\to\infty$, r.h.s.$\to\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$. Hence

$$\frac{\pi}{4}=\tan^{-1}\frac{2}{2^2}+\tan^{-1}\frac{2}{4^2}+\ldots+\tan^{-1}\frac{2}{4r^2}+\ldots$$

since $1+(2r-1)(2r+1)=4r^2$.

36. $\operatorname{sh}\theta-\sin\theta\simeq\left(\theta+\frac{\theta^3}{6}\right)-\left(\theta-\frac{\theta^3}{6}\right)=\frac{\theta^3}{3}$ with error $<\mathrm{A}\theta^7$,

$$\operatorname{ch}\theta-\cos\theta\simeq\left(1+\frac{\theta^2}{2}\right)-\left(1-\frac{\theta^2}{2}\right)=\theta^2 \text{ with error }<\mathrm{B}\theta^6,$$

compare VI. d, No. 20 ; $\therefore\ \frac{\operatorname{sh}\theta-\sin\theta}{\theta(\operatorname{ch}\theta-\cos\theta)}\to\frac{\theta^3}{3}\div\theta^3=\frac{1}{3}$.

37. As in No. 29, if $a=\operatorname{cis}2\alpha$, etc., $b-c=2i\sin(\beta-\gamma)\operatorname{cis}(\beta+\gamma)$; also $1+ab=1+\operatorname{cis}(2\alpha+2\beta)$

$$=2\cos^2(\alpha+\beta)+2i\sin(\alpha+\beta)\cos(\alpha+\beta)$$
$$=2\cos(\alpha+\beta)\,.\,\operatorname{cis}(\alpha+\beta)\,;$$

$$\therefore\ \text{l.h.s.}=\Sigma 2i\sin(\beta-\gamma)\operatorname{cis}(\beta+\gamma)$$
$$\times 2\cos(\alpha+\beta)\operatorname{cis}(\alpha+\beta)$$
$$\times 2\cos(\alpha+\gamma)\operatorname{cis}(\alpha+\gamma)$$
$$=8i\Sigma\sin(\beta-\gamma)\cos(\alpha+\beta)\cos(\alpha+\gamma)$$
$$\times\operatorname{cis}(2\alpha+2\beta+2\gamma)\,;$$

also $\text{r.h.s.}=2i\sin(\beta-\gamma)\operatorname{cis}(\beta+\gamma)$

$$\times 2i\sin(\gamma-\alpha)\operatorname{cis}(\gamma+\alpha)$$
$$\times 2i\sin(\alpha-\beta)\operatorname{cis}(\alpha+\beta)$$
$$=-8i\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\operatorname{cis}(2\alpha+2\beta+2\gamma)\,;$$

equate moduli.

38. $\cos x\operatorname{ch}y-i\sin x\operatorname{sh}y=\cos(x+iy)$

$$=\frac{\sin(\xi+i\eta)\cos(\xi-i\eta)}{\cos(\xi+i\eta)\cos(\xi-i\eta)}=\frac{\sin 2\xi+i\operatorname{sh}2\eta}{\cos 2\xi+\operatorname{ch}2\eta}\,;$$

put $\cos 2\xi+\operatorname{ch}2\eta=p$; then

$$\cos x\operatorname{ch}y=\frac{\sin 2\xi}{p}\text{ and }\sin x\operatorname{sh}y=-\frac{\operatorname{sh}2\eta}{p}\,;$$

$$\therefore\ \frac{\sin^2 2\xi\operatorname{sech}^2 y}{p^2}+\frac{\operatorname{sh}^2 2\eta\operatorname{cosech}^2 y}{p^2}=\cos^2 x+\sin^2 x=1\,;$$

$$\therefore\ \frac{2\sin^2 2\xi}{\operatorname{ch}2y+1}+\frac{2\operatorname{sh}^2 2\eta}{\operatorname{ch}2y-1}=p^2\,;$$

$$\therefore\ 2\operatorname{ch}2y(\sin^2 2\xi+\operatorname{sh}^2 2\eta)$$
$$-2(\sin^2 2\xi-\operatorname{sh}^2 2\eta)=p^2(\operatorname{ch}^2 2y-1),$$

which is equivalent to required relation.

39. $\cos 2n\theta=0$ is satisfied by $\cos\theta=\pm\cos\frac{(2r-1)\pi}{4n}$; by IX. e, No. 14, this gives the roots of

$$1-2n^2\cos^2\theta+\ldots+k\cos^{2n}\theta=0\,;$$

$$\therefore\ x=\cos^2\frac{(2r-1)\pi}{4n}\text{ satisfies }1-2n^2x+\ldots+kx^n=0\,;$$

expression $=$ sum of reciprocals of roots $=2n^2$.

40. Put $x=\tan\mathrm{A}$, $y=\tan\mathrm{B}$, $z=\tan\mathrm{C}$, then

$$\Sigma\tan\mathrm{A}=\tan\mathrm{A}\tan\mathrm{B}\tan\mathrm{C}\,;$$

$$\therefore\ \tan(\mathrm{A}+\mathrm{B}+\mathrm{C})=0\,;\quad\therefore\ \mathrm{A}+\mathrm{B}+\mathrm{C}=n\pi\,;$$

$$\therefore\ \tan(2\mathrm{A}+2\mathrm{B}+2\mathrm{C})=0\,;\quad\therefore\ \Sigma\tan 2\mathrm{A}=\prod\tan 2\mathrm{A}\,;$$

$$\therefore\ \sum\frac{2x}{1-x^2}=\prod\frac{2x}{1-x^2}\,;\text{ multiply each side by }\prod(1-x^2).$$

41. Objects P, Q ; the man starts from O and walks along OAB ; $\mathrm{OA}=c$, $\mathrm{AB}=d$, $\alpha=\angle\mathrm{POA}=\angle\mathrm{PAQ}=\angle\mathrm{PBQ}$, so that APQB is a cyclic quad. ;

$$\therefore\ \angle\mathrm{QAB}=\alpha+\angle\mathrm{PQA}=\alpha+\angle\mathrm{PBA}=\angle\mathrm{QBA}\,;$$

$\therefore$ QAB is isosceles ; $\therefore$ the perp. from Q to AB bisects AB ;

$\therefore$ $\mathrm{OQ}\cos\alpha=c+\frac{1}{2}d$; also APQB is a cyclic quadl. ;

$$\therefore\ c(c+d)=\mathrm{OP}\,.\,\mathrm{OQ}=\mathrm{OQ}(\mathrm{OQ}-\mathrm{PQ})$$
$$=(c+\tfrac{1}{2}d)\sec\alpha\{(c+\tfrac{1}{2}d)\sec\alpha-\mathrm{PQ}\}.$$

42. Draw XIY perp. to OI meeting AB, AC at X, Y ; and IP, IQ perp. to AB, AC ; and OM, ON perp. to IP, IQ, then by similar triangles $\mathrm{IX}:r=\mathrm{OI}:\mathrm{OM}$ and $\mathrm{IY}:r=\mathrm{OI}:\mathrm{ON}$;

$$\therefore\ \mathrm{AX}:\mathrm{AY}=\mathrm{IX}:\mathrm{IY}=\mathrm{ON}:\mathrm{OM}=\mathrm{AQ}-\mathrm{QC}:\mathrm{AP}-\mathrm{PB}$$
$$=\mathrm{AB}-\mathrm{BC}:\mathrm{AC}-\mathrm{CB}=\mathrm{AD}:\mathrm{AE}\,;$$

$\therefore$ DE is parallel to XY. **Or,** Projection of $\overline{\mathrm{OI}}=$ sum of projections of $\overline{\mathrm{OA}}$, $\overline{\mathrm{AI}}$, hence ratio of projection on BC to that on the perp. to BC is $\dfrac{\mathrm{R}\sin(\mathrm{C}-\mathrm{B})-r\operatorname{cosec}\frac{\mathrm{A}}{2}\sin\frac{\mathrm{C}-\mathrm{B}}{2}}{\mathrm{R}\cos(\mathrm{C}-\mathrm{B})-r\operatorname{cosec}\frac{\mathrm{A}}{2}\cos\frac{\mathrm{C}-\mathrm{B}}{2}}$

$$=\frac{2\sin\frac{\mathrm{C}-\mathrm{B}}{2}\cos\frac{\mathrm{C}-\mathrm{B}}{2}-4\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}\sin\frac{\mathrm{C}-\mathrm{B}}{2}}{\cos(\mathrm{C}-\mathrm{B})-4\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}\left(\cos\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2}+\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}\right)}$$

$$=\frac{2\sin\frac{\mathrm{C}-\mathrm{B}}{2}\cos\frac{\mathrm{C}+\mathrm{B}}{2}}{\cos(\mathrm{C}-\mathrm{B})-\sin\mathrm{B}\sin\mathrm{C}-(1-\cos\mathrm{B})(1-\cos\mathrm{C})}$$

$$=\frac{\sin\mathrm{C}-\sin\mathrm{B}}{\cos\mathrm{B}+\cos\mathrm{C}-1}=\frac{\mathrm{CE}\sin\mathrm{C}-\mathrm{BD}\sin\mathrm{B}}{\mathrm{BD}\cos\mathrm{B}+\mathrm{CE}\cos\mathrm{C}-\mathrm{BC}}$$

= − ratio of projections of ED on the perp. to BC and on BC.

43. Since $\sin 3\phi=3\sin\phi-4\sin^3\phi$, $\sin^3\phi=\frac{1}{4}(3\sin\phi-\sin 3\phi)$;

$$\therefore\ \text{series}=\tfrac{1}{4}\left\{\left(3^2\sin\frac{\theta}{3}-3\sin\theta\right)+\left(3^3\sin\frac{\theta}{3^2}-3^2\sin\frac{\theta}{3}\right)+\dots+\left(3^{n+1}\sin\frac{\theta}{3^n}-3^n\sin\frac{\theta}{3^{n-1}}\right)\right\}$$

$$=\tfrac{1}{4}\left(3^{n+1}\sin\frac{\theta}{3^n}-3\sin\theta\right),$$

this $\to\frac{1}{4}\left(3^{n+1}\cdot\frac{\theta}{3^n}-3\sin\theta\right)=\frac{1}{4}(3\theta-3\sin\theta)$ when $n\to\infty$.

44. Expression $=\Sigma\log(1-\alpha x)=-\left\{x\Sigma\alpha+\frac{x^2}{2}\Sigma\alpha^2+\frac{x^3}{3}\Sigma\alpha^3+\dots\right\}$.

If $\Sigma\alpha=p,\ \Sigma\alpha\beta=q,\ \Sigma\alpha\beta\gamma=r,\ \alpha\beta\gamma\delta=s,$

$$\begin{aligned}\text{expression}&=\log(1-px+qx^2-rx^3+sx^4)\\&=\log[1-x(p-qx+rx^2-sx^3)]\\&=-\{x(p-qx+rx^2-sx^3)\\&\qquad+\tfrac{1}{2}x^2(p-qx+rx^2-sx^3)^2+\dots\};\end{aligned}$$

equating coefficients of x^5, (*Bromwich, Infinite Series*, 1st edn., p. 67),

$$\tfrac{1}{5}\Sigma\alpha^5=-ps-qr+p^2r+pq^2-p^3q+\tfrac{1}{5}p^5.$$

$$\begin{aligned}\therefore\ \Sigma(\alpha^5)+5\Sigma\alpha\beta\,.\,\Sigma\alpha\beta\gamma&=\Sigma(\alpha^5)+5qr\\&=p^5-5p^3q+5p^2r+5p(q^2-s)\\&=p\{p^4-5p^2q+5pr+5q^2-5s\}.\end{aligned}$$

45. $s_n=1+z+z^2+\dots+z^{n-1}=\dfrac{1-z^n}{1-z}$, since $z\neq1$;

$$\begin{aligned}\therefore\ \frac{s_1+s_2+\dots+s_n}{n}&=\frac{1}{1-z}-\frac{1}{n(1-z)}\cdot(z+z^2+\dots+z^n)\\&=\frac{1}{1-z}-\frac{z(1-z^n)}{n(1-z)^2}\\&=\frac{1}{1-z}-\frac{z}{(1-z)^2}\times\frac{2\sin\frac{n\alpha}{2}\operatorname{cis}\left(\frac{n\alpha-\pi}{2}\right)}{n}\to\frac{1}{1-z}\end{aligned}$$

when $n\to\infty$,

since $\left|\sin\dfrac{n\alpha}{2}\right|\leqslant1$ and $\left|\operatorname{cis}\dfrac{n\alpha-\pi}{2}\right|=1.$

46. (i) $\dfrac{1}{(2n-1)^2(2n+1)^2}=\frac{1}{4}\left(\dfrac{1}{2n-1}-\dfrac{1}{2n+1}\right)^2$

$$=\tfrac{1}{4}\left\{\frac{1}{(2n-1)^2}+\frac{1}{(2n+1)^2}-\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]\right\};$$

$$\therefore\ 4s_n=2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots+\frac{1}{(2n-1)^2}\right]$$

$$1+\frac{1}{(2n+1)^2}-\left(1-\frac{1}{2n-1}\right)$$

$\therefore$ from XI. b, No. 16 (ii), when $n\to\infty$, $4s_n\to2\left(\dfrac{\pi^2}{8}\right)-1-1$;

(ii) $\dfrac{1}{(2n-1)^3(2n+1)^3}=\frac{1}{8}\left(\dfrac{1}{2n-1}-\dfrac{1}{2n+1}\right)^3$

$$=\tfrac{1}{8}\left\{\frac{1}{(2n-1)^3}-\frac{1}{(2n+1)^3}-\frac{3}{(2n-1)(2n+1)}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]\right\}$$

$$=\tfrac{1}{8}\left\{\frac{1}{(2n-1)^3}-\frac{1}{(2n+1)^3}-\frac{6}{(2n-1)^2(2n+1)^2}\right\};$$

but $\sum\left\{\dfrac{1}{(2n-1)^3}-\dfrac{1}{(2n+1)^3}\right\}=1-\dfrac{1}{(2n+1)^3}$;

$\therefore$ from (i), when $n\to\infty$, $s_n\to\frac{1}{8}\left\{1-6\left(\dfrac{\pi^2}{16}-\dfrac{1}{2}\right)\right\}$.

47. Let $\dfrac{\pi}{2n}=\theta$; angles in given series are $<\dfrac{\pi}{2}$, and writing

$$\begin{aligned}2\operatorname{cosec}\theta&=\operatorname{cosec}\theta+\operatorname{cosec}(\pi-\theta),\\-2\operatorname{cosec}3\theta&=\operatorname{cosec}(\pi+3\theta)+\operatorname{cosec}(2\pi-3\theta),\\2\operatorname{cosec}5\theta&=\operatorname{cosec}5\theta+\operatorname{cosec}(\pi-5\theta),\\-2\operatorname{cosec}7\theta&=\operatorname{cosec}(\pi+7\theta)+\operatorname{cosec}(2\pi-7\theta),\end{aligned}$$

etc., ... we get twice given sum $=\Sigma \operatorname{cosec}(4r+1)\theta$ where all angles $(4r+1)\theta$ occur which are $<2\pi$ except that $\frac{\pi}{2}$ is missing if $n=4p+1$, and $\frac{3\pi}{2}$ is missing if $n=4p+3$; hence given sum $=\frac{1}{2}\sum_0^{n-1}\operatorname{cosec}(4r+1)\theta-\frac{1}{2}(-1)^{\frac{1}{2}(n-1)}$. The values of $\sin(4r+1)\theta$ for $r=0$ to $n-1$ are the roots of $\sin n\phi=1$ expressed as an eqn. for $\sin\phi$; $\therefore$ from IX. e, No. 25, $\sum_0^{n-1}\operatorname{cosec}(4r+1)\theta$ is the sum of the reciprocals of the roots of $1-nx+\frac{n(n^2-1)}{3!}x^3-\ldots+kx^n=0$, and so equals n.

48. $\cos(\beta+\gamma+\theta)-\cos(\gamma+\alpha+\theta)=\cos\alpha-\cos\beta$;

$$\therefore\ 2\sin\left(\frac{\alpha+\beta}{2}+\gamma+\theta\right)\sin\frac{\alpha-\beta}{2}=2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2};$$

but $\sin\frac{\alpha-\beta}{2}\neq 0$; $\therefore\ \sin\left(\frac{\alpha+\beta}{2}+\gamma+\theta\right)=\sin\left(\pi+\frac{\alpha+\beta}{2}\right)$;

but $\gamma+\theta\neq(2n+1)\pi$;

$$\therefore\ \frac{\alpha+\beta}{2}+\gamma+\theta=(2n+1)\pi-\pi-\frac{\alpha+\beta}{2};$$

$\therefore\ \alpha+\beta+\gamma+\theta=2n\pi$; substitute for γ;

$\therefore\ \cos[2n\pi-\alpha]+\cos\beta+\cos[2n\pi-\alpha-\beta-\theta]=1$;

hence result.

49. Let AB cut the line of greatest slope OC at C. If the vertical height of C above O is h, $\mathrm{OC}=h\sec\gamma$, $\mathrm{OA}=h\sec\alpha$, $\mathrm{OB}=h\sec\beta$;

$\therefore\ h\sec\gamma\,.\,\mathrm{AB}=2\triangle\mathrm{OAB}=h\sec\alpha\,.\,h\sec\beta\,.\sin\theta$ where $\angle\mathrm{AOB}=\theta$; $\therefore\ \mathrm{AB}=h\sec\alpha\sec\beta\cos\gamma\sin\theta$;

$\therefore\ h^2\sec^2\alpha\sec^2\beta\cos^2\gamma\sin^2\theta$

$=\mathrm{AB}^2=h^2\sec^2\alpha+h^2\sec^2\beta-2h^2\sec\alpha\sec\beta\cos\theta$;

$\therefore\ \cos^2\gamma(1-\cos^2\theta)=\cos^2\beta+\cos^2\alpha-2\cos\alpha\cos\beta\cos\theta$.

50. By p. 25, $x^2+y^2=(ac+bd)\left\{\frac{ad+bc}{ab+cd}+\frac{ab+cd}{ad+bc}\right\}$

$$=xy\,.\left\{\frac{(ad+bc)^2+(ab+cd)^2}{(ab+cd)(ad+bc)}\right\}$$

$$\therefore\ \left(\frac{a}{c}+\frac{c}{a}\right)-\left(\frac{x}{y}+\frac{y}{x}\right)=\frac{a^2+c^2}{ac}-\frac{(a^2+c^2)(b^2+d^2)+4abcd}{bd(a^2+c^2)+ac(b^2+d^2)}$$

$$=\frac{bd(a^2+c^2)^2-4a^2c^2bd}{ac[bd(a^2+c^2)+ac(b^2+d^2)]};$$

the numerator $=bd(a^2-c^2)^2$; hence result.

51. $\operatorname{cis}\alpha+\cos\beta\operatorname{cis}(\alpha+\beta)+\ldots+\cos^n\beta\,.\operatorname{cis}(\alpha+n\beta)$

$=\operatorname{cis}\alpha[1+z+z^2+\ldots+z^n]$, where $z=\cos\beta\operatorname{cis}\beta$,

$$=\frac{\operatorname{cis}\alpha(1-z^{n+1})}{1-z}=\frac{\operatorname{cis}\alpha[1-\cos^{n+1}\beta\operatorname{cis}(n+1)\beta]}{1-\cos^2\beta-i\cos\beta\sin\beta};$$

denominator $=\sin\beta(\sin\beta-i\cos\beta)=\sin\beta\operatorname{cis}\left(\beta-\frac{\pi}{2}\right)$;

$$\therefore\ \text{series}=\operatorname{cosec}\beta\left[\operatorname{cis}\left(\alpha-\beta+\frac{\pi}{2}\right)-\cos^{n+1}\beta\operatorname{cis}\left(\alpha+n\beta+\frac{\pi}{2}\right)\right]$$

equate first parts.

52. $\log t=\int_1^t\frac{1}{y}\,dy>\int_1^t\frac{4}{(1+y)^2}\,dy$, since $(1+y)^2-4y=(1-y)^2\geqslant 0$,

$$=\left[\frac{-4}{1+y}\right]_1^t=\frac{-4}{1+t}+2=\frac{2(t-1)}{t+1}$$

also $\log t=\int_1^t\frac{1}{y}\,dy$

$<\frac{1}{2}\int_1^t\left(1+\frac{1}{y^2}\right)dy$,

since $\frac{2}{y}<1+\frac{1}{y^2}$,

$=\frac{t^2-1}{2t}$; $\therefore\ \frac{t^2-1}{2t}>\log t>\frac{2(t-1)}{t+1}$ for $t>1$,

put $t=\frac{x}{x-1}$ where $x>1$ and result follows.

Note.—If the tangents at points A, P on a rect. hyp., see p. 56, meet at T, and if AC, PN are perps. to an asymptote, the inequalities are equivalent to trapezium CAPN $>$ area under curve $>$ area CATPN.

53. By IX. e, No. 29, $f(n)=\cos\frac{n\pi}{3}$;

$$\therefore\ f(2n)=\cos\frac{2n\pi}{3}=\cos\left(n\pi-\frac{n\pi}{3}\right)$$

$$=\cos n\pi\cos\frac{n\pi}{3}=(-1)^n\,.\cos\frac{n\pi}{3}.$$

54. From Ch. IX. eqn. (8), p. 175,

$$x\cos\theta+x^2\cos 2\theta+\ldots+x^{n-1}\cos(n-1)\theta$$

$$=\frac{x\cos\theta-x^2-x^n\cos n\theta+x^{n+1}\cos(n-1)\theta}{1-2x\cos\theta+x^2}$$

Differentiate w.r.t. θ;

$$-\{x\sin\theta+2x^2\sin 2\theta+\ldots+(n-1)x^{n-1}\sin(n-1)\theta\}$$
$$=\frac{-x\sin\theta+nx^n\sin n\theta-(n-1)x^{n+1}\sin(n-1)\theta}{1-2x\cos\theta+x^2}$$
$$-\frac{2x\sin\theta\,[x\cos\theta-x^2+x^{n+1}\cos(n-1)\theta]}{(1-2x\cos\theta+x^2)^2};$$

$\therefore$ as on p. 175, when $n\to\infty$, for $|x|<1$,

$$\text{this}\to\frac{-x\sin\theta}{1-2x\cos\theta+x^2}-\frac{2x\sin\theta\,(x\cos\theta-x^2)}{(1-2x\cos\theta+x^2)^2};$$

$\therefore$ given series

$$=\frac{r\sin\theta(1-2r\cos\theta+r^2)+2r\sin\theta(r\cos\theta-r^2)}{(1-2r\cos\theta+r^2)^2}.$$

55. $x^n-1=\prod_0^{n-1}(x-t^r)$, see p. 219; $\therefore$ since $0<m<n$,

$$\frac{nx^{m-1}}{x^n-1}=\sum\frac{A_r}{x-t^r},\text{ see p. 231, where}$$

$$A_r=\lim_{x\to t^r}\frac{nx^{m-1}(x-t^r)}{x^n-1}=\lim\frac{nx^{m-1}}{nx^{n-1}}$$
$$=\lim x^{m-n}=t^{mr},\text{ since } t^n=1.$$

56. $\tan\alpha=\dfrac{\sin\theta}{\cos\theta}+\dfrac{\sin\phi}{\cos\phi}=\dfrac{\sin(\theta+\phi)}{\cos\theta\cos\phi}$; $\therefore\ \cos\theta\cos\phi=\dfrac{1}{c}\cot\alpha$;

similarly $\cot\beta=\dfrac{\sin(\theta+\phi)}{\sin\theta\sin\phi}$;

$\therefore\ \sin\theta\sin\phi=\dfrac{1}{c}\tan\beta$; $\therefore\ \cos(\theta+\phi)=\dfrac{1}{c}(\cot\alpha-\tan\beta)$;

$$\therefore\ 1-\frac{1}{c^2}=\cos^2(\theta+\phi)=\frac{1}{c^2}(\cot\alpha-\tan\beta)^2.$$

57. Flagstaff A, visible from B and C; then circular base, centre I, is in-circle of $\triangle$ABC;

$$\therefore\ \frac{r}{AI}=\sin\tfrac12 A=\cos\tfrac12(B+C)$$
$$=\sin\tfrac12 B\sin\tfrac12 C(\cot\tfrac12 B\cot\tfrac12 C-1)$$
$$=\frac{r^2}{BI\,.\,CI}\left(\frac{pq}{r^2}-1\right)=\frac{pq-r^2}{\sqrt{\{(p^2+r^2)(q^2+r^2)\}}}.$$

58. The lines AP, BQ, CR are such that $\angle APC=\angle BQA=\angle CRB=\theta$; BQ, CR meet at X, CR, AP meet at Y, and AP, BQ at Z. Elementary geometry shows that XYZ is equiangular to

ABC. From $\triangle$AYC, $\dfrac{CY}{\sin(\theta+C)}=\dfrac{b}{\sin B}=2R$, and from $\triangle$BXC, $\dfrac{CX}{\sin(\theta-C)}=\dfrac{a}{\sin A}=2R$, hence

$$XY=CY-CX=2R\{\sin(\theta+C)-\sin(\theta-C)\}$$
$$=2R\,.\,2\cos\theta\,.\,\sin C=2c\cos\theta;$$

as triangles are similar $\dfrac{\triangle'}{\triangle}=\dfrac{XY^2}{c^2}=4\cos^2\theta$.

59. If $\theta=k\pi$, $\sin r\theta=0$, $\cos^2 r\theta=1$; $\therefore$ limit $=0$.

If $\theta\neq k\pi$, $\sum_1^n\sin^2 r\theta=\frac12\Sigma(1-\cos 2r\theta)$

$$=\tfrac12\left[n-\frac{\sin n\theta\cos(n+1)\theta}{\sin\theta}\right]$$

and $\sum_1^n\cos^2 r\theta=\frac12\left[n+\dfrac{\sin n\theta\cos(n+1)\theta}{\sin\theta}\right]$

$$\therefore\ \text{ratio}=\frac{\sin\theta-\frac1n\sin n\theta\cos(n+1)\theta}{\sin\theta+\frac1n\sin n\theta\cos(n+1)\theta}\quad\text{and}\quad\sin\theta\neq0,\ \text{but}$$

$|\sin n\theta\cos(n+1)\theta|<1$; $\therefore$ ratio$\to1$ when $n\to\infty$.

60. Put $1+x=y$, to prove $\dfrac{(y-1)^2}{y}>(\log y)^2$ for $y>0$;

(i) For $y>1$, $\log y$ is positive;

$\therefore$ to prove $\log y<\dfrac{y-1}{\sqrt y}$,

$$\log y=\int_1^y\frac1t\,dt<\int_1^y\left(\frac{1}{2\sqrt t}+\frac{1}{2t\sqrt t}\right)dt,$$

since $\dfrac{1}{2\sqrt t}+\dfrac{1}{2t\sqrt t}-\dfrac1t=\dfrac{(\sqrt t-1)^2}{2t\sqrt t}>0$;

$$\therefore\ \log y<\left[\sqrt t-\frac{1}{\sqrt t}\right]_1^y=\sqrt y-\frac{1}{\sqrt y}=\frac{y-1}{\sqrt y};$$

Or, by the proof of No. 52,

$$\int_1^y\left(\frac{t^2-1}{t^2}-\frac{2\log t}{t}\right)dt>0,$$

for $y>1$, since the integrand is positive.

$$\therefore\ \left[t+\frac1t-(\log t)^2\right]_1^y>0;\quad\therefore\ y^2-2y+1>y(\log y)^2.$$

(ii) Put $y=\frac{1}{z}$, so that $z<1$, $\log\frac{1}{z}<\frac{\frac{1}{z}-1}{\sqrt{\frac{1}{z}}}$; $\therefore\ -\log z<\frac{1-z}{\sqrt{z}}$,

where $-\log z$ and $\frac{1-z}{\sqrt{z}}$ are each positive;

$$\therefore\ (\log z)^2<\frac{(1-z)^2}{z}=\frac{(z-1)^2}{z} \text{ also for } z<1.$$

61. $(x^3+1)^2+1=0$; $\therefore\ x^3=-1\pm i=\sqrt{2}\,\text{cis}\left(\pm\frac{3\pi}{4}\right)$;

$$\therefore\ x=2^{\frac{1}{6}}\cdot\text{cis}\left(\frac{2r\pi}{3}\pm\frac{\pi}{4}\right) \text{ for } r=0,\ 1,\ 2.$$

62. $\sin(\theta+i\phi)=\exp(\alpha+i\beta)=e^{\alpha}\,\text{cis}\,\beta$; $\therefore\ e^{2\alpha}=|\sin(\theta+i\phi)|^2$
$=\sin(\theta+i\phi)\sin(\theta-i\phi)=\frac{1}{2}(\cos 2i\phi-\cos 2\theta)$.

63. In No. 55, put $x=-1$ and suppose that n is odd, so that $x^n-1=-1-1=-2$. Then

$$\frac{n(-1)^{m-1}}{-2}=\sum_{0}^{n-1}\frac{\text{cis}\frac{2mr\pi}{n}}{-1-\text{cis}\frac{2r\pi}{n}};$$

$$\therefore\ \sum\frac{\text{cis}\frac{2mr\pi}{n}}{\cos\frac{r\pi}{n}\ \text{cis}\frac{r\pi}{n}}=n(-1)^{m-1};$$

$$\therefore\ \sum\frac{\text{cis}\frac{r(2m-1)\pi}{n}}{\cos\frac{r\pi}{n}}=n(-1)^{m-1}.$$

Put $2m-1=s$, so that $0<m<n$ requires $0<s<2n-1$. Then

$$\sum\frac{\text{cis}\frac{rs\pi}{n}}{\cos\frac{r\pi}{n}}=n(-1)^{\frac{1}{2}(s-1)}=n\sin\frac{s\pi}{2};$$

equate first parts and equate second parts.

64. $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ are the roots of

$$(t+\tan\alpha)(1-t^2)=2kt(1-t\,.\,\tan\alpha),$$

or $t^3+t^2\tan\alpha(1-2k)-t(1-2k)-\tan\alpha=0$;

$$\therefore\ \tan\Sigma(\theta)=\frac{s_1-s_3}{1-s_2}=\frac{\tan\alpha(2k-1)-\tan\alpha}{1+(1-2k)}=-\tan\alpha.$$

65. ABCD is square courtyard. From P, the buildings, height h, at A, B, C subtend 60°, 60°, 45°;

$$\therefore\ \text{PA}=h\cot 60^\circ=\tfrac{1}{3}h\sqrt{3}=\text{PB};\ \text{PC}=h;$$

PN is perpendicular to AB;

$$\therefore\ \text{PN}^2+\left(\frac{a}{2}\right)^2=(\tfrac{1}{3}h\sqrt{3})^2;\ \therefore\ \text{PN}^2=\tfrac{1}{12}(4h^2-3a^2);$$

$$\text{PC}^2=\text{PB}^2+\text{BC}^2-2\text{BC}\,.\,\text{PN};$$

$$\therefore\ h^2=(\tfrac{1}{3}h\sqrt{3})^2+a^2-2a\cdot\frac{1}{2\sqrt{3}}\sqrt{(4h^2-3a^2)};$$

$$\therefore\ \frac{a}{\sqrt{3}}\sqrt{(4h^2-3a^2)}=a^2-\tfrac{2}{3}h^2;$$

$$\therefore\ \frac{a^2}{3}(4h^2-3a^2)=a^4-\frac{4a^2h^2}{3}+\tfrac{4}{9}h^4;$$

$$\therefore\ 2h^4-12a^2h^2+9a^4=0;\ \therefore\ h^2=a^2(3\pm\tfrac{3}{2}\sqrt{2}).$$

But $h=\text{PC}<a\sqrt{2}$ since P is inside the square;

$$\therefore\ h^2\neq a^2(3+\tfrac{3}{2}\sqrt{2});\ \therefore\ h^2=a^2(3-\tfrac{3}{2}\sqrt{2}).$$

66. By pp. 24, 25, $\text{R}^2\sigma=(ab+cd)(ac+bd)(ad+bc)$, and

by Ex. II. a, No. 14, $\text{AO}\,.\,\text{OC}=\frac{abcd(ac+bd)}{(ab+cd)(ad+bc)}$;

thus $\sigma(ab+cd)(ad+bc)(\text{R}^2-\text{AO}\,.\,\text{OC})$
$=(ac+bd)\{(ab+cd)^2(ad+bc)^2-\sigma\,abcd\}$.

But the expression

$$\text{E}\equiv(ab+cd)^2(ad+bc)^2-(ac+bd)\{bd(a^2-c^2)^2+ac(b^2-d^2)^2\}$$

is equal to zero if $a=0$; $\therefore\ a$ is a factor of E, similarly for b, c, d; and when $a=b+c+d$,

$$\begin{aligned}\text{E}&=(b+d)^4(b+c)^2(c+d)^2\\&\quad-(b+c)(c+d)\{bd(b+d)^2(a+c)^2+ac(b+d)^2(b-d)^2\}\\&=(b+d)^2(b+c)(c+d)\{(b+d)^2(b+c)(c+d)\\&\quad-bd(a-c)^2-2abcd-ac(b+d)^2+2abcd\}\\&=(b+d)^4(b+c)(c+d)\{(b+c)(c+d)-bd-ac\}=0;\end{aligned}$$

$\therefore\ b+c+d-a$ is a factor of E; hence $\text{E}=\text{K}abcd\,\sigma$, where $-\text{K}=$ coefficient of a^5bcd in $\text{E}=-1$. Thus

$$\sigma(ab+cd)(ad+bc)(\text{R}^2-\text{AO}\,.\,\text{OC})$$
$$=(ac+bd)^2\{bd(a^2-c^2)^2+ac(b^2-d^2)^2\}.$$

Also $\text{R}^2-\text{AO}\,.\,\text{OC}=$ square of required distance.

67. Draw on the same axes the graphs of $\sin x$ and $\cos 2x$; for graph of $\sin x$, see *E.T.*, p. 104, graph of $\cos 2x$ crosses x-axis at $x=\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. Draw graph of $\sin x+\cos 2x$ by taking algebraic sum of ordinates of these two graphs. Read off values of x for which ordinate of composite graph $=\frac{1}{5}$.

68. If $y=\sin\theta-\theta\cos\theta-\frac{\theta^3}{3}$, $\frac{dy}{d\theta}=\theta\sin\theta-\theta^2<0$; $\therefore$ when θ increases from 0, y decreases; also $y=0$ when $\theta=0$.

If $z=\theta^3-\frac{\theta^5}{10}-3\sin\theta+3\theta\cos\theta$,

$$\frac{dz}{d\theta}=3\theta^2-\tfrac{1}{2}\theta^4-3\theta\sin\theta=-3\theta\left(\sin\theta-\theta+\frac{\theta^3}{6}\right)<0.$$

69. If $z=\sqrt{(3+4i)}\pm\sqrt{(3-4i)}$,

$$z^2=3+4i+3-4i\pm2\sqrt{(9+16)}=6\pm10=16 \text{ or } -4;$$

$\therefore z=\pm4$ or $\pm2i$.

Or, Let α be the positive acute angle such that $\tan\alpha=\frac{4}{3}$, $\cos\alpha=\frac{3}{5}$, then

$$z=\sqrt{[5\operatorname{cis}\alpha]}\pm\sqrt{[5\operatorname{cis}(-\alpha)]}=\sqrt{5}\left\{\pm\operatorname{cis}\left(\frac{\alpha}{2}\right)\pm\operatorname{cis}\left(-\frac{\alpha}{2}\right)\right\}$$

$$=\pm2\sqrt{5}\,.\cos\frac{\alpha}{2} \text{ or } \pm2i\sqrt{5}\,.\sin\frac{\alpha}{2}.$$

But $\cos\frac{\alpha}{2}=\frac{2}{\sqrt{5}}$ and $\sin\frac{\alpha}{2}=\frac{1}{\sqrt{5}}$; hence results as before.

70. Let $z=\operatorname{cis}\theta$ be represented by P and produce OP to C so that OC = 4OP; $\therefore$ C represents $4z$; to find the point Q to represent $4z+z^4$ make $\angle$OCQ $=\pi-3\theta$ and CQ = OP = 1 unit; $\therefore$ $\overline{OQ}=\overline{OC}+\overline{CQ}$, and CQ makes 4θ with Ox.
Circles centres O, C radii 3, 1 will touch, at T say; the first is fixed, and if it meets the negative x-axis at A, arc AT $=3(\pi-\theta)=2\pi+(\pi-3\theta)=2\pi+$ arc TQ of second circle; $\therefore$ second circle rolls on the first. Locus is a 3-cusped epicycloid.

71. $a=\alpha+\alpha^3+\alpha^4+\alpha^9+\alpha^{10}+\alpha^{12}$, $b=\alpha^2+\alpha^5+\alpha^6+\alpha^7+\alpha^8+\alpha^{11}$;

$$\therefore a+b=-1+(1+\alpha+\alpha^2+\alpha^3+\ldots+\alpha^{12})$$

$$=-1+\frac{1-\alpha^{13}}{1-\alpha}=-1 \text{ since } \alpha^{13}=\operatorname{cis}2\pi=1 \text{ and } \alpha\neq1.$$

Also writing down the product ab and replacing α^r by α^{r-13} when $r>13$, we have $ab=3\sum_1^{12}\alpha^r=3(-1)$ as before. But $a+b=-1$, $ab=-3$ are conditions that a, b are roots of $x^2+x-3=0$; $\therefore$ the values of a, b are $\frac{1}{2}(-1\pm\sqrt{13})$; equate "first parts," and use $\cos(2\pi-\theta)=\cos\theta$, thus

$$2\left(\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}\right)=\tfrac{1}{2}(-1\pm\sqrt{13})=\tfrac{1}{2}(-1+\sqrt{13})$$

because $\cos\frac{6\pi}{13}$ and $\cos\frac{2\pi}{13}+\cos\frac{8\pi}{13}$ are both positive;

$$\therefore \tfrac{1}{4}(3+\sqrt{13})=1+\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}$$

$$=2\cos^2\frac{\pi}{13}+2\cos\frac{\pi}{13}\cos\frac{7\pi}{13}=2\cos\frac{\pi}{13}\cdot\left(2\cos\frac{3\pi}{13}\cdot\cos\frac{4\pi}{13}\right).$$

72. $\dfrac{\tan\theta-\tan\phi}{\tan\theta+\tan\phi}=\dfrac{\sin\theta\cos\phi-\cos\theta\sin\phi}{\sin\theta\cos\phi+\cos\theta\sin\phi}=\dfrac{\sin(\theta-\phi)}{\sin(\theta+\phi)}$;

$$\therefore \text{ from } \frac{\tan(\alpha+\beta-\gamma)-\tan(\alpha-\beta+\gamma)}{\tan(\alpha+\beta-\gamma)+\tan(\alpha-\beta+\gamma)}=\frac{\tan\gamma-\tan\beta}{\tan\gamma+\tan\beta},$$

we have

$$\frac{\sin(2\beta-2\gamma)}{\sin2\alpha}=\frac{\sin(\gamma-\beta)}{\sin(\gamma+\beta)};$$

$\therefore$ either $\sin(\beta-\gamma)=0$ or $2\cos(\beta-\gamma)\sin(\gamma+\beta)=-\sin2\alpha$, that is $\sin2\beta+\sin2\gamma=-\sin2\alpha$.

73. Let A′, B′ be projections of A, B on horizontal plane through S, so that $\angle$A′SB′ $=\gamma$. Let SD′ be altitude of $\triangle$SA′B′. If AA′ $=h$, SA′ $=h\cot\alpha$, SB′ $=h\cot\beta$, SD′ $=h\cot\phi$;

$$\therefore h\cot\phi\,.\,A'B'=2\triangle SA'B'=h^2\cot\alpha\cot\beta\sin\gamma;$$

also

$$A'B'^2=h^2\cot^2\alpha+h^2\cot^2\beta-2h\cot\alpha\,.\,h\cot\beta\,.\cos\gamma.$$

Eliminate A′B′.

74. Make BCS′ $=60°$ with S′ on AB produced; let BS, CS′ cut at O; then TBCO is a kite, also $\angle$TBS $=180°-80°-60°=40°$; $\therefore$ $\angle$BTC $=50°$; $\therefore$ $\angle$BTO $=100°$; $\therefore$ $\angle$S′TO $=80°$; also $\angle$TOS′ $=180°-\angle$TOC $=180°-\angle$TBC $=80°$;

$$\therefore \text{S′T} = \text{S′O} = \text{S′S, but } \angle\text{TS′S}=80°;$$

$$\therefore \angle\text{TSS′}=\tfrac{1}{2}(180°-80°).$$

75. O is centre of given circle, diameter COD, CD $=2r$; let PQ be an arc, centre C which bisects the area of the semicircle CPD and meets the circle and its diameter at P, Q. Let POD $=\phi$, then PCD $=\frac{1}{2}\phi$, CP $=2r\cos\frac{1}{2}\phi$. Also sector C(PQ) $-\triangle$OPC $=$sector O(PD) $-$area bounded by QD and arcs PQ, PD, that is, sector O(PD) $-\frac{1}{4}\pi r^2$;

$$\therefore\ \tfrac{1}{4}\phi(2r\cos\tfrac{1}{2}\phi)^2-\tfrac{1}{2}r^2\sin\phi=\tfrac{1}{2}\phi r^2-\tfrac{1}{4}\pi r^2;$$

$$\therefore\ \phi(1+\cos\phi)-\sin\phi=\phi-\frac{\pi}{2};$$

$$\therefore\ \sin\phi-\phi\cos\phi=\frac{\pi}{2}.$$

The intersection of the graphs of $\sin\phi$ and $\phi\cos\phi+\frac{\pi}{2}$ gives $\phi\simeq 1{\cdot}9$; put $\phi=1{\cdot}9+\alpha$ then it is found by the method of p. 82 that $\alpha=0{\cdot}007$, hence

$$\phi=1{\cdot}907 \text{ radians}=109^\circ{\cdot}26. \quad \text{CP}=2r\cos\tfrac{1}{2}\phi=(1{\cdot}158)r.$$

76. $u_n-nu_{n-1}=u_{n-1}-(n-1)u_{n-2}$; write $n-1$ for n;

$$\therefore\ u_{n-1}-(n-1)u_{n-2}=u_{n-2}-(n-2)u_{n-3}$$
$$=\text{similarly } u_{n-3}-(n-3)u_{n-4}, \text{ etc.}\dots$$
$$=u_2-2u_1=k\text{(say)};$$

$$\therefore\ \frac{u_n}{n!}-\frac{u_{n-1}}{(n-1)!}=\frac{k}{n!};\ \text{write } n-1 \text{ for } n;$$

$$\therefore\ \frac{u_{n-1}}{(n-1)!}-\frac{u_{n-2}}{(n-2)!}=\frac{k}{(n-1)!},\ \text{etc.}\dots$$

down to $\frac{u_2}{2!}-\frac{u_1}{1!}=\frac{k}{2!}$; add;

$$\therefore\ \frac{u_n}{n!}-\frac{u_1}{1!}=k\left(\frac{1}{n!}+\frac{1}{(n-1)!}+\dots+\frac{1}{2!}\right);$$

$$\therefore\ \lim_{n\to\infty}\frac{u_n}{n!}=u_1+k(e-2)=u_1+(u_2-2u_1)(e-2).$$

77. Let $\overline{\text{OA}}$, $\overline{\text{OB}}$ represent the complex numbers α, β. On OA, OB take points U, V representing $t\alpha$ and $t^2\beta$. Complete parallelogram UOVP, then $\overline{\text{OP}}$ represents $\alpha t+\beta t^2$. But $\frac{\text{PV}^2}{\text{OV}}=\frac{t^2\,.\,(|\alpha|)^2}{t^2\,.\,|\beta|}=$constant; $\therefore$ locus of P is a parabola, touching OA at O and having OB as diameter.

Note.—If $\alpha=\pm i\beta$, $\angle$AOB $=90^\circ$ and O is the vertex and OB the principal axis of the parabola.

78. $$\frac{1-x^4}{1-x^6}=(1-x^4)\left(1+x^6+x^{12}+\dots+x^{6r-6}+\frac{x^{6r}}{1-x^6}\right)$$
$$=1-x^4+x^6-x^{10}+\dots+x^{6r-6}-x^{6r-2}+k,\ \text{where}$$
$$k=\frac{x^{6r}(1-x^4)}{1-x^6};$$

$$\therefore\ \int_0^1\frac{1-x^4}{1-x^6}dx=1-\tfrac{1}{5}+\tfrac{1}{7}-\dots+\frac{1}{6r-5}-\frac{1}{6r-1}+\int_0^1 k\,dx.$$

Now

$$\int_0^1\frac{1-x^4}{1-x^6}dx=\int_0^1\frac{1+x^2}{1+x^2+x^4}dx=\tfrac{1}{2}\int_0^1\frac{dx}{1-x+x^2}+\tfrac{1}{2}\int_0^1\frac{dx}{1+x+x^2}$$
$$=\tfrac{1}{2}\,\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2x-1}{\sqrt{3}}+\tan^{-1}\frac{2x+1}{\sqrt{3}}\right]_0^1=\frac{1}{\sqrt{3}}\left(\frac{\pi}{6}+\frac{\pi}{3}\right)=\frac{\pi}{2\sqrt{3}},$$

and $$\int_0^1 k\,dx=\int_0^1\frac{x^{6r}(1+x^2)}{1+x^2+x^4}dx<\int_0^1 x^{6r}\,dx=\frac{1}{6r+1}$$

which $\to 0$ as $r\to\infty$;

$$\therefore\ \lim_{r\to\infty}\left(1-\tfrac{1}{5}+\tfrac{1}{7}-\dots-\frac{1}{6r-1}\right)=\frac{\pi}{2\sqrt{3}}.$$

Or, from p. 248, for $-\pi<\theta<\pi$,

$$\sin\theta-\tfrac{1}{2}\sin 2\theta+\tfrac{1}{3}\sin 3\theta-\dots \text{ equals } \tfrac{1}{2}\theta. \quad \text{Put } \theta=\frac{\pi}{3};$$

then $$\frac{\pi}{6}=\frac{\sqrt{3}}{2}(1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{8}+\tfrac{1}{10}-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{14}+\dots);$$

$$\therefore\ \frac{\pi}{3\sqrt{3}}=(1-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{17}+\dots)$$
$$-\tfrac{1}{2}(1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}\dots)$$
$$=\text{given series}-\tfrac{1}{2}\cdot\frac{\pi}{3\sqrt{3}};$$

$$\therefore\ \text{given series}=\frac{\pi}{3\sqrt{3}}+\frac{\pi}{6\sqrt{3}}.$$

79. Expression $=\Sigma\sec\frac{(2r-1)\pi}{11}$ for $r=1$ to 5, since

$$-\sec\frac{2\pi}{11}=\sec\frac{9\pi}{11},\ \text{etc.}$$

Now $\dfrac{1+\cos 11\theta}{1+\cos\theta}$ is the square of a polynomial in $\cos\theta$ of degree 5, since it

$$=\frac{2\cos^2\frac{11\theta}{2}}{2\cos^2\frac{\theta}{2}}=\left\{\frac{2\cos\frac{11\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right\}^2=\left(\frac{\cos 6\theta+\cos 5\theta}{1+\cos\theta}\right)^2;$$

$\therefore$ the roots of $\cos 11\theta+1=0$ are $\cos\theta=-1$ and

$$\cos\theta=\cos\frac{(2r-1)\pi}{11}\text{ for } r=1 \text{ to } 5,$$

the last five roots being repeated. From IX. e, No. 15, p. 183, $1+\cos 11\theta=1-11x+ax^3+\ldots+bx^{11}$, where $x=\cos\theta$,

$$\therefore \frac{1+\cos 11\theta}{1+\cos\theta}=1-12x+kx^2+\ldots+lx^{10};$$

$\therefore$ given expression $=$ half sum of reciprocals of roots $=\frac{1}{2}$ of 12.

80. x and y are the negative roots of $t^3=9(t+1)$; put $t=2\sqrt{3}\,.\cos\theta$, then $2\sqrt{3}\,.\cos\theta(12\cos^2\theta-9)=9$;

$$\therefore \cos 3\theta=\frac{\sqrt{3}}{2}=\cos 30^\circ;$$

$$\therefore \cos\theta=\cos 10^\circ,\ -\cos 50^\circ,\ \text{or}\ -\cos 70^\circ;$$

but $x<y<0$; $\therefore x=-2\sqrt{3}\,.\cos 50^\circ$, $y=-2\sqrt{3}\,.\cos 70^\circ$. Also $\cos 70^\circ=-\cos(60^\circ+50^\circ)=\frac{\sqrt{3}}{2}\sin 50^\circ-\frac{1}{2}\cos 50^\circ$, thus

$$7+x=7-2\sqrt{3}\,.\cos 50^\circ=7+4\sqrt{3}\left(\cos 70^\circ-\frac{\sqrt{3}}{2}\cos 40^\circ\right)$$

$$=7+4\sqrt{3}\,.\cos 70^\circ-6\cos 40^\circ$$

$$=1+4\sqrt{3}\cos 70^\circ+12\sin^2 20^\circ$$

$$=(1+2\sqrt{3}\cos 70^\circ)^2=(1-y)^2.$$